COMPLETELY SYMMETRIC CONFIGURATIONS FOR σ-GAMES ON GRID GRAPHS

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Abstract. The paper deals with σ-games on grid graphs (in dimension 2 and more) and conditions under which any completely symmetric configuration of lit vertices can be reached – in particular the completely lit configuration – when starting with the all-unlit configuration. The answer is complete in dimension 2. In dimension ≥ 3, the answer is complete for the σ^+-game, and for the σ^--game if at least one of the sizes is even. The case σ^-, dimension ≥ 3 and all sizes odd remains open.

Introduction

A nice combinatorial game is the following. Suppose you have a graph whose vertices can be *lit* or *unlit* (equivalently *on* or *off*). When you push on a vertex, its state as well as the state of its neighbors changes. This kind of game is called a σ^+*-game*. A *configuration* of such a game played on a graph $G = (V, E)$ is an element of $\mathbb{F}_2^V$, whose v-th coordinate is 0 if the vertex v is unlit and 1 otherwise.

You start with the all-off configuration. Can you find a sequence of pushes such that you get the all-on configuration? The rather unexpected answer is that it is always possible to find such a sequence. Indeed Sutner proved [6]

Theorem 0.1 (Sutner's theorem)**.** *The all-on configuration can always be achieved starting from the all-off configuration for a σ^+-game on any graph $G = (V, E)$.*

It is possible to define a similar game, the σ^-*-game*, for which pushing on a vertex changes the state of all its neighbors but not its own state. In this case, things become harder since it is not always possible to find a sequence achieving the all-on configuration when starting from the all-off configuration. Simple examples are provided by complete graphs with an odd number of vertices, paths of odd length, etc.

σ-games have been intensively studied, and it not possible to give here the whole list of references on this topic (see the article [5] for an extensive bibliography). Here we focus on the case when the graph is a grid graph. Note that σ-games on grid graphs have already been studied ([4] or [1], among many others) but for other questions (for instance, the number of distinct configurations that can be reached from a given one). Usually, two kinds of neighborhood are considered for the grid graph: if the grid graph is seen as a chessboard (the squares being the vertices), two squares sharing a common edge are neighbors; depending whether two squares in contact by their corners are or are not declared to be neighbors, we get one or the other kind of neighborhood. The first kind of neighborhood is denoted by $\square$ and the second one by $\boxtimes$. See Figure 1. We will consider these kinds of neighborhood, but also many others.

In 2002, the French magazine "Pour la Science" published an article written by Jean-Paul Delahaye and dealing with the σ^--game on grid graphs [2] (see also an updated version of this article

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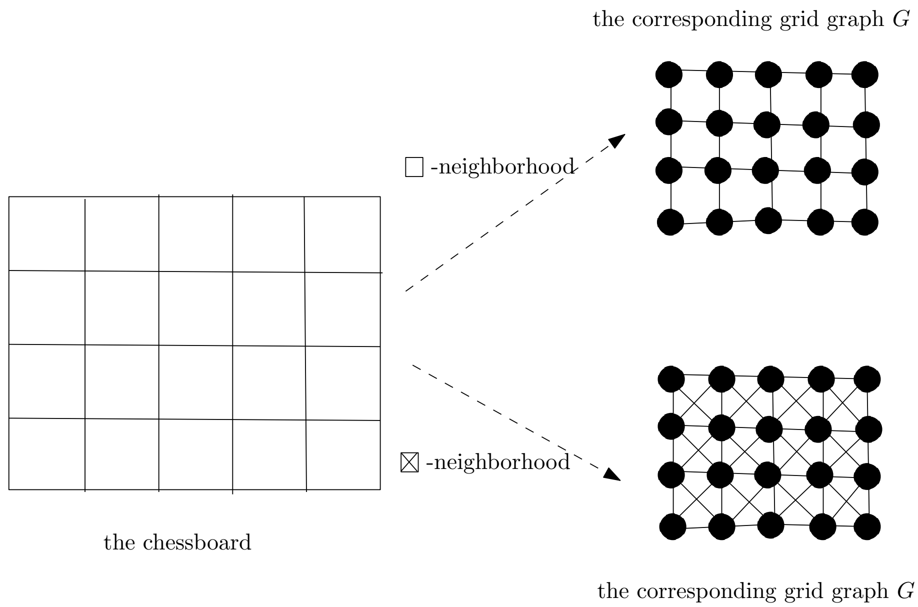

FIGURE 1. The two usual kinds of neighborhood used for playing a σ-game on a chessboard – they lead to two distinct grid graphs.

in the book [3]). The game was defined on a chessboard and the neighbors of a square were the adjacent squares, having a corner in common being enough to be neighbor. Hence a square could have 8, 5 or 3 neighbors, depending whether the square was or was not on the border or in the corner of the chessboard (except if one dimension of the chessboard is 1, in which case the number of neighbors is 2 or 1). We are here precisely in the case of the ⊠-neighborhood.

In this article, a conjecture of a reader – Nicolas Vaillant – was proposed. Recall that the 2-valuation of a number n is the largest j such that 2^j is a divisor of n.

Conjecture (Vaillant's conjecture): The all-on configuration cannot be achieved starting from the all-off configuration for a σ^--game played on an $n \times m$ chessboard if and only if n and m are odd and such that $m+1$ and $n+1$ have the same 2-valuation.

In the present paper, we prove a general theorem (Theorem 2.8, Section 2) that gives a necessary and sufficient condition for a σ-game played on an $n \times m$ chessboard to be such that any doubly symmetric configuration can be achieved. The approach will be purely algebraic. Vaillant's conjecture is a consequence of this theorem. In particular, we get a sufficient and necessary condition for the all-on configuration to be achieved by a σ^--game played on the grid. Such existence results, common for σ^+-games, are extremely rare for σ^--games.

As another application, we obtain

Any doubly symmetric configuration can be achieved starting from the all-off configuration for a σ^+-game played on a chessboard for both the □- and the ⊠-neighborhoods.

A doubly symmetric configuration is a configuration that is invariant by the symmetries with respect to the two medians of the sides of the chessboard. Let us be more precise. Each vertex is identified with a $(i,j) \in \{0,\dots,n-1\} \times \{0,\dots,m-1\}$.

Definition 0.2. *A configuration $Y = (y_{i,j})$ on a grid graph $n \times m$ is said to be* doubly symmetric *if*

$$y_{i,j} = y_{n-1-i,j} = y_{i,m-1-j} = y_{n-1-i,m-1-j} \quad \textit{for all } i,j.$$

The statements above (Vaillant's conjecture and the one concerning doubly symmetric configurations for σ^+-games) are not only true for the two usual kinds of neighborhood ($\square$ and $\boxtimes$), but also for many others.

The paper also deals with the case of $n_1 \times n_2 \times \ldots \times n_d$ grids, where $d \geq 3$. We then speak about completely symmetric configurations. Each vertex is identified with a $(i_1, \ldots, i_d) \in \Pi_{j=1}^{d}\{0, \ldots, n_j - 1\}$.

Definition 0.3. *A configuration* $Y = (y_{j_1,\ldots,j_d})$ *on a d-dimensional grid graph is said to be* completely symmetric *if*

$$y_{j_1,\ldots,j_{i-1},j_i,j_{i+1},\ldots,j_d} = y_{j_1,\ldots,j_{i-1},n_i-1-j_i,j_{i+1},\ldots,j_d} \qquad \text{for all } i, j_1, j_2, \ldots, j_d.$$

We will then prove the following result in Section 3 (in a slightly more general form, Theorem 3.1), but with a different approach than that of Section 2:

Any completely symmetric configuration can be achieved starting from the all-off configuration for a σ^+-game played on a d-dimensional grid for both the $\square$- and the $\boxtimes$-neighborhoods..

It is also proved for many other kinds of neighborhoods. The result also holds in the case of σ^--game when at least one of the n_i is even. Note that the question whether there is a simple condition for the existence of a completely symmetric configuration when all dimensions are odd for the σ^--game remains unsettled (when $d \geq 3$, since the $d = 2$ case is settled in this paper). Maybe this is due to the lack of an algebraic approach for this case.

1. Basic notions and notation

Throughout this paper, we shall denote by k a field. It will be of characteristic 2 starting from Subsection 2.2. We denote by $\overline{\mathbb{F}}_2$ an algebraic closure of $\mathbb{F}_2$. For $n \geq 1$, denote by J_n the $n \times n$ matrix (with coefficients in k)

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & \ddots & \ddots & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

with 1's directly above and under the diagonal, and 0's everywhere else.

A *game* G on the $n \times m$ grid (the squares of which can be lit or unlit) is given by the following. To each vertex v of the grid, we associate a set of vertices whose state change if we push on the vertex v. Equivalently, one may give a $nm \times nm$ matrix M with coefficients in $\mathbb{F}_2$ (the field with two elements), defined by the following property: let $1 \leq i, k \leq n$ and $1 \leq j, l \leq m$ be integers. Then the coefficient of the (i, j)-th column and the (k, l)-th line of M is 0 if pressing on the vertex (i, j) does not change the state of the vertex (k, l), and 1 otherwise. We call M the *generalized adjacency matrix* of the game G. We will often assume that M commutes with $J_n \otimes I_m$ and $I_n \otimes J_m$. The matrix M can then be written as a sum of scalar multiples of $J_n^r \otimes J_m^s$, where r and s are in $\mathbb{N}$ (cf. Lemma 2.5). For instance a σ^--game played on a $n \times m$ chessboard with a $\boxtimes$-neighborhood has a generalized adjacency matrix $M = J_n \otimes J_m + J_n \otimes I_m + I_n \otimes J_m$ (which of course commutes with $J_n \otimes I_m$ and $I_n \otimes J_m$). Similarly, a σ^+-game played on a $n \times m$ chessboard with a $\square$-neighborhood has a generalized adjacency matrix $M = I_n \otimes I_m + J_n \otimes I_m + I_n \otimes J_m$ (which of course commutes with $J_n \otimes I_m$ and $I_n \otimes J_m$). In the sequel, we will often denote the matrix $J_n \otimes I_m$ simply by J_n when no confusion can arise.

All games will be assumed to be symmetric, i.e. that pressing on the vertex (i, j) changes the state of the vertex (i', j') if and only if pressing on the vertex (i', j') changes the state of the vertex (i, j).

Using the generalized adjacency matrix M, to say that a configuration can be achieved by the game G starting from the all-off configuration is equivalent to say that this configuration (or more precisely the column vector of size nm associated to it) is in the image of M. We shall repeatedly use this elementary remark without further mention.

We extend all these notions also for grids of dimensions ≥ 3.

Let us also recall the classical proof of Sutner's theorem (Theorem 0.1), which is valid for any kind of graph. The following lemma is an elementary result.

Lemma 1.1. *Let U be a finite dimensional linear space over k, endowed with a symmetric non-degenerate bilinear form, and let ϕ be a self-adjoint endomorphism $U \to U$. We have*

$$\mathrm{Im}\ \phi = (\mathrm{Ker}\ \phi)^{\perp}.$$

Theorem 0.1 is a straightforward consequence of Lemma 1.1, applied to $k = \mathbb{F}_2$, $U = \mathbb{F}_2^V$ and to ϕ being the adjacency matrix of G plus the identity matrix. Indeed, it is then enough to prove that if we push on a subset S of vertices that keeps the configuration in the all-off state, then S has cardinality even. But this is obvious since each vertex of $G[S]$ must be of odd degree (otherwise some vertices of S would be on) and since the number of odd degree vertices in any graph is always even.

We can reformulate this last sentence as a lemma, which will be useful in the proof of Theorem 3.1, in the particular case of grid graphs. The matrix M is the 'generalized adjacency matrix' of the game, defined in the beginning of this section.

Lemma 1.2. *In the case of a σ^+-game, the number of nonzero entries of any element of* Ker M *is even.*

2. Doubly symmetric configurations on chessboards (or 2-dimensional grids)

2.1. **Some algebra.** In this section, we introduce the technical material needed in the proof of the main theorem (Theorem 2.8, Section 2).
Let us begin with a key lemma.

Let $\mathcal{A}$ be a factorial ring, and $p \in \mathcal{A}$ a prime element. For any nonzero $x \in \mathcal{A}$, we denote by $v_p(x)$ the highest power of p dividing x.

Lemma 2.1. *Let p, q, r and s be nonnegative integers. Consider the (local) k-algebras $A = k[X]/X^p$ and $B = k[Y]/Y^q$. Denote by x (resp. by y) the class of X (resp. of Y) in A (resp. in B). In $A \otimes_k B$, we still denote by x the element $x \otimes 1$, and similarly for y. Let $u = x - y \in A \otimes_k B$. Then the element $x^r y^s$ is divisible by u if and only if $r + s \geq \inf\{p, q\}$. What is more, the same statement holds if we replace u by $cx + dy +$ terms of order at least 2, where c, d are nonzero elements of k.*

Proof. Assume that $p \leq q$. Suppose that r is positive. From the relation $(x - y)(x^r y^{s-1}) = x^{r+1}y^{s-1} - x^r y^s$, we deduce that $x^r y^s$ is divisible by $x - y$ if and only if $x^{r+1}y^{s-1}$ is so. Thus, we are reduced to the case where $s = 0$. If $r \geq \inf\{p, q\} = p$, then $x^r = 0$, so one implication of the statement is obvious. Conversely, assume that x^r is divisible by u. Write $x^r = (x - y)v$, for some v in $A \otimes_k B$. We have a morphism

$$f : A \otimes_k B \longrightarrow A,$$

$$x \mapsto x,$$

$$y \mapsto x.$$

Applying f to the previous equality, we get $x^r = 0$, hence $r \geq p$, qed. For the last assertion, assume that $u = cx + dy + \lambda x + \mu y$, where λ, μ lie in the maximal ideal M of $A \otimes_k B$. Clearly, we may assume that $c = d = 1$. Put $x' = x(1+\lambda)$ and $y' = -y(1+\mu)$. Then $u = x' - y'$ and $k[x', y']$ equals $A \otimes_k B$. Indeed, the obvious map

$$\begin{aligned} f : A \otimes_k B &\longrightarrow k[x', y'], \\ x &\mapsto x', \\ y &\mapsto y' \end{aligned}$$

is injective, hence an isomorphism by dimension reasons. Note that injectivity can be seen the following way: if a is an element of M^n (where n is a positive integer, and M denotes as before the maximal ideal of $A \otimes_k B$), we have $f(a) = a$ modulo M^{n+1}. The second result of the lemma now follows by an application of the preceding statement to x' and y'. Indeed, $x^r y^s$ is divisible by u if and only if $x'^r y'^s$ is so.

□

We can now state and prove the main proposition of this subsection.

Proposition 2.2. *Assume that k is algebraically closed. Let P, Q, R, S be four polynomials (in $k[X]$). Consider the k-algebras $A = k[X]/P$ and $B = k[Y]/Q$. Denote by x (resp. by y) the class of X (resp. of Y) in A (resp. in B). Let U be an element of $k[X, Y]$. Put $u = U(x, y) \in A \otimes_k B$. Assume the following: for every $\alpha, \beta \in k$ such that $P(\alpha) = Q(\beta) = 0$ and $U(\alpha, \beta) = 0$, we have that $\frac{\partial U}{\partial X}(\alpha, \beta) \neq 0$ and that $\frac{\partial U}{\partial Y}(\alpha, \beta) \neq 0$. Then u divides $R(x)S(y)$ if and only if the following holds: for every α, β as above, denote by p (resp. q, r, s) the multiplicity of α (resp. β, α, β) as a root of P (resp. Q, R, S). Then $r + s \geq inf\{p, q\}$.*

Proof. Write $P = (X - \alpha_1)^{m_1} ... (X - \alpha_d)^{m_d}$. The Chinese Remainder Theorem ensures that the natural morphism

$$A \longrightarrow k[X]/(X - \alpha_1)^{m_1} \times ... \times k[X]/(X - \alpha_d)^{m_d}$$

is an isomorphism. Using the similar isomorphism for B, we are immediately reduced to the case where $P = (X - \alpha)^p$ and $Q = (Y - \beta)^q$. If $U(\alpha, \beta) \neq 0$, then u is invertible in $A \otimes_k B$, hence the proposition is true in this case. If $U(\alpha, \beta) = 0$, then replacing P by $P(X+\alpha)$ and Q by $Q(Y+\beta)$, we may assume that $\alpha = \beta = 0$, i.e. that $P = X^p$ and $Q = Y^q$. We may also assume that R and S are powers of X (indeed, if T is a polynomial such that $T(0) \neq 0$, then $T(x)$ (resp. $T(y)$) is invertible in A (resp. in B)). The content of the proposition then boils down to that of Lemma 2.1, since the hypothesis about partial derivatives ensures that u is of the form $cx + dy +$ higher order terms. □

Lemma 2.3. *Let $P, Q \in k[X]$ be two polynomials. Put $A = k[X]/P(X)$ and $B = k[Y]/Q(Y)$. Denote by x (resp. by y) the class of X (resp. of Y) in A (resp. in B). Let u be an element of $A \otimes_k B$. Let $U \in k[X, Y]$ be a polynomial such that $u = U(x, y)$. Assume that $\alpha \in k$ is a root of multiplicity ≥ 2 of P, and let $\beta \in k$ be any root of Q. Then the partial derivative $\frac{\partial U}{\partial X}(\alpha, \beta)$ is independent of the choice of U.*

Proof. Indeed, any other U' satisfying $u = U'(x, y)$ is of the form $U' = U + R(X, Y)P(X) + S(X, Y)Q(Y)$, and the hypothesis about α implies that $\frac{\partial U}{\partial X}(\alpha, \beta) = \frac{\partial U'}{\partial X}(\alpha, \beta)$. □

Definition 2.4. *Under the hypothesis of Lemma 2.3, we shall denote $\frac{\partial U}{\partial X}(\alpha, \beta)$, which is independent of the choice of U, by $\frac{\partial u}{\partial x}(\alpha, \beta)$.*

2.2. **Preliminaries on Chebychev polynomials.** Chebychev polynomials modulo 2 are classical tools in the study of σ-games on grid graphs in dimension 2 (see [7] for instance, or [4], where they are called Fibonacci polynomials). We recall in this subsection their definition and some of their properties.

2.2.1. *Classical Chebychev polynomials.* The usual Chebychev polynomials are elements of $\mathbb{Z}[X]$ defined as follows.
Set $P_0 = 2$ and $P_1 = X$. Then, define P_n inductively by the formula

$$P_{n+1} = XP_n + P_{n-1}.$$

This formula will be called the *Chebychev relation.*
The P_n's satisfy the following well-known properties, valid for all nonnegative integers n and m:

i) $P_n(X + X^{-1}) = X^n + X^{-n}$,
ii) $P_n P_m = P_{n+m} + P_{|n-m|}$.

Property i) in fact characterizes the Chebychev polynomials, and ii) is an easy consequence of i).

2.2.2. *Chebychev polynomials modulo* 2. From now on, k will be assumed to have characteristic 2. It is readily seen that all P_n's are divisible by X modulo 2. For a nonnegative integer n, we thus define Q_n to be the reduction modulo 2 of $\frac{P_{n+1}}{X}$, seen as an element of $k[X]$. Note that the Q_n's also satisfy the Chebychev relation. We shall now study some elementary divisibility properties of these polynomials. First of all, an easy induction shows that Q_n is divisible by X if and only if n is odd. From point ii) of the preceding subsection, we have that $Q_{2n-1} = XQ_{n-1}^2$. This implies a formula useful in the sequel. Take an odd positive integer n. Write $n + 1 = 2^j\eta$, with η odd. The preceding formula, applied several times, then yields $Q_n = X^{2^j-1}Q_{\eta-1}^{2^j}$. Since $\eta - 1$ is even, we have that $Q_{\eta-1}$ is not divisible by X, hence the relation:

$$v_X(Q_n) = 2^j - 1.$$

We also get the following. Let n be odd, and $R \neq X$ be a (monic) prime polynomial dividing Q_n. From the relation $Q_n = XQ_{\frac{n-1}{2}}^2$, we infer that

$$v_R(Q_n) = 2v_R(Q_{\frac{n-1}{2}})$$

and

$$v_X(Q_n) = 1 + 2v_X(Q_{\frac{n-1}{2}}).$$

Those relations are basically the only facts we shall need about Chebychev polynomials.

2.3. **Statement and proof of the main theorem.** It is an elementary exercise to check that the characteristic polynomial of J_n is Q_n. Let e_i $(i = 0, \dots, n-1)$ denote the i'th basis vector of k^n. Consider the linear map

$$\begin{array}{rccc} \Phi_n : & k[X]/Q_n & \longrightarrow & k^n \\ & X^i + (Q_n) & \longmapsto & J_n^i(e_0). \end{array}$$

This map is well-defined by the Cayley-Hamilton theorem, since Q_n is the characteristic polynomial of J_n. It is readily checked that it is surjective, hence an isomorphism (this amounts to saying that the characteristic and minimal polynomials of J_n coincide). In the sequel, we will *identify* k^n with $k[X]/Q_n$ using Φ_n. We shall denote by x_n the class of X in $k[X]/Q_n$. One sees that, under the isomorphism given by Φ_n, e_i corresponds to $Q_i(x_n)$. Furthermore, the action of J_n on $k[X]/Q_n$ is simply given by multiplication by x_n.

Lemma 2.5. *Let f be an endomorphism of the k-vector space $k[X]/Q_n \otimes_k k[Y]/Q_m$ commuting with multiplication by x_n and y_m (more precisely, $x_n \otimes 1$ and $1 \otimes y_m$). Then f is given by multiplication by $f(1)$.*

Proof. Easy verification. □

Definition 2.6. *(central configuration) The central configuration of the $n \times m$ grid is defined the following way: put $c_n := Q_{\frac{n-1}{2}}(x_n)$ if n is odd, $c_n := Q_{\frac{n}{2}}(x_n) + Q_{\frac{n}{2}-1}(x_n)$ if n is even, and define d_m similarly with respect to y_m. Then the central configuration is $c = c_n d_m$. It consists in the central square if n and m are both odd, in the 2 central squares if exactly one of the two integers n and m is odd, and in the 4 central squares if n and m are both even.*

The central configuration is of course doubly symmetric.

Lemma 2.7. *Every doubly symmetric configuration in $k[X]/Q_n \otimes_k k[Y]/Q_m$ is divisible by the central one. Moreover, if n and m are both odd, then the central configuration is divisible by the all-on configuration.*

Proof. Let us prove the first assertion. It suffices to show that $e_{n-1-i}+e_i = Q_{n-1-i}(x_n)+Q_i(x_n)$ is divisible by c_n in $k[X]/Q_n(X)$ for any $n \geq 1$ and any integer i satisfying $0 \leq i \leq \lceil n/2 \rceil - 1$. This is an easy descending induction on i, using the relation $Q_{k+1} = XQ_k + Q_{k-1}$. Let us now handle the second assertion. It suffices to prove it for the 1-dimensional case. The all-on configuration is then $\sum_{i=0,\ldots,n-1} Q_i(x_n)$. Let us compute this sum. Put $R_i(X) = Q_0(X) + Q_1(X) + \ldots + Q_i(X)$. One has the relation $R_{i+1} = XR_i + R_{i-1} + 1$. From this, we infer that the polynomials $S_i := XR_i + 1$ satisfy the Chebychev relations. Hence S_i can be expressed as a linear combination of Q_i and Q_{i+1}. Since $S_0 = X + 1 = Q_0 + Q_1$ and $S_1 = X^2 + X + 1 = Q_1 + Q_2$, we find that $S_i = Q_i + Q_{i+1}$ for all i. We thus have

$$R_{n-1} = \frac{Q_{n-1} + Q_n + 1}{X} = \frac{P_n + P_{n+1} + X}{X^2} = \frac{P_{\frac{n-1}{2}} P_{\frac{n+1}{2}} + P_{\frac{n+1}{2}} P_{\frac{n+1}{2}}}{X^2},$$

where the last equality follows from property ii) of Subsection 2.2.1. Rewriting the last expression in terms of the Q_i's, we have $R_{n-1} = Q_{\frac{n-3}{2}} Q_{\frac{n-1}{2}} + Q^2_{\frac{n-1}{2}}$. Hence the all-on configuration equals $Q_{\frac{n-1}{2}}(x_n)(Q_{\frac{n-1}{2}}(x_n) + Q_{\frac{n-3}{2}}(x_n)) = c_n(c_n + Q_{\frac{n-3}{2}}(x_n))$. It is enough to show that $Q_{\frac{n-1}{2}}(x_n) + Q_{\frac{n-3}{2}}(x_n)$ is invertible in $k[X]/Q_n(X)$, i.e. that $Q_{\frac{n-1}{2}}(X) + Q_{\frac{n-3}{2}}(X)$ and $Q_n(X) = XQ^2_{\frac{n-1}{2}}(X)$ are coprime. Let T be a monic prime polynomial dividing these two polynomials. Certainly T is not X since the constant term of $Q_{\frac{n-1}{2}}(X) + Q_{\frac{n-3}{2}}(X)$ is 1. But then T divides both $Q_{\frac{n-1}{2}}(X)$ and $Q_{\frac{n-3}{2}}(X)$, which are coprime. □

Theorem 2.8. *Let n and m be integers. Let G be a game on the $n \times m$ grid (identified with $\mathbb{F}_2[X]/Q_n \otimes_{\mathbb{F}_2} \mathbb{F}_2[Y]/Q_m$) which commutes with the elementary games J_n and J_m. Let f be the endomorphism of $\mathbb{F}_2[X]/Q_n \otimes_{\mathbb{F}_2} \mathbb{F}_2[Y]/Q_m$ given by G – it corresponds to the generalized adjacency matrix M of G. By Lemma 2.5, f is then given by multiplication by $u := f(1)$. For any $\alpha \in \overline{\mathbb{F}}_2$ (resp. $\beta \in \overline{\mathbb{F}}_2$), which is a root of Q_n (resp. Q_m), such that $u(\alpha, \beta) = 0$, we assume the following. If α (resp. β) is a root of multiplicity ≥ 2 of Q_n (resp. Q_m), then $\frac{\partial u}{\partial x}(\alpha, \beta) \neq 0$ (resp. $\frac{\partial u}{\partial y}(\alpha, \beta) \neq 0$) (these quantities are well-defined thanks to Lemma 2.3).*
Then any doubly symmetric configuration can be achieved starting from the all-off configuration for the game G if and only if the three following conditions do not simultaneously hold: n and m are both odd, $u(0, 0) = 0$ and $v_2(n + 1) = v_2(m + 1)$.

Remark 2.9. In the case where n and m are both odd, the fact that any doubly symmetric configuration can be achieved is equivalent to the fact that the all-on configuration can be achieved; this is the content of Lemma 2.7.

Remark 2.10. Let us be more precise concerning how the theorem implies Vaillant's conjecture, stated in the beginning of the paper. This is the $\boxtimes$-case

$$u = y_m + x_n + y_m x_n.$$

The condition about partial derivatives is here obvious: indeed, put $U = X + Y + XY$. We have $\frac{\partial U}{\partial X} = 1 + Y$, so that the condition may fail only for $\beta = 1$. But $U(X, 1) = 1$ has no root in $\overline{\mathbb{F}}_2$.

For the $\square$-case

$$u := y_m + x_n,$$

we see that the conclusion is identical.

In the case of a σ^+-game with the usual neighborhoods defined in the introduction (the $\square$ and the $\boxtimes$ neighborhoods), i.e. when $u := y_m + x_n + 1$ or $u = y_m + x_n + y_m x_n + 1$, we have $u(0, 0) = 1$. Hence, any doubly symmetric configuration can always be achieved.

Proof.[Proof of Theorem 2.8] Put $k = \mathbb{F}_2$. Let $A = k[X]/Q_n$ and $B = k[Y]/Q_m$. By Lemma 2.7, it is enough to show that the central configuration can be obtained. Over fields, the formation of the image of a linear map commutes with scalar extension. Thus, there is no harm in replacing k by an algebraic closure of k. Let $U \in k[X, Y]$ be such that $U(x_n, y_m) = u$. Assume the hypothesis about partial derivatives holds. If α is a simple root of Q_n, then $Q_n'(\alpha) \neq 0$. Because k is infinite, we can then replace U by $U + \lambda Q_n(X)$, for a suitable $\lambda \in k$, in such a way that the partial derivatives of U with respect to X are nonzero when evaluated at (α, β), where α runs through the simple roots of Q_n and β through the roots of Q_m, submitted to the condition $u(\alpha, \beta) = 0$. We can then do the same for partial derivatives with respect to Y. By doing so, we get a $U \in k[X, Y]$ such that $U(x_n, y_m) = u$ and $\frac{\partial U}{\partial X}(\alpha, \beta) \neq 0$, $\frac{\partial U}{\partial Y}(\alpha, \beta) \neq 0$ for any $\alpha \in k$ (resp. β) which is a root of Q_n (resp. Q_m) and such that $u(\alpha, \beta) = 0$. We now want to apply Proposition 2.2. Let α, $\beta \in k$ be as before. Put $P := Q_n$, $Q := Q_m$, $p := v_{X-\alpha}(P)$, $q := v_{Y-\beta}(Q)$. Put $R := Q_{\frac{n}{2}}(X) + Q_{\frac{n}{2}-1}(X)$ if n is even, $R := Q_{\frac{n-1}{2}}(X)$ if n is odd. Similarly, put $S := Q_{\frac{m}{2}}(Y) + Q_{\frac{m}{2}-1}(Y)$ if m is even, $S := Q_{\frac{m-1}{2}}(Y)$ if m is odd. Put $r := v_{X-\alpha}(R)$ and $s := v_{Y-\beta}(S)$. If n is even, we compute:

$$XR^2 = XQ_{\frac{n}{2}}(X)^2 + XQ_{\frac{n}{2}-1}(X)^2 =$$

$$Q_{n+1}(X) + Q_{n-1}(X) = XQ_n(X).$$

Hence $p = 2r$. If n is odd, we then have $1 + 2r = p$ if $\alpha = 0$ and $2r = p$ otherwise, as proved in Section 2.2. Similarly, we get relations between s and q. To finish the proof, we have to show that the relation $r + s \geq \inf\{p, q\}$ (for every α and β) is equivalent to the fact that the three conditions of the theorem do not simultaneously hold. Assume that n is even. If $s \geq r$, then $r + s \geq 2r = \inf\{p, q\}$. If $s < r$, then $r + s \geq 2s + 1 \geq q = \inf\{p, q\}$. Hence the relation is valid in this case. In the same way, it is valid if m is even. Assume now that n and m are both odd. If $\alpha \neq 0$, then $2r = p$, and we conclude as before that the relation holds. In the same way, it holds if $\beta \neq 0$. Assume now that $\alpha = \beta = 0$ (hence that $u(0, 0) = 0$). Then $1 + 2r = p$ and $1 + 2s = q$, hence the relation $r + s \geq \inf\{p, q\}$ holds if and only if p and q are distinct, which in view of Section 2.2 amounts to saying that $v_2(n + 1) \neq v_2(m + 1)$.

$\square$

3. Completely symmetric configurations on d-dimensional grids

We extend in this section some of the previous results.

Theorem 3.1. *Let G be a game on the $n_1 \times \ldots \times n_d$ grid that commutes with the elementary games J_{n_i} for $i = 1, \ldots, d$. If G is a σ^+-game, then any completely symmetric configuration can be achieved starting from the all-off configuration.*

Note that the dimension 2 case is also covered, giving an alternative proof of some statements already proved in the previous section for σ^+-games.

The following lemma plays a crucial role in the proof. In a sense, it proves the theorem above for the 1-dimensional case. Denote by S the map

$$\begin{array}{ccll} \mathbb{F}_2^n & \longrightarrow & \mathbb{F}_2^{\lceil n/2 \rceil} & \\ (y_0,\ldots,y_{n-1}) & \longmapsto & \begin{array}{c}(y_0+y_{n-1}, y_1+y_{n-2},\ldots,y_{n/2-1}+y_{n/2}) \\ (y_0+y_{n-1}, y_1+y_{n-2},\ldots,y_{(n-3)/2}+y_{(n+1)/2}, y_{(n-1)/2})\end{array} & \begin{array}{l}\text{if } n \text{ is even,}\\ \text{if } n \text{ is odd,}\end{array} \end{array}$$

and by c the map

$$\begin{array}{ccc} \mathbb{F}_2^n & \rightarrow & \mathbb{F}_2 \\ (y_0,\ldots,y_{n-1}) & \mapsto & \sum_{i=0}^{n-1} y_i \end{array}$$

Lemma 3.2. *Let F be a linear subspace of $\mathbb{F}_2^n$ such that $J_nF \subseteq F$ and $F \subseteq \text{Ker } c$. Then $F \subseteq \text{Ker } S$.*

$F \subseteq \text{Ker } c$ reads also *any element of F has an even number of nonzero entries.* Note the similarity with the statement of Lemma 1.2. Indeed, we will apply Lemma 3.2 to some linear spaces constructed from M.

Proof.[Proof of Lemma 3.2] Suppose that there is $\boldsymbol{x}$ in F with a i such that $x_i \neq x_{n-1-i}$. Since $J_n^r\boldsymbol{x} \in F$ for any positive integer r, we can assume that $i = 0$.

We have $J_n\boldsymbol{x} \in F$, hence $c(J_n\boldsymbol{x}) = 0$ and

$$x_1 + (x_0 + x_2) + \ldots + (x_{n-3} + x_{n-1}) + x_{n-2} = 0,$$

whence $x_0 + x_{n-1} = 0$, a contradiction.

Therefore, for all $\boldsymbol{x} \in F$ and all $i = 0,\ldots,n-1$, we have $x_i = x_{n-1-i}$.

When n is odd, $x_{\frac{n-1}{2}} = 0$ is then a direct consequence of the fact that any element of F has an even number of nonzero entries.

□

We now restate Lemma 3.2 in a slightly more general form, which will suit the proof scheme of Theorem 3.1.

Lemma 3.3. *Let V be any $\mathbb{F}_2$-vector space. Let F be a linear subspace of $V \otimes \mathbb{F}_2^n$ such that $(Id \otimes J_n)(F) \subseteq F$ and $F \subseteq \text{Ker } (Id \otimes c)$. Then $(Id \otimes S)(F) \subseteq V \otimes \mathbb{F}_2^{\lceil n/2 \rceil}$ is zero.*

Proof. Let ϕ be any linear form on V. The lemma is a direct consequence of Lemma 3.2 once we have noticed that $(\phi \otimes Id)(F) \subseteq \mathbb{F}_2^n$ satisfies the conditions of Lemma 3.2 for F, and that the following diagram commutes:

$$\begin{tikzcd} V \otimes \mathbb{F}_2^n \arrow[r, "Id\otimes S"] \arrow[d, "\phi\otimes Id"] & V \otimes \mathbb{F}_2^{\lceil n/2 \rceil} \arrow[d, "\phi\otimes Id"] \\ \mathbb{F}_2^n \arrow[r, "S"] & \mathbb{F}_2^{\lceil n/2 \rceil}. \end{tikzcd}$$

The vector space $(Id \otimes S)(F)$ is thus killed by all linear forms on V, hence is zero.

□

Proof.[Proof of Theorem 3.1] Let M be the generalized adjacency matrix of the game G. The hypothesis that M commutes with all J_{n_i}'s ensures that M is a linear combination of matrices of the form $J_{n_1}^{i_1} \otimes \ldots \otimes J_{n_d}^{i_d}$ – the proof of this fact is the same as that of Lemma 2.5. Define $N := \text{Ker } M$. Note that N is stable by all J_{n_i}. Denote by $S_i : \mathbb{F}_2^{n_i} \longrightarrow \mathbb{F}_2^{\lceil n_i/2 \rceil}$ (resp. $c_i : \mathbb{F}_2^{n_i} \longrightarrow \mathbb{F}_2$) the map defined in the same way as S (resp. c), for $n = n_i$.

We have the following property.

Let $i \in \{0, 1, \ldots, d\}$. Then

$$S_1 \otimes S_2 \otimes \ldots \otimes S_i \otimes c_{i+1} \otimes c_{i+2} \otimes \ldots \otimes c_d(N) = \{0\}.$$

Indeed, it is true for $i = 0$, according to Lemma 1.2.

The other cases are true by induction. Suppose that the property is true for $i \geq 0$. Define

$$F := S_1 \otimes S_2 \otimes \ldots \otimes S_i \otimes Id \otimes c_{i+2} \otimes c_{i+3} \otimes \ldots \otimes c_d(N).$$

F is a linear subspace of $V \otimes \mathbb{F}_2^{n_{i+1}}$ where $V = \mathbb{F}_2^{\lceil n_1/2 \rceil} \otimes \ldots \otimes \mathbb{F}_2^{\lceil n_i/2 \rceil}$. We have $(Id_V \otimes J_{n_{i+1}})F \subseteq F$, and, by induction, $(Id_V \otimes c_{i+1})(F) = \{0\}$. We apply Lemma 3.3 and get that $(Id_V \otimes S_{i+1})(F) = \{0\}$, which means exactly

$$S_1 \otimes S_2 \otimes \ldots \otimes S_{i+1} \otimes c_{i+2} \otimes c_{i+3} \otimes \ldots \otimes c_d(N) = \{0\}.$$

The statement of the theorem for the σ^+-game is a direct consequence of the property above for $i = d$: apply Lemma 1.1 to get that any completely symmetric configuration is in the image of M. □

The previous theorem has a nice corollary concerning σ^--games in any dimension.

Corollary 3.4. *Let G be a d-dimensional σ^--game on a $n_1 \times \ldots \times n_d$ grid, with n_1 even. Assume it can be written as a finite sum $\sum_{i_1,\ldots,i_d} \lambda_{i_1,\ldots,i_d} J_{n_1}^{i_1} \otimes \ldots \otimes J_{n_d}^{i_d}$ ($\lambda_{i_1,\ldots,i_d} \in \mathbb{F}_2$), where $\lambda_{1,0,\ldots,0} = 1$ and where all other $\lambda_{i_1,\ldots,i_d}$ equal 0 except possibly when at least one of the i_j, $j \geq 2$, equals 1. Then every completely symmetric configuration can be achieved starting from the all-off configuration.*

Proof. Define the game $M' := J_{n_1}^{-1} M$. The hypothesis of the corollary ensures that M' is the generalized adjacency matrix of a σ^+-game. Theorem 3.1 applies to M'. The result follows, for the space of completely symmetric configurations is stable by J_{n_1}. □

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