

On composition of torsors

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Abstract

Let K be a field, let X be a connected smooth K -scheme and let G, H be two smooth connected K -group schemes. Given $Y \rightarrow X$ a G -torsor and $Z \rightarrow Y$ an H -torsor, we study whether one can find an extension E of G by H so that the composite $Z \rightarrow X$ is an E -torsor. We give both positive and negative results, depending on the nature of the groups G and H .

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1. Introduction

Consider a field K , a smooth¹ connected K -scheme X and two smooth connected K -group schemes G and H . In the present article, we are interested in the following question about compositions of torsors:

Question 1.1. Let $Y \rightarrow X$ be a G -torsor, and let $Z \rightarrow Y$ be an H -torsor. Can one find an extension of K -group schemes

$$1 \rightarrow H \rightarrow E \rightarrow G \rightarrow 1,$$

together with an E -torsor structure on the composite $Z \rightarrow X$, such that the following holds.

- The action of E on Z extends that of H .
- The G -torsors $Z/H \rightarrow X$ and $Y \rightarrow X$ are isomorphic.

Of course, one does not expect to get a positive answer to this question in full generality. The goal of the article is to give both positive and negative results, depending on the nature of the groups G and H .

Particular cases of Question 1.1 have been considered in [HS05], [BD13], [BDLM20] and [ILA21], as well as [Bri12], [Bri13] and [Bri20]. In [HS05], [BD13] and [ILA21], compositions of torsors are used to study obstructions to the local-global principle and to weak approximation over various arithmetically interesting fields, while in [BDLM20] they are used to study invariants of reductive groups. In [Bri12], [Bri13] and [Bri20], vector bundles (usual and projective) over abelian varieties, which are essentially compositions of torsors, are studied as interesting geometrical objects in their own right.

¹In this article, we follow Hartshorne's definition of smoothness, which in particular implies that the scheme is of finite type (this is *not* an assumption taken by the Stacks Project, for instance).

In the present article, we study compositions of torsors in a systematic way, at least in the case where K has zero characteristic. The main positive result in this direction goes as follows.

Theorem 1.2. *Let K be a field of characteristic 0. Let X be a connected smooth K -scheme. Let G, H be smooth connected K -group schemes. Let $Y \rightarrow X$ be a G -torsor and let $Z \rightarrow Y$ be an H -torsor. Assume one of the following:*

- *H is an abelian variety.*
- *H is a semi-abelian variety and G is linear.*

Then there exists a canonical extension of K -group schemes

$$1 \rightarrow H \rightarrow E \rightarrow G \rightarrow 1,$$

together with a canonical structure of an E -torsor on the composite $Z \rightarrow X$ such that the following holds.

- *The action of E on Z extends that of H .*
- *The G -torsors $Z/H \rightarrow X$ and $Y \rightarrow X$ are isomorphic.*

In order to prove this result, we first give in Section 2 an abstract statement (Theorem 2.1) for torsors and groups satisfying certain technical conditions. In Section 3, we prove that these conditions are met in the cases given in Theorem 1.2. In Section 4, we present a weaker version of Theorem 1.2 that works over arbitrary fields (cf. Theorem 4.1).

Theorem 1.2 covers a certain number of the previously known results in the literature: [BD13, Lem. 2.13] deals with the case where $H = \mathbb{G}_m$ and G is linear, [BDLM20, Thm. A.1.5] deals with the case where $X = \text{Spec}(K)$, H is a special torus and G is reductive, while [ILA21, Thm. A.1] deals with the case where H is a torus and $\text{Pic}(\bar{G}) = 0$. In [Bri12], [Bri13] and [Bri20], Brion studies homogeneous bundles over abelian varieties, getting results that are related to our main theorem in the case where $X = \text{Spec}(K)$ and G is an abelian variety, although they do not deal directly with compositions of torsors with such G (see however [Bri12, Cor. 3.2] and compare with our Theorem 2.1 and Proposition 2.6).

Theorem 1.2 does not cover all cases dealt with by Harari and Skorobogatov in [HS05, Prop. 1.4], since they consider H to be of multiplicative type, and hence it may be non-connected. However, using a variant of the abstract Theorem 2.1 (cf. Theorem 2.4), we recover their result. Since they also provide an abstract result in their article (cf. [HS05, Thm. 1.2]), we compare this result with ours at the end of Section 2 (cf. Remark 2.7).

Finally, in Section 5, we present a certain number of counterexamples to Question 1.1. Table 1 summarizes both the positive and negative results we obtain in characteristic zero.

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	G				
H		t.	u.	s.s.	a.v.
t.		✓	✓	✓	✗
u.		✗	✗	✗	✗
s.s.		✗	✗	✗	✗
a.v.		✓	✓	✓	✓

t. :	torus
u. :	unipotent
s.s. :	semisimple
a.v. :	abelian variety
✓ :	positive answer
✗ :	negative answer

Table 1: Answer to Question 1.1 for several types of groups G and H over a field of characteristic zero.

2. Abstract results

In this section, unless otherwise stated, K is an arbitrary field. For a K -scheme W , we denote by X_W, Y_W, Z_W, H_W, G_W the W -schemes obtained by base change from X, Y, Z, H, G respectively. We start by proving the following abstract theorem, which will be the key tool to settle Theorem 1.2.

Theorem 2.1. *Let K be a field. Let X be a smooth K -scheme. Let G, H be smooth connected K -group schemes with H abelian. Let $Y \rightarrow X$ be a G -torsor and let $Z \rightarrow Y$ be an H -torsor. Finally, let \mathcal{M} be the sheaf over the small smooth site over K associated to the presheaf given by $(W \mapsto H(Y_W)/H(W))$. Assume the following:*

- (i) *The class of $Z_\Omega \rightarrow Y_\Omega$ in $H^1(Y_\Omega, H_\Omega)$ is $G(\Omega)$ -invariant for every separably closed field Ω/K .*
- (ii) *The sheaf \mathcal{M} is étale-locally isomorphic to the constant sheaf \mathbb{Z}^n for a certain $n \in \mathbb{N}$. In particular, it is representable by a K -group-scheme M .*

Then there exists a canonical extension of K -group schemes

$$1 \rightarrow H \rightarrow E \rightarrow G \rightarrow 1,$$

together with a canonical structure of an E -torsor on the composite $Z \rightarrow X$ such that the following holds.

- *The action of E on Z extends that of H .*
- *The G -torsors $Z/H \rightarrow X$ and $Y \rightarrow X$ are isomorphic.*

We start with a technical lemma, which might have an interest of its own.

Lemma 2.2. *Let K be a field and let G be a connected (resp. smooth connected) K -group scheme. Denote by $K(G)/K$ the function field of G/K , and by Ω an algebraic (resp. separable) closure of $K(G)$. Let \mathcal{E} be a contravariant group functor over the fppf (resp. small smooth) site of K , equipped with a K -homomorphism $\pi : \mathcal{E} \rightarrow G$. Assume the following:*

- (a) *The functor \mathcal{E} , on K -algebras, commutes with filtered direct limits.*
- (b) *The arrow $\pi(\Omega) : \mathcal{E}(\Omega) \rightarrow G(\Omega)$ is surjective.*

Then, the following hold.

- (1) The arrow $\mathcal{E}(\bar{K}) \rightarrow G(\bar{K})$ is surjective, where \bar{K} is the algebraic (resp. separable) closure of K .
- (2) There exists a finite set I and an fppf (resp. smooth) cover $(V_i \rightarrow G)_{i \in I}$ such that, for each $i \in I$, the arrow $V_i \rightarrow G$, considered as an element of $G(V_i)$, lifts via $\pi(V_i) : \mathcal{E}(V_i) \rightarrow G(V_i)$. As a consequence, the arrow $\pi : \mathcal{E} \rightarrow G$ is surjective.

Proof. We prove (1). To do so, we may change the base field from K to \bar{K} , reducing us to the case $K = \bar{K}$. Pick a point $g \in G(K) \subset G(\Omega)$. Let $e \in \mathcal{E}(\Omega)$ be a lift of g . Let $U_0 = \text{Spec}(A_0)$ be a non-empty affine open subscheme of G . Write Ω as the direct limit (union) $\varinjlim A_j$, of its flat (resp. smooth) and finitely presented A_0 -subalgebras A_j . By condition (a), e belongs to $\mathcal{E}(A_j)$ for some j . Since fppf (resp. smooth) morphisms are open (cf. [SP18, Tag 01UA]), in geometric terms, there exists a non-empty affine open $U \subset U_0 \subset G$, and an fppf (resp. smooth) morphism $V := \text{Spec}(A_j) \rightarrow U$, such that e belongs to $\mathcal{E}(V)$. Since V is a non-empty (resp. smooth) K -scheme of finite-type and $K = \bar{K}$, there exists a closed point $v : \text{Spec}(K) \rightarrow V$. Then, the image of e via the morphism $\mathcal{E}(V) \rightarrow \mathcal{E}(K)$ induced by v is the desired lift of g .

We prove (2). Denote by $g \in G(\Omega)$ the generic point of G . Let $e \in \mathcal{E}(\Omega)$ be a lift of g , which exists by condition (b). The same limit argument used in (1), produces an affine open $U \subset G$, and an fppf (resp. smooth) morphism $V \rightarrow U$, such that the composite $V \rightarrow U \rightarrow G$ lifts via π , to an element of $\mathcal{E}(V)$. Since $G(\bar{K})$ is Zariski-dense in G , the translates $\gamma \cdot V$, for $\gamma \in G(\bar{K})$, form an fppf (resp. smooth) cover of $G_{\bar{K}}$, from which we may extract a finite cover. Since these γ 's lift to $\mathcal{E}(\bar{K})$ by (1), there exists a finite (resp. finite separable) field extension L/K , such that a cover of G_L exists, with the required property. Composing with the projection $G_L \rightarrow G$, which is finite (resp. finite separable) and locally free, hence fppf (resp. smooth), gives such a cover over K .

For the last assertion, consider an arbitrary morphism of K -schemes (resp. smooth K -schemes) $\phi : S \rightarrow G$ and define the fppf (resp. smooth) cover $(S \times_G V_i \rightarrow S)_{i \in I}$ of S by pullback. Since the map $\phi_i : S \times_G V_i \rightarrow G$ induced by ϕ factors through $V_i \rightarrow G$, which lifts to $\mathcal{E}(V_i)$, we see that ϕ_i lifts to $\mathcal{E}(S \times_G V_i)$ and the surjectivity follows. \square

The following is an easy exercise given the actual literature, but we state it here since it is used several times in what follows.

Lemma 2.3. *Let K be a field and let G, H be K -group schemes. Let G^0, H^0 denote the corresponding neutral connected components. Assume that G^0, H^0 are smooth and that $F := H/H^0$ is étale-locally isomorphic to \mathbb{Z}^n for some $n \in \mathbb{N}$. Let*

$$1 \rightarrow H \rightarrow \mathcal{E} \rightarrow G \rightarrow 1,$$

be an exact sequence of sheaves over the fppf (resp. small smooth) site of K . Then \mathcal{E} is representable by a K -group scheme E .

Proof. Since H is normal in \mathcal{E} and H^0 is characteristic in H , we obtain that H^0 is normal in \mathcal{E} . We may then quotient \mathcal{E} by H^0 in order to get an exact sequence

$$1 \rightarrow F \rightarrow \mathcal{E}' \rightarrow G \rightarrow 1.$$

Since F is locally isomorphic to \mathbb{Z}^n we know by [SGA7, Exp. 8, Prop. 5.1] that every F -torsor over any connected component of G is locally trivial, hence representable. In particular, this is the case for \mathcal{E}' . Thus we have an exact sequence of K -group schemes

$$1 \rightarrow F \rightarrow E' \rightarrow G \rightarrow 1,$$

which gives us the following exact sequence of sheaves

$$1 \rightarrow H^0 \rightarrow \mathcal{E} \rightarrow E' \rightarrow 1. \quad (1)$$

Now, by Chevalley's Theorem (cf. [Con02, Thm. 1.1] or [BLR90, §9.2, Thm. 1]), there is an exact sequence

$$1 \rightarrow L \rightarrow H^0 \rightarrow A \rightarrow 1,$$

with L linear and A an abelian variety. Since L is a characteristic subgroup of H^0 and H^0 is normal in \mathcal{E} , we see that L is normal in \mathcal{E} . We may then quotient \mathcal{E} by L in order to get an exact sequence

$$1 \rightarrow A \rightarrow \mathcal{E}'' \rightarrow E' \rightarrow 1.$$

Then \mathcal{E}'' is representable by [Mil80, III, Thm. 4.3.(c)] since A is smooth, proper and connected over K . Thus we have an exact sequence of K -group schemes

$$1 \rightarrow A \rightarrow E'' \rightarrow E' \rightarrow 1,$$

which gives us the following exact sequence of sheaves

$$1 \rightarrow L \rightarrow \mathcal{E} \rightarrow E'' \rightarrow 1.$$

Then \mathcal{E} is representable by [Mil80, III, Thm. 4.3.(a)] since L is affine.

Finally, note that Milne's results [Mil80, III, Thm. 4.3.(a),(c)] are stated over the fppf site. However, if we are over the small smooth site, since H^0 is smooth and \mathcal{E} is an H^0 -torsor (cf. sequence (1)), we deduce that it corresponds to a unique H^0 -torsor over the fppf site (cf. [Gro68b, Thm. 11.7.1, Rem. 11.8.3]), which is then representable. This implies the representability of \mathcal{E} over the small smooth site (by the same K -scheme). ☺

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. For a K -scheme W , consider the group $\text{Aut}_{X_W}^H(Z_W)$ of X_W -automorphisms φ of Z_W that are compatible with the action of H in the sense that the following diagram commutes:

$$\begin{array}{ccc} H \times Z_W & \xrightarrow{a_W} & Z_W \\ \text{id} \times \varphi \downarrow & & \downarrow \varphi \\ H \times Z_W & \xrightarrow{a_W} & Z_W, \end{array}$$

where a denotes the morphism defining the action of H on Z and a_W the corresponding morphism after base change. The functor $W \mapsto \text{Aut}_{X_W}^H(Z_W)$ defines a group presheaf over the small smooth site over K . Denote by $\underline{\text{Aut}}_X^H(Z)$ the corresponding sheaf and consider the subsheaf $\underline{\text{Aut}}_Y^H(Z)$ defined by taking the subgroup $\text{Aut}_{Y_W}^H(Z_W)$ of $\text{Aut}_{X_W}^H(Z_W)$

for each W . Since every element in $\text{Aut}_{X_W}^H(Z_W)$ induces an X_W -automorphism of Y_W , we have an exact sequence of sheaves

$$1 \rightarrow \underline{\text{Aut}}_Y^H(Z) \rightarrow \underline{\text{Aut}}_X^H(Z) \xrightarrow{\pi} \underline{\text{Aut}}_X(Y),$$

where $\underline{\text{Aut}}_X(Y)$ denotes the sheaf of X -automorphisms of Y .

Note that G is naturally a subgroup of $\underline{\text{Aut}}_X(Y)$. Taking the pullback via π , we get an exact sequence of sheaves

$$1 \rightarrow \mathcal{A} \rightarrow \mathcal{E}' \xrightarrow{\pi} G,$$

where $\mathcal{A} = \underline{\text{Aut}}_Y^H(Z)$. Now, the functor $V/Y \mapsto \text{Aut}_V^H(Z \times_Y V)$ over the small smooth site of Y is represented, as a Y -scheme, by H_Y (cf. for instance [Gir71, III.§1.5]). In other words, $\mathcal{A}(W) = \text{Aut}_{Y_W}^H(Z_W) = H(Y_W)$ and hence $\mathcal{M} = \mathcal{A}/H$. Thus, by (ii), we get an exact sequence of group sheaves over K

$$1 \rightarrow H \rightarrow \mathcal{A} \rightarrow M \rightarrow 1,$$

where M is a K -group scheme that is étale-locally isomorphic to \mathbb{Z}^n for some $n \in \mathbb{N}$. By Lemma 2.3, it follows that \mathcal{A} is represented by a K -group scheme A . Moreover, since H is connected, it corresponds to the neutral connected component of A . In particular, H is a characteristic subgroup of A .

On the other hand, since X , Z , H and G are of finite type over K , the functor $\mathcal{E}' \subset \underline{\text{Aut}}_K(Z)$ commutes with direct limits. Moreover, by (i), if we set $\Omega := \overline{K(G)}$, we know that Z_Ω is isomorphic to g^*Z_Ω as an H -torsor over G_Ω for every $g \in G(\Omega)$, and thus the arrow $\mathcal{E}'(\Omega) \rightarrow G(\Omega)$ is surjective. Then, by Lemma 2.2, we get that the arrow $\pi : \mathcal{E}' \rightarrow G$ is surjective. In particular, we get an exact sequence of sheaves

$$1 \rightarrow A \rightarrow \mathcal{E}' \xrightarrow{\pi} G \rightarrow 1.$$

And since $M = A/H$ is locally isomorphic to \mathbb{Z}^n by (ii), we see by Lemma 2.3 that \mathcal{E}' is representable. Thus we have an exact sequence of K -group *schemes*

$$1 \rightarrow A \rightarrow E' \xrightarrow{\pi} G \rightarrow 1. \tag{S}$$

Since A is normal in E' and H is characteristic in A , we obtain that H is normal in E' . We may then quotient by H in order to get an exact sequence

$$1 \rightarrow M \rightarrow F \xrightarrow{\bar{\pi}} G \rightarrow 1. \tag{\bar{S}}$$

Since M is discrete and torsion-free by (ii), and since G is connected, we see that the neutral connected component $F^0 \subset F$, is mapped isomorphically to G by $\bar{\pi}$. This provides a *canonical* splitting of extension (\bar{S}) . As a consequence, extension (S) is the pushout of an extension of group schemes (obtained as a pullback via the canonical splitting)

$$1 \rightarrow H \rightarrow E \rightarrow G \rightarrow 1.$$

As a subgroup of $\underline{\text{Aut}}_X^H(Z)$, it acts on Z , and it is immediate to check then that $Z \rightarrow X$ is an E -torsor, which enjoys the required properties. To conclude, note that the construction above is canonical. \square

In the previous theorem, the assumptions that H is abelian and that G and H are both connected can be removed when M is the trivial group. In that way, one gets the following result, which implies [HS05, Prop. 1.4].

Theorem 2.4. *Let K be a field. Let X be a smooth K -scheme. Let G, H be smooth K -group schemes. Let $Y \rightarrow X$ be a G -torsor and let $Z \rightarrow Y$ be an H -torsor. Assume the following:*

- (i) *The class of $Z_\Omega \rightarrow Y_\Omega$ in $H^1(Y_\Omega, H_\Omega)$ is $G(\Omega)$ -invariant for every separably closed field Ω/K .*
- (ii') *The sheaf of sets \mathcal{M} over the small smooth site over K associated to the presheaf given by $(W \mapsto H(Y_W)/H(W))$ is trivial.*

Then there exists a canonical extension of K -group schemes

$$1 \rightarrow H \rightarrow E \rightarrow G \rightarrow 1,$$

together with a canonical structure of an E -torsor on the composite $Z \rightarrow X$ such that the following holds.

- *The action of E on Z extends that of H .*
- *The G -torsors $Z/H \rightarrow X$ and $Y \rightarrow X$ are isomorphic.*

Proof. The proof starts exactly as the one above, except for the following modification. Instead of considering the groups $\text{Aut}_{X_W}^H(Z_W)$ and $\text{Aut}_{Y_W}^H(Z_W)$ for smooth $W \rightarrow K$ and the corresponding sheaves $\underline{\text{Aut}}_X^H(Z)$ and $\underline{\text{Aut}}_Y^H(Z)$, we consider the groups $\text{Aut}_{X_W}^{H'}(Z_W)$ and $\text{Aut}_{Y_W}^{H'}(Z_W)$ and the corresponding sheaves $\underline{\text{Aut}}_X^{H'}(Z)$ and $\underline{\text{Aut}}_Y^{H'}(Z)$, where H' is the Y -group scheme obtained by twisting H_Y by the torsor $Z \rightarrow Y$ (H' is actually H_Y when H is abelian). This group scheme acts naturally on Z on the left compatibly with the right action of H (cf. [Gir71, III.§1.5]). In particular, we still have the equality $\text{Aut}_{Y_W}^{H'}(Z_W) = H(Y_W)$ by *loc. cit.* and an exact sequence

$$1 \rightarrow \mathcal{A}' \rightarrow \mathcal{E} \xrightarrow{\pi} G \rightarrow 1,$$

with $\mathcal{A}' = \underline{\text{Aut}}_Y^{H'}(Z)$, where the surjectivity of π is given once again by Lemma 2.2. The assumption (ii') on the sheaf \mathcal{M} tells us then that \mathcal{A}' is actually H , and hence the exact sequence becomes

$$1 \rightarrow H \rightarrow \mathcal{E} \rightarrow G \rightarrow 1.$$

Thus \mathcal{E} is an H -torsor, which is then representable by a K -group scheme E by Lemma 2.3. And again, since E is by definition a subgroup of $\underline{\text{Aut}}_X^H(Z)$, it is immediate to check that E acts on Z and that $Z \rightarrow X$ is an E -torsor. The fact that the construction is canonical is once again easy to see. ☺

Remark 2.5. As a referee pointed out, the proofs of Theorems 2.1 and 2.4 do not use the fact that $Y \rightarrow X$ is a G -torsor, but rather that G acts on the X -scheme Y . One may extend thus the statements of both theorems to a more general setting (for instance, one can consider projective bundles or Severi-Brauer schemes over X , which have natural actions by forms of PGL_n). However, we were not able to come up with new applications in this setting.

In Theorems 2.1 and 2.4, the $G(\Omega)$ -invariance of the H_Ω -torsor $Z_\Omega \rightarrow Y_\Omega$ is, in a wide variety of cases, a strictly necessary hypothesis in order to get a positive answer to Question 1.1. More precisely:

Proposition 2.6. *Let K be an algebraically closed field of characteristic 0. Let*

$$1 \rightarrow H \rightarrow E \rightarrow G \rightarrow 1,$$

be an extension of smooth K -group schemes with G connected. Assume that the unipotent radical of H is trivial. Let X be a smooth K -scheme, let $Z \rightarrow X$ be an E -torsor and let $Y := Z/H$, so that $Z \rightarrow Y$ is an H -torsor and $Y \rightarrow X$ is a G -torsor. Then the class of $Z \rightarrow Y$ in $H^1(Y, H)$ is $G(K)$ -invariant.

Proof. Define C to be the centralizer of H in E . We claim that C surjects onto G via the projection. Since C is the kernel of the natural arrow $E(K) \rightarrow \text{Aut}(H)$ given by conjugation, the claim amounts to proving that the induced morphism $G(K) \rightarrow \text{Out}(H)$ is trivial, where $\text{Out}(H) := \text{Aut}(H)/\text{Inn}(H)$.

By Lemma 3.3, which we prove in the following section and uses the hypothesis on the characteristic of K , we know that $G(K)$ is generated by its infinitely divisible elements, while $\text{Out}(H)$ has no such elements. Indeed, this group is finite for reductive groups (cf. [Dem65, Thm. 5.2.3]), while it is a subgroup of $\text{GL}_n(\mathbb{Z})$ for abelian varieties (as follows from [Mil86, Thm. 10.15]). In the general case, our hypothesis on H and Chevalley's Theorem (cf. [Con02, Thm. 1.1]) ensure that H is an extension

$$1 \rightarrow L \rightarrow H \rightarrow A \rightarrow 1,$$

of an abelian variety A and a reductive linear group L . Since L is a characteristic subgroup of H and scheme morphisms from an abelian variety to a linear group are constant, one easily sees that $\text{Aut}(H)$ is isomorphic to a subgroup of $\text{Aut}(L) \times \text{Aut}(A)$. We deduce the same property for $\text{Out}(H)$, which implies the claim.

Now consider $g \in G(K)$ and let us prove that the torsor $g^*Z \rightarrow Y$, defined as the left vertical arrow of the fiber product

$$\begin{array}{ccc} g^*Z & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y & \xrightarrow{g} & Y, \end{array}$$

is isomorphic to the torsor $Z \rightarrow Y$. Let $c \in C(K) \subset E(K)$ be a preimage of g . Then we have a commutative square

$$\begin{array}{ccc} Z & \xrightarrow{c} & Z \\ \downarrow & & \downarrow \\ Y & \xrightarrow{g} & Y. \end{array}$$

Then, by the universal property of the fiber product, we get a Y -morphism $Z \rightarrow g^*Z$, which we claim it is H -equivariant. This is a straightforward computation that uses the fact that $c \in C(K)$ commutes with H . This proves that the class of $Z \rightarrow Y$ is g -invariant and hence $G(K)$ -invariant. \square

Remark 2.7. A result similar to Theorem 2.4 can be found in [HS05, Thm. 1.2]. However, the assumptions are slightly different:

Harari and Skorobogatov assume that every morphism $Z_{\bar{K}} \rightarrow H_{\bar{K}}$ is trivial. This is easily seen to imply the triviality of \mathcal{M} and hence our assumption (ii').

On the other hand, they assume that every automorphism of $Y_{\bar{K}}$ given by an element $g \in G(\bar{K})$ can be lifted to an automorphism of $Z_{\bar{K}}$. Our assumption (i) implies this, of course, but it is not clear whether they are equivalent assumptions, even though ours seems to be always necessary, as it can be seen from Proposition 2.6.

In any case, assumption (ii') of Theorem 2.4 is met for instance when H is affine, G is anti-affine and X is connected and proper. Indeed, in this case $\mathcal{O}(Y) = K$ and hence $H(Y \times W) = H(W)$ for geometrically integral W by [Bri21, Lem. 5.2]. This implies the triviality of \mathcal{M} . These are milder hypotheses than those considered by Harari and Skorobogatov in [HS05, Prop. 1.4], who deal for instance with the case of H of multiplicative type and Y proper.

3. Proof of Theorem 1.2

It will suffice to prove that the assumptions (i) and (ii) of Theorem 2.1 are met under each of the hypotheses of Theorem 1.2. We fix then a field K of characteristic 0 and keep the other notations as above: X is a connected smooth K -scheme; G, H are smooth connected K -group schemes; $Y \rightarrow X$ is a G -torsor and $Z \rightarrow Y$ is an H -torsor; \mathcal{M} is the sheaf over the small smooth site over K associated to the presheaf given by $(W \mapsto H(Y_W)/H(W))$.

We prove (ii) first. By étale descent, we may assume that K is algebraically closed and we need to prove that \mathcal{M} is representable and isomorphic to \mathbb{Z}^n . This is a direct consequence of a result of Rosenlicht, which we prove in the appendix in the context of separably closed fields (cf. Lemma A.1).

We are then left with the proof of (i), which is clearly implied by the following result.

Proposition 3.1. *Let K be an algebraically closed field of characteristic 0. Let G and H be smooth connected K -algebraic groups and make one of the following assumptions:*

- (a) H is an abelian variety.
- (b) G is linear and H is a semi-abelian variety.

Let X be a smooth K -scheme and let $Y \rightarrow X$ be a G -torsor. Then the action of $G(K)$ on $H^1(Y, H)$ is trivial.

Proof of Proposition 3.1.(a). According to [Ray70, Cor. XIII.2.4, Prop. XIII.2.6], the group $H^1(Y, H)$ is torsion. Hence, given an element $a \in H^1(Y, H)$, it comes from $H^1(Y, H[n])$ for some $n > 0$. By [BD13, Th. 5.2], we deduce that a is fixed by $G(K)$. ☺

We prove now Proposition 3.1.(b). For that purpose, we first need to prove some lemmas on the structure of the groups involved. In all of them, we keep the notations of Proposition 3.1.

Lemma 3.2. *Let A be an abelian variety over K . Then the group $H^1(Y, A)$ is torsion of cofinite type, i.e. its m -torsion subgroup is finite for every $m \in \mathbb{N}$.*

Proof. As it was already noted in the proof of Proposition 3.1.(a), the group $H^1(Y, A)$ is torsion. Moreover, by [SGA4, Th. 5.2 of Exp. XVI], for each integer $n > 0$, the group $H^1(Y, A[n])$ is finite, and hence so is its quotient $H^1(Y, A)[n]$. ☺

Lemma 3.3. *The group $G(K)$ is spanned by its divisible subgroups.*

Proof. Write $G = G_{\text{aff}}G_{\text{ant}}$ where G_{aff} is the largest connected affine subgroup of G and G_{ant} is the largest anti-affine subgroup of G (cf. [BSU13, Thm. 1.2.4]). Every element g of $G_{\text{aff}}(K)$ can be written as:

$$g = su_1 \dots u_r$$

where s is a semisimple element of $G_{\text{aff}}(K)$ and each u_i is contained in a subgroup of G_{aff} isomorphic to \mathbb{G}_a . Hence $G_{\text{aff}}(K)$ is spanned by its divisible subgroups. Moreover, the anti-affine group G_{ant} is connected commutative (cf. [BSU13, Thm. 1.2.1]), and hence $G_{\text{ant}}(K)$ is a divisible group. We deduce that $G(K)$ is spanned by its divisible subgroups. \square

Lemma 3.4. *Let Γ be a profinite group. Then, Γ has no non-trivial infinitely divisible elements.*

Proof. The statement is clear if Γ is finite. It thus also holds for inverse limits of finite groups. \square

Proof of Proposition 3.1.(b). We have an exact sequence:

$$0 \rightarrow T \rightarrow H \rightarrow A \rightarrow 0,$$

where T is a torus and A is an abelian variety. It induces a cohomology exact sequence:

$$H^1(Y, T) \xrightarrow{f} H^1(Y, H) \xrightarrow{g} H^1(Y, A),$$

whose arrows are clearly $G(K)$ -equivariant since the action is on Y . Put $M := \text{im}(g)$ and $N := \text{im}(f)$, so that we have the exact sequence of $G(K)$ -groups

$$0 \rightarrow N \rightarrow H^1(Y, H) \rightarrow M \rightarrow 0.$$

By Lemma 3.2, $H^1(Y, A)$ is torsion of cofinite type, and hence so is M . Moreover, by Proposition 3.1.(a), the group $G(K)$ acts trivially on $H^1(Y, A)$, and hence on M . On the other hand, note that $H^1(Y, T) \cong \text{Pic}(Y)^{\dim(T)}$. Since G is linear, a result of Sumihiro (cf. [Bri18, Thm. 5.2.1]) tells us that the action of $G(K)$ on $H^1(Y, T)$ is trivial, hence also its action on N .

Thus, the action of $G(K)$ on $H^1(Y, H)$ corresponds to a morphism from $G(K)$ to $\text{Hom}(M, N)$. The abelian group M is torsion, and hence $\text{Hom}(M, N) = \text{Hom}(M, N_{\text{tors}})$. Moreover, M is of cofinite type, and so is the group N_{tors} since it is isomorphic to a quotient of $\text{Pic}(Y)^{\dim(T)}$. We can therefore write:

$$M \cong \bigoplus_p (F_p \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^{r_p}),$$

$$N_{\text{tors}} \cong \bigoplus_p (F'_p \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^{r'_p}),$$

where p runs through the set of all prime numbers, F_p and F'_p are finite abelian p -groups and $r_p, r'_p \geq 0$. It follows that $\text{Hom}(M, N_{\text{tors}})$ is a profinite group. Hence, it has no non-trivial infinitely divisible elements by Lemma 3.4. Thus, every morphism from $G(K)$ to $\text{Hom}(M, N_{\text{tors}})$ is trivial by Lemma 3.3. We deduce that the action of $G(K)$ on $H^1(Y, H)$ is trivial. \square

We finish this section with an example that shows that one really needs to assume K to be algebraically closed in Proposition 3.1.(b) (and hence separably closed in assumption (i) of Theorems 2.1 and 2.4).

Example 3.5. Let K be a field and let L/K be a separable quadratic field extension such that the norm $N_{L/K} : L^\times \rightarrow K^\times$ is not surjective. Consider the extension of algebraic K -groups

$$1 \rightarrow R_{L/K}^1(\mathbb{G}_m) \rightarrow R_{L/K}(\mathbb{G}_m) \xrightarrow{N_{L/K}} \mathbb{G}_m \rightarrow 1,$$

where $R_{L/K}$ denotes Weil scalar restriction and $N_{L/K}$ is the norm of L/K . Set $G := \mathbb{G}_m$, $H := R_{L/K}^1(\mathbb{G}_m)$ and $X := \text{Spec}(K)$. The extension above provides a class

$$x_0 := [R_{L/K}(\mathbb{G}_m) \rightarrow \mathbb{G}_m] \in H^1(Y, H).$$

This class is not invariant under the action of $G(K)$. Indeed, the action of $G(K)$ is described as follows:

$$\lambda \cdot x = x + p^* \delta(\lambda),$$

where $\lambda \in G(K)$, $x \in H^1(Y, H)$, $p : Y \rightarrow K$ is the structure morphism and $\delta : G(K) \rightarrow H^1(K, H)$ is the connecting map in Galois cohomology. In particular, since the arrow

$$p^* : H^1(K, H) \rightarrow H^1(Y, H),$$

is injective, we have $\lambda \cdot x = x$ if and only if $\lambda \in N_{L/K}(L^\times)$, which does not hold in general.

4. Positive characteristic

In this section, we present a weaker version of Theorem 1.2 that works in positive characteristic.

Theorem 4.1. *Let K be a field. Let X be a connected smooth K -scheme. Let G, H be smooth connected K -group schemes. Let $Y \rightarrow X$ be a G -torsor and let $Z \rightarrow Y$ be an H -torsor. Assume one of the following:*

- H, G are abelian varieties and X is proper.
- H is a torus and G is linear.

Then there exists a canonical extension of K -group schemes

$$1 \rightarrow H \rightarrow E \rightarrow G \rightarrow 1,$$

together with a canonical structure of an E -torsor on the composite $Z \rightarrow X$.

Proof. The proof of this result is given once again by Theorem 2.1, which is valid over an arbitrary field. We need to prove then that assumptions (i) and (ii) of Theorem 2.1 are met. The proof of (ii) is exactly as before: By étale descent, we may assume that K is separably closed and we need to prove that \mathcal{M} is representable and isomorphic to \mathbb{Z}^n . This is a direct consequence of Lemma A.1, which is valid over separably closed fields.

Thus, we are only left with (i). In the second case, this is a direct consequence of Sumihiro's result we used before (cf. [Bri18, Thm. 5.2.1]). In the first case, (i) is implied by Proposition 4.2 here below. ☺

Proposition 4.2. *Let K be a separably closed field. Let X be a smooth proper K -scheme. Let G and H be abelian varieties and let $Y \rightarrow X$ be a G -torsor. Then the action of $G(K)$ on $H^1(Y, H)$ is trivial.*

Proof. Without loss of generality, we can assume that X (and hence Y) is connected. Moreover, since H is smooth, we may and do assume that K is algebraically closed. Note that $H^1(Y, H)$ is torsion by [Ray70, Cor. XIII.2.4, Prop. XIII.2.6]. As in the proof of Proposition 3.1, [BD13, Th. 5.2] implies that $G(K)$ acts trivially on the q -primary part of $H^1(Y, H)$ for every prime $q \neq p$. It is therefore enough to prove that $G(K)$ also acts trivially on $H^1(Y, H)[p^n]$ for every $n \in \mathbb{N}$. We proceed by induction on n .

For $n = 1$, we know that $H^1(Y, H)[p]$ is a quotient of $H_{\text{fppf}}^1(Y, H[p])$ and $H[p] \cong (\mathbb{Z}/p\mathbb{Z})^a \times (\mu_p)^b \times (\alpha_p)^c$ for some integers a, b, c (cf. [Sha86]). It is therefore enough to prove that the action of $G(K)$ on $H_{\text{fppf}}^1(Y, \mathbb{Z}/p\mathbb{Z})$ and $H_{\text{fppf}}^1(Y, \mu_p)$ and $H_{\text{fppf}}^1(Y, \alpha_p)$ is trivial.

Since Y is proper over K , [Mil80, Cor. VI.2.8] ensures the finiteness of $H_{\text{fppf}}^1(Y, \mathbb{Z}/p\mathbb{Z})$, which implies the triviality of the action in this case.

Now by Kummer theory, we have an exact sequence:

$$0 \rightarrow K[Y]^\times / (K[Y]^\times)^p \rightarrow H_{\text{fppf}}^1(Y, \mu_p) \rightarrow \text{Pic}(Y)[p] \rightarrow 0.$$

Using the properness of Y once more, we have $K[Y] = K$, and hence the quotient $K[Y]^\times / (K[Y]^\times)^p$ is trivial. Moreover, the group $\text{Pic}(Y)[p]$ is always finite. Hence $H_{\text{fppf}}^1(Y, \mu_p)$ is finite, which implies the triviality of the action in this case.

We deal now with $H_{\text{fppf}}^1(Y, \alpha_p)$. Let A be a K -algebra. Then $H^0(Y_A, \mathcal{O}_{Y_A}) = A$ and $H^1(Y_A, \mathcal{O}_{Y_A}) = H_{\text{fppf}}^1(Y, \mathcal{O}_Y) \otimes_K A$, so that, after taking cohomology of the extension of fppf sheaves (over Y_A)

$$0 \rightarrow \alpha_p \rightarrow \mathbb{G}_a \xrightarrow{\text{Frob}} \mathbb{G}_a \rightarrow 0,$$

we get an exact sequence

$$0 \rightarrow A/A^p \rightarrow H_{\text{fppf}}^1(Y_A, \alpha_p) \rightarrow H^1(Y, \mathcal{O}_Y) \otimes_K A.$$

Taking $A = K$, we get an inclusion of finite-dimensional K -vector spaces

$$H_{\text{fppf}}^1(Y, \alpha_p) \subset H^1(Y, \mathcal{O}_Y).$$

It then suffices to show that $G(K)$ acts trivially on $H_{\text{fppf}}^1(Y, \mathcal{O}_Y)$. To do so, observe that the G -action on Y , induces an action of the abstract group $G(A)$ on the A -scheme Y_A , and hence an A -linear action of $G(A)$ on $H_{\text{fppf}}^1(Y, \mathcal{O}_Y) \otimes_K A$. Being functorial in A , it arises from a morphism of algebraic K -groups $\rho : G \rightarrow \text{GL}(H_{\text{fppf}}^1(Y, \mathcal{O}_Y))$, which is trivial because G is an abelian variety. This concludes the proof for $n = 1$.

Consider now the following exact sequence

$$0 \rightarrow H^1(Y, H)[p] \rightarrow H^1(Y, H)[p^{n+1}] \rightarrow H^1(Y, H)[p^n],$$

and let I be the image of the rightmost arrow. By the inductive assumption, the group $G(K)$ acts trivially on $H^1(Y, H)[p]$ and on I . Hence the action of $G(K)$ on $H^1(Y, H)[p^{n+1}]$ corresponds to a morphism $G(K) \rightarrow \text{Hom}(I, H^1(Y, H)[p])$. But this morphism is trivial since $G(K)$ is divisible and $\text{Hom}(I, H^1(Y, H)[p])$ is p -torsion. We deduce that $G(K)$ acts trivially on $H^1(Y, H)[p^{n+1}]$, as wished. \square

5. Counterexamples

In this section, we provide examples of towers of torsors that do not admit a torsor structure under an extension of the two involved groups. We treat every negative case considered in Table 1.

5.1 Examples where H is a torus

As it is suggested by Lemma A.1, when H is a torus, assumption (ii) of Theorem 2.1 is satisfied in all generality. According to Table 1, it is then assumption (i), on the $G(\Omega)$ -invariance of the H -torsor $Z \rightarrow Y$ that must fail in order to get counterexamples.

Example 5.1. Assume that K is algebraically closed of characteristic 0, $X = \text{Spec}(K)$, $G = Y$ is an elliptic curve, and $H = \mathbb{G}_m$. Then the group $G(K)$ acts on $H^1(Y, H) = \text{Pic}(G)$ via the following formula:

$$Q \cdot [D] = [D] + \deg(D) \cdot ([Q] - [O]), \quad Q \in G(K), [D] \in \text{Pic}(G).$$

This action is not trivial, and hence one can find a class in $\text{Pic}(G)$ that is not $G(K)$ -invariant. By Proposition 2.6, this class represents an H -torsor $Z \rightarrow Y$ such that the composition $Z \rightarrow K$ is not a torsor under an extension E of G by H .

5.2 Examples where H is unipotent

We continue with the case in which H is unipotent. The following example covers the cases in which G is either a torus, a unipotent group or a semisimple group.

Example 5.2. Let $H = \mathbb{G}_a$, X an elliptic curve over an algebraically closed field K of characteristic zero and Y the trivial G -torsor with G either \mathbb{G}_a , \mathbb{G}_m or SL_n (with $n \geq 2$).

On the one hand, by Künneth's formula we have

$$H^1(Y, \mathbb{G}_a) = H^1(X, \mathbb{G}_a) \otimes_K \mathcal{O}(G) = \mathcal{O}(G),$$

which is an infinite-dimensional K -vector space.

On the other hand, every extension E of G by \mathbb{G}_a is split by the basic theory of linear groups. Even more, if $G = \mathbb{G}_a$ or $G = \text{SL}_n$, then the extension is simply the direct product, while if $G = \mathbb{G}_m$, then it corresponds to the semi-direct product $E_k := \mathbb{G}_a \rtimes_k \mathbb{G}_m$ with \mathbb{G}_m acting on \mathbb{G}_a by a character of the form

$$\begin{aligned} \chi_k : \mathbb{G}_m &\rightarrow \text{Aut}(\mathbb{G}_a) = \mathbb{G}_m, \\ x &\mapsto x^k, \end{aligned}$$

for some $k \in \mathbb{Z}$. In particular, these extensions are parametrized by \mathbb{Z} .

Thus, in the former two cases (where $G = \mathbb{G}_a$ or $G = \text{SL}_n$), we get that E -torsors lifting $Y \rightarrow X$ are classified by the one-dimensional vector space $H^1(X, \mathbb{G}_a)$, while H -torsors $Z \rightarrow Y$ are classified by the infinite dimensional vector space $H^1(X, \mathbb{G}_a) \otimes_K \mathcal{O}(G)$. This tells us that there exist H -torsors $Z \rightarrow Y$ such that the composite $Z \rightarrow Y \rightarrow X$ is not a torsor under an extension of G by H .

In the latter case where $G = \mathbb{G}_m$ and $E_k = \mathbb{G}_a \rtimes_k \mathbb{G}_m$, we have a split exact sequence:

$$1 \rightarrow H^1(X, \mathbb{G}_a) \rightarrow H^1(X, E_k) \rightarrow H^1(X, G) \rightarrow 1.$$

Since the torsor $Y \rightarrow X$ is trivial, we deduce that E_k -torsors lifting $Y \rightarrow X$ are classified by the one-dimensional vector space $H^1(X, \mathbb{G}_a)$. Consider then the composite

$$H^1(X, \mathbb{G}_a) \rightarrow H^1(X, E_k) \rightarrow H^1(Y, \mathbb{G}_a),$$

where the first arrow is induced by the injection $\mathbb{G}_a \subset E_k$ and the second is the arrow that sends an E_k -torsor $Z \rightarrow X$ to the \mathbb{G}_a -torsor $Z \rightarrow Z/\mathbb{G}_a = Y$. One can easily check that its image in $H^1(Y, \mathbb{G}_a) = \mathcal{O}(G)$ is the one-dimensional subspace generated by χ_k . We deduce that H -torsors over Y that may be lifted to an E_k -torsor over X for some $k \in \mathbb{Z}$ correspond, inside $H^1(Y, \mathbb{G}_a) = \mathcal{O}(G)$, to the union of the one-dimensional subspaces generated by the different $\chi_k \in \mathcal{O}(G)$. In particular, there exist H -torsors $Z \rightarrow Y$ such that the composite $Z \rightarrow Y \rightarrow X$ is not a torsor under an extension of G by H .

This example leads to a more general construction for towers of \mathbb{G}_a -torsors over curves of genus ≥ 2 .

Example 5.3. Let X/K be a smooth projective curve, of genus $g \geq 2$. Then, the K -vector space $H^1(X, \mathcal{O}_X)$ is g -dimensional. Let $Y \rightarrow X$ be a non-trivial \mathbb{G}_a -torsor, whose class in $H^1(X, \mathbb{G}_a)$ we denote by y . Using the correspondence between \mathbb{G}_a -torsors over X and extensions of vector bundles of \mathcal{O}_X by itself (both objects are classified by the group $H^1(X, \mathbb{G}_a) = H^1(X, \mathcal{O}_X)$ by étale descent), Y corresponds to an extension

$$\mathcal{E} : \quad 0 \rightarrow \mathcal{O}_X \xrightarrow{s} E \xrightarrow{\pi} \mathcal{O}_X \rightarrow 0.$$

More precisely, we have

$$Y = \text{Spec} \left(\varinjlim_n \text{Sym}^n E \right),$$

where the transition morphisms $\text{Sym}^n E \rightarrow \text{Sym}^{n+1} E$ are given by multiplication by s , and hence

$$H^1(Y, \mathcal{O}_Y) = H^1(X, \varinjlim_n \text{Sym}^n E) = \varinjlim_n H^1(X, \text{Sym}^n E).$$

Now, to compute this direct limit, one can use the symmetric powers of \mathcal{E} :

$$\text{Sym}^n \mathcal{E} : \quad 0 \rightarrow \text{Sym}^{n-1} E \xrightarrow{\times s} \text{Sym}^n E \xrightarrow{\pi^n} \mathcal{O}_X \rightarrow 0,$$

where

$$\pi^n(e_1 \otimes \dots \otimes e_n) := \pi(e_1) \dots \pi(e_n).$$

Denote by $y^n \in H^1(X, \text{Sym}^n E)$ the class of $\text{Sym}^n \mathcal{E}$. We have a commutative diagram of extensions

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Sym}^n(E) & \xrightarrow{\times s} & \text{Sym}^{n+1}(E) & \xrightarrow{\pi^{n+1}} & \mathcal{O}_E \longrightarrow 0 \\ & & \downarrow \pi^n & & \downarrow g & & \downarrow \times(n+1) \\ 0 & \longrightarrow & \mathcal{O}_E & \xrightarrow{s} & E & \xrightarrow{\pi} & \mathcal{O}_E \longrightarrow 0, \end{array}$$

where g is given by the formula

$$g(e_0 \dots e_n) = \sum_0^n \pi(e_0) \dots \widehat{\pi(e_i)} \dots \pi(e_n) e_i,$$

where $\hat{}$ denotes an omitted variable. Since E is non-split, and since $(n+1) \in K^\times$, we get that (the class of) $\text{Sym}^n \mathcal{E}$ does not belong to the image of $H^1(X, \text{Sym}^{n-1} E) \rightarrow H^1(X, \text{Sym}^n E)$; in particular, it is non-split.

Now, the cohomology exact sequence associated to $\text{Sym}^n \mathcal{E}$ gives:

$$K \rightarrow H^1(X, \text{Sym}^{n-1} E) \xrightarrow{s_n} H^1(X, \text{Sym}^n E) \rightarrow H^1(X, \mathcal{O}_X) \rightarrow 0,$$

where the image of the leftmost arrow is precisely the subspace generated by the class of $\text{Sym}^n \mathcal{E}$. This tells us that, if we set $V_n := H^1(X, \text{Sym}^n E) / \langle y^n \rangle$, we have an exact sequence

$$0 \rightarrow V_n \rightarrow V_{n+1} \rightarrow H^1(X, \mathcal{O}_X) / \langle y \rangle \rightarrow 0.$$

Since $\dim(H^1(X, \mathcal{O}_X)) = g \geq 2$, we see that the direct limit of the V_n 's has infinite dimension, so that the same holds for $H^1(Y, \mathcal{O}_Y) = \varinjlim_n H^1(X, \text{Sym}^n E)$.

Assume now, that for every \mathbb{G}_a -torsor $Z \rightarrow Y$, we can find an extension of X -group schemes

$$1 \rightarrow \mathbb{G}_a \rightarrow \Gamma \rightarrow \mathbb{G}_a \rightarrow 1,$$

such that $Z \rightarrow X$ can be equipped with the structure of a Γ -torsor. Since K has characteristic zero, Γ is the affine space of a vector bundle F over X , fitting into an extension of vector bundles over X

$$\mathcal{F}: \quad 0 \rightarrow \mathcal{O}_X \xrightarrow{s} F \xrightarrow{\pi} \mathcal{O}_X \rightarrow 0.$$

Using the same computation as above, we get that $H^1(X, F)$ is $(2g-1)$ -dimensional. Thus, the moduli space of torsors under extensions of \mathbb{G}_a by itself is $(3g-1)$ -dimensional. This contradicts the fact that $H^1(Y, \mathcal{O}_Y)$ is infinite-dimensional.

Remark 5.4. Note that in Examples 5.2 and 5.3 the base scheme X is always proper. On the opposite side, when the base is affine, we get a particular case where the answer to Question 1.1 is positive with H unipotent as follows:

Assume that H is a *split* unipotent group, that G is linear and that X is *affine*. Let Y be a G -torsor over X and let Z be an H -torsor over Y . Then Y is affine, and hence $H^1(Y, \mathbb{G}_a) = 0$. We deduce that $H^1(Y, H)$ is trivial, so that $Z = Y \times H$. In particular, Z is a $(G \times H)$ -torsor over X .

In contrast with the last remark, if G is not linear, we can also provide an example in which the base scheme X is not proper over K .

Example 5.5. Let $G = A$ be an abelian variety, $H = \mathbb{G}_a$ and let X be the spectrum of a function field L over K . Then $\text{Ext}(A, \mathbb{G}_a) \simeq H^1(A, \mathcal{O}_A)$ by [Ser75, VII.17, Thm. 7], which is a K -vector space for group extensions over K and an L -vector space of the same dimension if we do the corresponding base change (and the restriction arrow is the obvious injection). This tells us immediately that there are extensions of A by \mathbb{G}_a over L that do not come from extensions over K . In particular, these extensions are towers of torsors over L that cannot have a torsor structure under an extension defined over K (if an extension were a torsor under another extension, they would have the same underlying variety and hence define the same element in $H^1(A, \mathcal{O}_A)$). Obviously, these extensions can be built over a suitable (smooth affine) K -scheme with function field L , if one wants X to be more than just a single point.

5.3 Examples where H is semisimple

We finish this section with examples in which H is semisimple. This completes the study of all cases in Table 1.

Example 5.6. Let $H = \mathrm{PGL}_n$ with $n \geq 2$ and let G be either \mathbb{G}_a^m , \mathbb{G}_m^m , PGL_m , or an abelian variety A . Consider an H -torsor $Z \rightarrow G$, and the trivial G -torsor $G \rightarrow K$ below it. If Question 1.1 had a positive answer for this tower, then Z would be an E -torsor for some extension E of G by H . However, in all four cases for G (assuming $n \gg m$ if $G = \mathbb{G}_a^m$ and assuming K algebraically closed if $G = A$) we have that the only possible extension is the direct product $E = G \times H$. Indeed, the first three cases are elementary results from the theory of linear groups, and the case $G = A$ comes from [BSU13, Prop. 3.1.1]. This implies in particular that the class in $H^1(G, H)$ of $Z \rightarrow G$ must come from $H^1(K, H)$. Thus, any class in $H^1(G, H)$ which does not come from $H^1(K, H)$ gives a negative answer to Question 1.1. Now, recall that classes in $H^1(G, H)$ classify Azumaya algebras over G , which correspond also to classes in the Brauer group $\mathrm{Br}(G)$ (cf. for instance [CTS21, Thm. 3.3.2]).

Assume that $G = \mathbb{G}_a^m$ and that $\mathrm{Br}(K) \neq 0$. Then by [OS71, Prop. 2], there exist non-constant Azumaya algebras over \mathbb{G}_a^2 . These correspond to elements in $H^1(G, H)$ that do not come from $H^1(K, H)$.

Assume that $G = \mathbb{G}_m^m$. Then a simple computation using residue maps with respect to the irreducible divisors in $\mathbb{P}^m \setminus \mathbb{G}_m^m$ (cf. for instance [CTS21, Thm. 3.7.2]) tells us that $\mathrm{Br}(G)/\mathrm{Br}(K) \neq 0$. A class in $\mathrm{Br}(G) \setminus \mathrm{Br}(K)$ corresponds then to a class in $H^1(G, H)$ which does not come from $H^1(K, H)$.

Assume that $G = \mathrm{PGL}_m$ and that $H^1(K, \mathbb{Z}/m\mathbb{Z}) \neq 0$. Since the subgroup of algebraic classes in $\mathrm{Br}(G)/\mathrm{Br}(K)$ is isomorphic to $H^1(K, \mathbb{Z}/m\mathbb{Z})$ (cf. [San81, Lem. 6.9(iii)]), one can find non-constant classes as well in this case.

Finally, assume that $G = A$ and that K is algebraically closed. Then it is well-known that $\mathrm{Br}(A)/\mathrm{Br}(K)$ is non-trivial in general (cf. [Ber72, p. 182]). We conclude as before.

Remark 5.7. Given that all the examples above use the adjoint group $H = \mathrm{PGL}_n$, one could wonder whether Question 1.1 has a positive answer when H is semi-simple and simply connected. This question remains open.

A. An elementary proof of Rosenlicht's Lemma

We prove the following lemma, due to Rosenlicht in the case of an algebraically closed field (cf. [Ros61]).

Lemma A.1. *Let H be a semi-abelian variety over a field K . Let V and W be geometrically integral K -varieties. Then, the following holds.*

1. *The abelian group $H(W)/H(K)$ is finitely generated and free.*
2. *If K is separably closed, the sequence*

$$0 \rightarrow H(K) \xrightarrow{h \mapsto (h, -h)} H(V) \times H(W) \xrightarrow{\pi_V^* + \pi_W^*} H(V \times_K W) \rightarrow 0,$$

is exact, where

$$\begin{aligned} \pi_V^* : H(V) &\rightarrow H(V \times_K W) \\ h &\mapsto h \circ \pi_V. \end{aligned}$$

Proof. In both statements, W and V can be replaced by a non-empty open subvariety. In particular, by generic smoothness, we can thus assume that V and W are smooth over K .

Let us prove the first assertion. Denoting by \bar{K} a separable closure of K , the natural arrow

$$H(W)/H(K) \rightarrow H(\bar{W})/H(\bar{K})$$

is injective, so that we may assume $K = \bar{K}$. By definition, there is an exact sequence

$$1 \rightarrow T \rightarrow H \xrightarrow{\pi} A \rightarrow 1,$$

with T a torus and A an abelian variety. Since K is separably closed and T is smooth, the snake lemma gives an exact sequence

$$1 \rightarrow T(W)/T(K) \rightarrow H(W)/H(K) \xrightarrow{\pi} A(W)/A(K).$$

It will suffice then to treat the cases $H = A$, or $H = \mathbb{G}_m$.

Up to shrinking W , we can assume there is a smooth K -morphism $W \rightarrow U$ of relative dimension one and with geometrically connected fibers, where $U = D(f)$ is a principal open subset of some affine space. Denote by $K' = K(U)$ the field of functions of U , and set $W' := W \times_U K'$. Then W' is a smooth K' -curve, and there is an exact sequence

$$0 \rightarrow H(U)/H(K) \rightarrow H(W)/H(K) \rightarrow H(W')/H(K').$$

Thus, the problem is further reduced to two particular cases: $W = D(f)$ is a principal open subset of some affine space, or W is a smooth curve over K . The first case is trivial for abelian varieties (a morphism from a rational variety to an abelian variety is constant). For \mathbb{G}_m , it is dealt with by a straightforward direct computation. It remains to treat the case of a smooth affine curve W/K . The case $H = \mathbb{G}_m$ is once again a straightforward computation, using the smooth proper curve C compactifying W and the fact that $H(C) = H(K)$. For H an abelian variety, using [Mil86, Thm. 6.1], the statement is equivalent to $\text{Hom}_{\text{gp}}(\text{Jac}(C), H)$ being a free abelian group of finite rank, which holds by [Mil86, Thm. 10.15].

In order to establish the second assertion, we only need to check the surjectivity of $\pi_V^* + \pi_W^*$. Pick rational points $v_0 \in V(K)$ and $w_0 \in W(K)$, which exist since K is separably closed and V, W are smooth over K . For $f \in H(V \times_K W)$, set

$$\tilde{f}(v, w) := f(v, w) - f(v_0, w) - f(v, w_0) + f(v_0, w_0).$$

Then, the composite

$$\pi \circ \tilde{f} : V \times_K W \rightarrow A$$

vanishes on $\{v_0\} \times_K W$ and on $V \times_K \{w_0\}$. Using [Mil86, Thm. 3.4], we get $\pi \circ \tilde{f} = 0$. In other words, \tilde{f} takes values in T . Thus, in order to conclude, it suffices to prove the exactness when $H = \mathbb{G}_m$. Fix $f \in H(V \times_K W) = \mathcal{O}_{V \times_K W}^\times$. Replacing f by $(v, w) \mapsto f(v, w)f(v_0, w)^{-1}$, we may assume that $f = 1$ on $\{v_0\} \times_K W$. To conclude, we have to prove that f factors through the projection $\pi_V : V \times W \rightarrow V$, i.e. that f does not depend on W . Using that the smooth K -variety W is covered by smooth K -curves (for instance, by Bertini's Theorem), we easily reduce to the case where W is a curve.

If W is an open subset of \mathbb{G}_m , this is once again a straightforward computation. In general, for a given $v_1 \in V(K)$, set

$$\begin{aligned} g_1 : W &\rightarrow \mathbb{G}_m, \\ w &\mapsto f(v_1, w). \end{aligned}$$

We have to show that g_1 is constant. Assume it is not. Then, it is finite of degree $d \geq 1$ over its image. Up to shrinking W , we may assume that g_1 is a composite arrow $W \rightarrow W' \rightarrow U \subset \mathbb{G}_m$ with $W \rightarrow W'$ purely inseparable and $W' \rightarrow U$ finite and étale.

Assume first that $W = W'$. Denote by $\tilde{W} \rightarrow U$ the Galois closure of g_1 . There is a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{V \times W}^\times & \xrightarrow{\rho_1} & \mathcal{O}_{\tilde{W}}^\times \\ \downarrow N & & \downarrow N \\ \mathcal{O}_{V \times U}^\times & \xrightarrow{\rho_1} & \mathcal{O}_U^\times, \end{array}$$

where N is the (multiplicative) norm with respect to the finite étale morphism g_1 and ρ_1 denotes the restrictions to the fiber above v_1 (in particular, it maps f to g_1). We claim that $N(f) \in \mathcal{O}_{V \times_K U}^\times$ is trivial on $\{v_0\} \times U$. Indeed, we have $N(f) = f_1 f_2 \dots f_d$, where $f = f_1, f_2, \dots, f_d$ are the images of f with respect to the different embeddings of $\mathcal{O}_{V \times_K W}$ in $\mathcal{O}_{V \times_K \tilde{W}}$. These are trivial on $\{v_0\} \times \tilde{W}$, whence the claim. Since we know the conclusion of the Lemma for $W = U$, we get that $N(f) \in \mathcal{O}_{V \times U}^\times$ does not depend on U . Using commutativity of the diagram above, we compute:

$$\rho_1(N(f)) = N(g_1) = g_1^d.$$

Thus, g_1^d is constant. Hence so is g_1 , which finishes the proof when $W = W'$. In general, for a finite purely inseparable morphism of degree p^r , the norm is given by $N(x) = x^{p^r}$, so that a straightforward variant of the proof above applies. \square

Remark A.2. The second statement of Lemma A.1, over a non-separably closed K , is false in general. Indeed, surjectivity fails when $V = W$ is a non-trivial H -torsor. This is essentially the only counterexample, as surjectivity holds whenever $H^1(K, H) = 0$ (e.g. for $H = \mathbb{G}_m$).

Remark A.3. When H is a torus, the proof of the second statement of Lemma A.1 that we provide above uses affine geometry, combined with a norm argument. In this sense, it is an “inner” proof. This is a more elementary approach than the use of a normal compactification of W in Rosenlicht’s original proof.

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