A short proof of Klyachko's theorem about rational algebraic tori

Mathieu Florence

Abstract

In this paper, we give another proof of a theorem by Klyachko ([**?**]), which asserts that Zariski's conjecture holds for a special class of tori over an arbitrary ground field.

1 Introduction

The main purpose of this paper is to give a much simpler proof of a theorem due to Klyachko ([**?**]; see also [**?**], chap. 2, 6.3), which is here theorem **??**. To achieve this, we first prove a generalization of a theorem due to Voskresenskii ([**?**], chap. 2, 5.1, corollary). To be more precise, we show stable rationality for a certain class of algebraic tori over a given field k, strictly containing the cyclotomic ones. What is more, we give an effective way of presenting the character module of these tori as the kernel of a surjection between permutation modules (that is, lattices that contain a basis which is permuted by the action of the absolute Galois group of k). Recall that, according to *loc. cit.*, chap. 2, 4.7, theorem 2 , the existence of such a surjection is a necessary and sufficient condition for a torus to be stably rational. All the basic material concerning algebraic tori and rationality questions related to these is contained in *loc. cit.*, chap.2; we shall assume that the reader is familiar with this reference.

In the following section, the symbol $\otimes$ alone means $\otimes_{\mathbb{Z}}$. If k is a field with separable closure k_s, we denote by Γ_k the profinite group $\mathrm{Gal}(k_s/k)$. Let Γ be a profinite group. By a Γ-lattice, we mean a free $\mathbb{Z}$-module of finite rank, endowed with a continuous action of Γ. We will say simply 'exact sequence' instead of 'exact sequence of Γ-lattices'.

2 Stably rational and rational algebraic tori

To begin this section, we prove an elementary but crucial lemma.

Lemma 2.1 *Let Γ be a profinite group. Let A_i, B_i, C_i, $i = 1, 2$ be Γ-lattices, fitting into two exact sequences*

$$0 \longrightarrow A_i \xrightarrow{j_i} B_i \xrightarrow{\pi_i} C_i \longrightarrow 0.$$

Assume we are given $s_i : C_i \longrightarrow B_i$, and d_1, d_2 two coprime integers, such that $\pi_i \circ s_i = d_i Id$, $i = 1, 2$. Let

$$A_3 = A_1 \otimes A_2,$$

$$B_3 = (B_1 \otimes B_2) \oplus (C_1 \otimes C_2),$$

and

$$C_3 = (C_1 \otimes B_2) \oplus (B_1 \otimes C_2).$$

Then there is an exact sequence

$$0 \longrightarrow A_3 \xrightarrow{j_3} B_3 \xrightarrow{\pi_3} C_3 \longrightarrow 0,$$

together with a morphism $s_3 : C_3 \longrightarrow B_3$, satisfying $\pi_3 \circ s_3 = d_1 d_2 Id$.

Proof. We have an exact sequence

$$0 \longrightarrow A_1 \otimes A_2 \longrightarrow B_1 \otimes B_2 \stackrel{(\pi_1 \otimes Id) \oplus (Id \otimes \pi_2)}{\longrightarrow} (C_1 \otimes B_2) \oplus (B_1 \otimes C_2) \xrightarrow{\pi} C_1 \otimes C_2 \longrightarrow 0,$$

where $\pi = Id \otimes \pi_2 - \pi_1 \otimes Id$.
Select integers u, v such that $vd_2 - ud_1 = 1$. Then the map

$$s : C_1 \otimes C_2 \longrightarrow (C_1 \otimes B_2) \oplus (B_1 \otimes C_2),$$

$$c_1 \otimes c_2 \mapsto (vc_1 \otimes s_2(c_2), us_1(c_1) \otimes c_2)$$

is a splitting of π. Hence we have an exact sequence

$$0 \longrightarrow A_3 \xrightarrow{j_3} B_3 \xrightarrow{\pi_3} C_3 \longrightarrow 0$$

as stated, where

$$\pi_3 : (B_1 \otimes B_2) \oplus (C_1 \otimes C_2) \stackrel{((\pi_1 \otimes Id) \oplus (Id \otimes \pi_2), s)}{\longrightarrow} (C_1 \otimes B_2) \oplus (B_1 \otimes C_2).$$

The last assertion is obvious: if $r_i : B_i \longrightarrow A_i$ $(i = 1, 2)$ are such that $r_i \circ j_i = d_i Id$, then

$$r_3 := (r_1 \otimes r_2, 0) : B_3 \longrightarrow A_3$$

satisfies $r_3 \circ j_3 = d_1 d_2 Id$.□

From this we can derive the following

Theorem 2.2 *Let k be a field, and $X_1, ..., X_r$ be finite Γ_k-sets. For $i = 1, ..., r$, denote by J_i the kernel of the canonical surjection $\mathbb{Z}^{X_i} \xrightarrow{\pi_i} \mathbb{Z}$. Let $J = \otimes_i J_i$. If the orders of the X_i are two by two coprime, then we have an exact sequence*

$$0 \longrightarrow J \longrightarrow \bigoplus_{I \in \mathfrak{I}_0} \mathbb{Z}^{\Pi_{i\in I} X_i} \xrightarrow{\pi} \bigoplus_{I \in \mathfrak{I}_1} \mathbb{Z}^{\Pi_{i\in I} X_i} \longrightarrow 0,$$

where $\mathfrak{I}_i$ is the set of subsets of $\{1, ..., r\}$ whose cardinality is congruent to $r - i$ mod 2. In particular, a k-torus with character module isomorphic to J is stably rational over k. What is more, let d denote the product of the orders of the X_i, $i = 1, ..., r$. Then there exists

$$s : \bigoplus_{I \in \mathfrak{I}_1} \mathbb{Z}^{\Pi_{i\in I} X_i} \longrightarrow \bigoplus_{I \in \mathfrak{I}_0} \mathbb{Z}^{\Pi_{i\in I} X_i}$$

such that $\pi \circ s = dId$.

Proof. For $i = 1, ..., r$, we have a canonical map

$$s_i : \mathbb{Z} \longrightarrow \mathbb{Z}^{X_i},$$

$$1 \mapsto \sum_{x \in X_i} x,$$

which satisfies $\pi_i \circ s_i = d_i Id$, where d_i is the order of X_i. The proof is then an easy induction using the previous lemma and the obvious isomorphism $\mathbb{Z}^X \otimes \mathbb{Z}^Y \simeq \mathbb{Z}^{X \times Y}$, for any two finite sets X and Y. □

As a particular case of this theorem, we recover a result due to Voskresenskii ([**?**], chap. 2, 5.1 corollary).

Corollary 2.3 *Let k be a field, and l/k a Galois extension with cyclic Galois group G of ordrer $n = p_1 ... p_r$, where the p_i are prime numbers. Let σ be a generator of this Galois group, and T/k the n^{th} cyclotomic torus, i.e. the torus with character group isomorphic to $\mathbb{Z}[X]/\phi_n(X)$, where $\phi_n(X)$ is the n^{th} cyclotomic polynomial, the action of σ being given by multiplication by X (in other words, the character group of T is isomorphic to the ring of integers of the n^{th} cyclotomic extension of $\mathbb{Q}$, with the action of σ being given by multiplication by a primitive n^{th} root of unity). Then T is stably rational over k.*

Proof. For $i = 1, ..., r$, let X_i be the unique quotient of G isomorphic to $\mathbb{Z}/p_i$. With the notations of the preceding theorem, the Γ_k-module J is isomorphic to the character module of T (this is just the fact that the ring of integers of the n^{th} cyclotomic extension of $\mathbb{Q}$ is naturally isomorphic to the tensor product of the rings of integers of the p_i^{th} cyclotomic extensions of $\mathbb{Q}$), whence the claim.□

We are now able to give a simple proof of the following theorem.

Theorem 2.4 *(Klyachko) Let k be a field, and X, Y two finite Γ_k-sets, of coprime orders p and q, respectively. Consider the two basic exact sequences*

$$0 \longrightarrow J_X \longrightarrow \mathbb{Z}^X \longrightarrow \mathbb{Z} \longrightarrow 0,$$

$$0 \longrightarrow J_Y \longrightarrow \mathbb{Z}^Y \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Then, a k-torus T with character module isomorphic to $J := J_X \otimes J_Y$ is rational over k.

Proof. Select integers u, v such that $up - vq = 1$. Theorem **??** gives the following presentation of J:

$$0 \longrightarrow J \longrightarrow \mathbb{Z}^{X\times Y} \oplus \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}^X \oplus \mathbb{Z}^Y \longrightarrow 0,$$

where $\pi(x \otimes y, 0) = (x, y)$ and $\pi(0, 1) = (u \sum_{x\in X} x, v \sum_{y\in Y} y)$. Let E/k (resp. F/k) be the etale extension of k corresponding to X (resp. to Y). Then, in terms of tori, this exact sequence reads as

$$1 \longrightarrow \mathrm{R}_{E/k}(\mathbb{G}_m) \times \mathrm{R}_{F/k}(\mathbb{G}_m) \xrightarrow{i} \mathrm{R}_{E\otimes_k F/k}(\mathbb{G}_m) \times \mathbb{G}_m \longrightarrow T \longrightarrow 1,$$

where R denotes Weil scalar restriction. The map i is given on the k-points of the considered tori by the following formula:

$$i(x, y) = (x \otimes y, N_{E/k}(x)^u N_{F/k}(y)^v), x \in E^*, y \in F^*.$$

Thus, we have a generically free action of the algebraic k-group $H := \mathrm{R}_{E/k}(\mathbb{G}_m) \times \mathrm{R}_{F/k}(\mathbb{G}_m)$ on the k-vector space $V := (E \otimes_k F) \oplus k$, such that T is birational to the quotient V/H (of course, such a quotient is defined up to birational equivalence only).
Assume that $p < q$. Let G/k be the algebraic k-group $\mathrm{GL}_k(E) \times \mathrm{R}_{F/k}(\mathbb{G}_m)$ (E being viewed as a k-vector space). I claim that the action of H on V can be naturally extended to an action of G on V (H being viewed as a

subgroup of G the obvious way). Indeed, this new action is given on the k-points by the formula, for $g = (\phi, y) \in G(k), v = (e \otimes f, \lambda) \in V$:

$$g.v = (\phi(e) \otimes yf, \det(\phi)^u N_{F/k}(y)^v \lambda).$$

This action is generically free. Indeed, this is an easy consequence of the equality $up - vq = 1$ and of the following lemma.

Lemma 2.5 *Let G act on $E \otimes_k F$ the obvious way. Then the stabilizer of a generic element is the subgroup $\mathbb{G}_m$ of G given, on the level of k-points, by elements of the form $(x, x^{-1}) \in \mathrm{GL}_k(E) \times F^*$, for $x \in k^*$.*

We postpone the proof until the end of this section. Assuming this lemma, we have a birational G-equivariant isomorphism $V \simeq (V/G) \times G$, where the action of G on the right is given by translation. Indeed, this is a direct consequence of Hilbert's theorem 90, asserting that $H^1(l, G) = 1$ for any field extension l of k. Hence we have birational isomorphisms

$$T \simeq V/H \simeq V/G \times G/H.$$

It is clear that the k-variety $G/H = \mathrm{GL}_k(E)/\mathrm{R}_{E/k}(\mathbb{G}_m)$ is k-rational. As in Klyachko's original proof, the key point is here that the k-variety (defined up to birational equivalence) V/G is independent of E (seen as an etale k-algebra). Hence, the birational equivalence class of T is independent of E; we may therefore assume that E is split, i.e. that the action of Γ_k on X is trivial. But then J is isomorphic to ${J_Y}^{p-1}$, hence T is birational to $(\mathrm{R}_{F/k}(\mathbb{G}_m)/\mathbb{G}_m)^{p-1}$, which is a rational variety (it is an open subvariety of $(\mathbb{P}_k^{q-1})^{p-1}$). □

Proof of lemma **??**. We may assume that F is split, i.e. $F = k^q$ as an etale k-algebra. Let f_i, $i = 1, ..., q$ denote the canonical k-basis of F. Consider an element $w = \sum_i e_i \otimes f_i \in E \otimes_k F$ in general position. Let $g = (\phi, (\lambda_1, ..., \lambda_q)) \in \mathrm{GL}_k E \times F^*$ be such that $g.w = w$. This amounts to saying that $\phi(e_i) = \lambda_i^{-1} e_i$ for all i. Since $p < q$ and since w is in general position, $e_1, ..., e_p$ form a basis of E with respect to which the i'th component of e_{p+1} is non zero for all $i = 1, ..., p$. This readily implies that the λ_i are all equal to some scalar λ and that $\phi = \lambda^{-1} Id$, thus proving the claim. □

References

[Vo] V. E. Voskresenskii. — *Algebraic Groups and their Birational Invariants*, Translations of Mathematical Monographs **179** (1991), Amer. Math. Soc.

[Kl] A. A. Klyachko. — *On rationality of tori with a cyclic splitting field*, Arithmetic and Geometry of Varieties, Kuibyshev Univ. Press (1988), 73-78 (russian).