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Mémoire d'Habilitation à Diriger des Recherches

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Dynamical singularities

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Jury

Contents

Introduction

Short Curriculum Vitae

I was born the 30 April 1984 in Bonn (Germany), italian nationality, married.

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Current situation

Since 1st September 2013, I am maître de conférences at the Université Paris Cité, UFR de Mathématiques, research unit IMJ-PRG, group Géométrie et Dynamique. This position is equivalent to Associate Professor.

(Selected) education and diplomas

- A.Y. 2002-2007: Student in Mathematics at Scuola Normale Superiore (Pisa), and at the Università di Pisa.
- 01.2008-12.2010: PhD student in Mathematics at Scuola Normale Superiore (Pisa, Italy).
- 15.03.2011: PhD degree in Mathematics at Scuola Normale Superiore di Pisa, with thesis: "The valuative tree, rigid germs and Kato varieties", advisor M. Abate, vote 70/70 with honors.
- 11.11.2020-2029: National Scientific Qualification (Italy), Associate Professor.

Employment

- 05-06.2011: Internship by Scuola Normale Superiore (Pisa, Italy) at the École Polytechnique de Paris, CMLS lab (Palaiseau, France).
- 09.2011-08.2013: Post-doc of the FMJH at the École Polytechnique, CMLS lab (Palaiseau, France).
- 09.2013-present: Maître de conférences at the Université Paris Cité (ex Université de Paris, ex Université Paris-Diderot), UFR de Mathématiques, research unit IMJ-PRG, group Géométrie et Dynamique.

Long visits abroad

- 19.04-08.07.2018: ICL-CNRS fellowship, Academic visitor at Imperial College, London, UK.
- 26.08-29.12.2018: Visiting professor at BICMR, PKU, Beijing, China.
- 26.08-27.09.2019: Visiting professor at BICMR, PKU, Beijing, China.
- 09.10.2023-07.01.2024: Visiting professor at BICMR, PKU, Beijing, China.

Summary scientific supervision

2 Phd students, 2 Master students.

Summary organization of events

9 Conferences, 2 Seminars, 1 Lecture group.

Summary of talks

18 talks and 1 mini-course in conferences,
45 talks and 2 mini-courses in research seminars,
6 talks and 1 mini-course in lecture groups,
3 popularization talks.

Summary of scientific evaluation

2 hiring committées, referee for 9 journals and 1 research project.

Research supervision activities

Courses and minicourses

- 04,11,18.09.2019 (5h) : Beijing (China), BICMR PKU, "Local dynamical systems and valuations".
- 06-09.11.2019 (4h30) : Istanbul (Turkey), Galatasaray University, Minischool on singularities of surfaces, "Intersection theory on valuation spaces and applications".
- 2019-20 and 2020-21 classes "Holomorphic Dynamical Systems", 5th year Maths, at Université Paris Diderot (Paris, France).
- 25.10, 1-8-15-22.11, 6-13.12.2013 (10h30) : Beijing (China), BICMR PKU, "Resolution of vector fields and applications to parabolic dynamics".

Master thesis mentoring

- 2020: Damien Coll, "Réduction des singularités d'un feuilletage holomorphe en dimension 2".
- 2021: Kémo Morvan, "Classification des automorphismes contractants sur les singularités normales de surfaces complexes".

PhD students mentoring

- 2022-2025: Kémo Morvan, École Doctorale 386 Université Paris Cité, in codirection with André Belotto da Silva.
- 2022-2028: Bilal Balo, École Doctorale 386 Université Paris Cité, in codirection with Hussein Mourtada.

List of publications

We present here my list of publications: you find in blue the publications spawned from my phd thesis, or written during the doctorate, and in red the publications presented in this manuscript.

Research papers

[1] M.Ruggiero: "Rigidification of holomorphic germs with non-invertible differential". Michigan Mathematical Journal, Volume 61 Issue 1, pp. 161–185, 2012. http://projecteuclid.org/.

[2] M.Ruggiero: "Contracting rigid germs in higher dimensions". Annales de l'Institut Fourier, Volume 63 Issue 5, pp. 1913–1950, 2013. http://www.numdam.org/.

[3] W.Gignac and M.Ruggiero: "Growth of attraction rates for iterates of a superattracting germ in dimension two". Indiana University Mathematics Journal, Volume 63, no.4, pp. 1195–1234, 2014. http://www.iumj.indiana.edu/.

[4] C.Favre and M.Ruggiero: "Normal surface singularities admitting contracting automorphisms". Annales Mathématiques de la faculté des sciences de Toulouse, Volume 23, no. 4, pp. 797–828, 2014. http://afst.cedram.org/.

[5] M.Ruggiero: "Classification of one dimensional superattracting germs in positive characteristic". Ergodic Theory and Dynamical Systems, Volume 35, Issue 7, pp. 2242–2268, 2015. http://journals.cambridge.org/.

[6] M.Ruggiero and K.Shaw: "Tropical Hopf manifolds and contracting germs". Manuscripta Mathematica, Volume 152, Issue 1-2, pp. 1–60, 2017. http://link.springer.com/.

[7] W.Gignac and M.Ruggiero: "Local dynamics of non-invertible maps near normal surface singularities". Memoirs of the AMS 272, no. 1337, xi+100 pages, 2021. https://bookstore.ams.org/.

[8] E.García Barroso, P. González Pérez, P. Popescu-Pampu and M.Ruggiero: "Ultrametric properties for valuation spaces of normal surface singularities". Transactions of the AMS, Volume 372, Issue 12, pp. 8423–8475, 15 December 2019. https://www.ams.org/.

[9] L.Fantini, C.Favre and M.Ruggiero: "Links of sandwiched surface singularities and self-similarity". Manuscripta Mathematica, Volume 162, Issue 1-2, pp. 23–65, 2020. https://link.springer.com/.

[10] N.Istrati, A.Otiman, M.Pontecorvo and M.Ruggiero: "Toric Kato manifolds". Journal de l'École polytechnique, Volume 9, pp. 1347–1395, 2022.
https://jep.centre-mersenne.org/.

[11] S.Mongodi and M.Ruggiero: "Birational properties of tangent to the identity germs without non-degenerate singular directions". Journal of the London Mathematical Society, pp. 1–55, 2023.
https://onlinelibrary.wiley.com/.

Preprints

[12] R.Dujardin, C.Favre and M.Ruggiero: "On the dynamical Manin-Mumford conjecture for plane polynomial maps". Preprint, pp. 20, 2023.
https://arxiv.org/.

Book chapters

[c1] M.Ruggiero: "Dynamics of foliations in the Siegel domain", inside M.Abate (editor): "Local dynamics of singular holomorphic foliations". Edizioni ETS, Dipartimento di Matematica dell'Università di Pisa, Dottorato di ricerca in matematica.

Other publications

[o1] M.Ruggiero: Rigid germs, the valuative tree, and applications to Kato Varieties. PhD thesis defended the 15/03/2011. Edizioni della Normale, Volume 20, pp. XXVI-170, 2015.
http://edizioni.sns.it/.

Presentation of my research activity

Topics presented in this memoir

Introduction

Dynamical systems modelize motion, the changing of the state of a given system with time. In a *discrete* dynamical system the time is measured by the number of seconds elapsed. Maps $f_t\colon X \circlearrowleft$ describe the change of configuration from x to $f_t(x)$ that occurs from an instant $t \in \mathbb{N}$ (or $t \in \mathbb{Z}$) to the next. When the maps $f_t =: f$ do not depend on t, we say that the system is *autonomous*, and a particle starting at configuration x_0 at time $t = 0$ will be at configuration $x_n = f^n(x_0)$ at time $t = n$, where f^n is the n-th iterate of the selfmap $f\colon X \circlearrowleft$. Discrete dynamical systems have a *continuous* counterpart, where time is modelized by real (or complex) numbers, and the object under study is the flow defined by a family of maps $f^t\colon X \circlearrowleft$ satisfying $f^{t+t'} = f^t \circ f^{t'}$ for any t, t'.

The goal of dynamical systems is to understand the behavior of the orbits of points in X, and the asymptotic behavior when $t \to \infty$. Often the configuration space X supports additional geometrical structure, which is preserved by the maps acting on X. Depending on the structure, the goals and techniques involved may vary, and give theories with very peculiar flavors. For example, when X is a topological space, we study the asymptotic properties of orbits, such as topological entropy or topological mixing; when we consider measures on X, we are interested in the ergodic properties of the system. Differential, analytic or algebraic structures on X add rigidity on the maps we consider, that can be exploited to study the corresponding dynamical systems.

In my research activity, I am mainly interested in the local dynamics of analytic germs $f\colon (X, Z) \circlearrowleft$, where X is a (possibly singular) analytic variety, Z is a proper subvariety of X, and f is an analytic map defined on a neighborhood of Z and leaving Z invariant (i.e., $f(Z) \subseteq Z$; often Z is simply a fixed point for f). I mostly work over the field $\mathbb{C}$ of complex numbers; nevertheless, in several situations such as deformations of dynamical systems or specialization over subfields, we are led to work over more general metrized fields, and in particular over non-archimedean fields.

In the analytic setting, some of the natural objects of study (spaces, maps, vector fields, foliations, etc.) can be seen as sections of coherent sheaves that are locally free in a Zariski-open set, and degenerate along (proper) analytic subvarieties, called *singularities* of the given object. The structure of these objects at regular points is rather simple: regular points of complex analytic varieties look locally as the germ $(\mathbb{C}^d, 0)$ of the complex space at the origin; holomorphic maps are local diffeomorphisms outside their critical locus; holomorphic vector fields in $(\mathbb{C}^d, 0)$ are locally equivalent to ∂_{z_1}, and regular foliations of dimension r locally look like the horizontal plane distribution $\{(z_{r+1}, \ldots, z_d) = \text{const}\}$, where $(z_1, \ldots, z_d)$ are local coordinates at 0.

The singularities of an analytic map $f\colon (X, x_0) \to (Y, y_0)$ correspond to its critical set $\mathcal{C}(f)$, where f is not locally invertible. The geometry of $\mathcal{C}(f)$ in general quite complicated: by blowing-up the source space, one can solve the singularities of $\mathcal{C}(f)$; by further blow-ups at the source and target space, one can obtain local monomialization results (see [Cut99, Cut17, BB23]).

For analytic selfmaps $f\colon (X,x_0)\circlearrowleft$, we need to consider simultaneous changes of coordinates at (X,x_0), which is both the source and the target space of the map f. The added rigidity of the allowed transformations makes the problem of classification of dynamical systems up to change of coordinates much harder. The local structure of the *dynamical singularities* (the fixed or periodic points of a map f) can be described either by finding *normal forms*, i.e., simpler expressions in local coordinates, or by *resolution* of the singularities via blow-ups or more general bimeromorphic transformations; more often than not, we need a combination of the two approaches: a *reduction* of the dynamical singularities to simpler families, for which it is possible to find normal forms.

Notice that for germs $f\colon (\mathbb{C}^d,0)\circlearrowleft$, the linear part (and more precisely, its Jordan normal form) is an invariant of (formal and holomorphic) conjugacy, which gives a first approximation of the dynamical behavior of f. While generic analytic diffeomorphisms are analytically linearizable, non-generic situations often come out, both when studying global dynamics and pathological behaviors (see for example the construction of a quadratic polynomial with a Julia set with positive area [BC12], or the construction of wandering domains for polynomial endomorphisms [ABD+16]), or when studying bifurcation phenomena in families of dynamical systems (see for example the parabolic and semi-parabolic implosion of [Dou94, Shi00, BSU17]); or again in applications in geometry (see, e.g., [KS22]), physics and celestial mechanics.

My research focuses principally in two classes of selfmaps, where the linear part gives the least informations on the dynamics: non-invertible endomorphisms (for example, superattracting germs, where the linear part vanishes), and local automorphisms that are tangent to the identity. In both cases, infinitely many (linearly independent) obstructions to the reduction to simple (polynomial) normal forms occur, and explicit normal forms cannot be found. This suggests the richness of the moduli space of these classes of holomorphic germs up to conjugacy, and the necessity of reducing the singularities via bimeromorphic transformations.

For germs $f\colon (\mathbb{C}^d,0)\circlearrowleft$ obtained as high order perturbations of the identity, this process realizes via the reduction of singularities of their associated *infinitesimal generator*: a (possibly non-convergent) vector field χ whose flow at time 1 gives f. In this case the singularities of f are simply the singularities of χ, and the classical reduction of singularities (when available) of χ and its associated foliation by curves allows to describe the very intricate geometry of the basin of attraction of f.

In the superattracting case, the rough dynamical behavior is quite clear: orbits converge exponentially fast to the origin. But when studying finer dynamical properties (speed of convergence of a given orbit, structure of invariant curves, etc.) the situation becomes much more intricated. In the search for bimeromorphic models that resolve the dynamical singularities given by periodic orbits, we often create indeterminacy points, that in general cannot be eliminated. In dimension 2, the indeterminacy set is finite: the new type of dynamical singularities that we wish to eliminate via blow-ups consists of periodic indeterminacy points. Models without such dynamical singularities are called *algebraically stable* (in the sense of Fornaess and Sibony); algebraic stability allows a good control of the dynamics, sufficient for the construction of invariant objects such as measures and currents. In order to find algebraically stable models, it is sometimes necessary to allow (mild) singularities on the ambient space. This phenomenon is parallel to the necessity to allow cyclic quotient singularities for the resolution of singularities of vector fields in dimension 3, or other classes of singularities in the Minimal Model Program for projective varieties.

The appearence of singularities on algebraically stable models motivated my interest in local dynamical systems defined on singular varieties (germs of analytic varieties will be simply called *singularity* from now on; they will often supposed to be normal). Another reason is given by a very interesting phenomenon: the mere existence of dynamical systems with special properties forces the singularity itself to be relatively mild. This is somehow natural, since one would expect the existence of lots of endomorphisms on a singularity to reflect its symmetries. Some of these dynamical data can be used to construct special compact complex manifolds (as compactifications of orbit spaces), which do not admit Kähler forms. Non-Kähler manifolds remain quite mysterious, and the interplay between geometry and dynamics allows a better understanding of both aspects for these classes of manifolds.

Most of my contributions in these topics rely on the study of dynamical systems induced on valuation spaces, an approach initiated by Favre and Jonsson around 20 years ago. Valuation spaces have been intensively studied since the works of Zariski, following his local uniformization approach to resolution of singularities. After a slow decline in prestige due to the absence of valuations in Hironaka's proof of resolution of singularities of varieties, valuation spaces saw a comeback, due to their application in the resolution of singularities in positive characteristic, to their relationship with rigid geometry and analytic structures initiated by Berkovich, and to their use in the study of birational properties of local and global analytic dynamical systems. Valuation and Berkovich spaces are also related to non-archimedean dynamics, deformation of algebraic structures, as well as toric and tropical geometry.

Dynamics of non-invertible germs

As mentioned above, normal forms for non-invertible selfmaps are hard to find. There are several natural candidate normal forms, for example monomial maps, or the (quasi-)homogeneous parts of smallest degree, but these are too special to model even generic dynamical systems. See for example the monomialization results (on wedges) by Ueda and Ushiki [Ued86, Ush92], or the more recent results by Ueno (see, e.g., [Uen16]); or again the classification of superattracting germs conjugated to their homogeneous part of smallest degree, by [BEK12].

As remarked in [HP94], one of the main obstructions to finding normal forms is given by the complicated structure of the *generalized critical set* $C(f^\infty)$, defined as the union of the critical sets of the iterates of f. This remark leads to dividing the task of styding non-invertible germs into two steps.

Step 1: to find normal forms for special families, where the formal obstructions do not appear (hence, where the generalized critical set is relatively simple); among other contributors to this step, we mention Buff, Epstein, Favre, Hubbard, Koch, Papadopol, Ueda, Ueno, Ushiki (see, e.g., [Ued86, Ush92, HP94, Fav00, BEK12, Rug12, Rug13, Uen16, Uen19b, Uen19a]).

Step 2: to reduce any non-invertible germ to one belonging to the special classes of *Step* 1 via *modifications*, i.e., proper birational maps $\pi : X_\pi \to (X, x_0)$; among other contributors to this step, we mention Diller, Favre, Gignac, Hasselblatt, Jonsson, Lin, Propp, Wulcan (see, e.g., [DF01, Fav02, HP07, FJ07, FJ11, JW11, Rug12, Lin12, LW14, GR14, GR21]).

Both these steps are essentially achieved in dimension 2. One of the main contributions of my works with William Gignac on *Step 2* is a generalization of the works of [FJ07] to the singular setting, and an adaptation to the local setting of techniques developed in [FJ11]. We can summarize the state of the art with the following statement.

Theorem A ([FJ07, Rug12, GR14, GR21])**.** *Let $f : (X, x_0) \circlearrowleft$ be a (dominant) non-invertible selfmap at a normal surface singularity. Assume that f is not a finite germ at a cusp singularity. Then for any modification $\pi : X_\pi \to (X, x_0)$, there exists another modification $\pi' : X_{\pi'} \to (X, x_0)$ dominating π (i.e., $\pi^{-1} \circ \pi'$ is regular), so that the model $X_{\pi'}$ is* algebraically stable*: for any irreducible curve $E \subset \pi'^{-1}(0)$, $f^n_{\pi'}(E)$ is not an indeterminacy point of $f_{\pi'}$ for $n \gg 0$.*

Moreover, one can pick π' so that there exists a fixed point $p \in X_{\pi'}$ where $f_{\pi'}$ is rigid*, i.e., $\mathcal{C}(f^\infty_{\pi'})$ has simple normal crossings at p.*

Algebraic stability is a property of global algebraic dynamics, introduced by Fornaess and Sibony (see [Sib99, Section 1.4]). It is crucial to control the action in cohomology defined by the pullback of f, which allows the construction of invariant objects, such as measures and currents.

By [Fav00], *contracting* rigid germs of $(\mathbb{C}^2, 0)$ have explicit holomorphic normal forms, where a germ is "contracting" if the eigenvalues of its linear part are in the unit disk. Notice that, even when starting with a superattracting germ $f : (\mathbb{C}^2, 0) \circlearrowleft$, the reduction to rigid germs of [FJ07, Rug12] does not necessarily give contracting rigid germs: one could get rigid germs whose linear part has eigenvalues $(0, \lambda)$ with λ any complex number. The non-contracting case is very subtle: it presents Stokes phenomena when λ is a root of unity, which prevent the convergence of the formal conjugacy (but allow convergence along sectors); when $|\lambda| > 1$ we have even poorer summability properties. As contributions of my work to *Step 1*, one can find in [Rug12] formal normal forms and comments on the summability of the conjugacy maps for non-contracting rigid germs in dimension 2. In [Rug13] there is a partial generalization of Favre's result in higher dimensions, where we highlight new resonance phenomena for rigid germs, in the interaction between the linear part and the monomial part of the rigid germ.

Topics around Theorem A are developed mainly in Chapter 2.

Singularities admitting special dynamical systems

In Theorem A we can remark the notable exception of finite germs on cusp singularities. This is an instance of a broader phenomenon, which relates the geometry of a singularity, and the type of dynamcal systems that it can support. It is easy to show that any singularity (X, x_0) supports non-finite, high topological degree selfmaps, and infinite-order automorphisms that are high order perturbations of the identity by [Mül87]. However, when we impose some special dynamical properties of the selfmap $f :: (X, x_0) \circlearrowleft$, it turns out that the singularity (X, x_0) has to be rather special. We collect in the next statement known results for normal surface singularities.

Theorem B. *A normal surface singularity (X, x_0) admits:*

(a) [Wah90] a non-trivial finite selfmap if and only if it is log-canonical.

(b) [FR14] a contracting automorphism if and only if it is quasi-homogeneous.

(c) [FFR20] a non-finite map of topological degree 1 if and only if it is sandwiched.

In the case of finite maps, Favre reproves Wahl's theorem and shows the existence of cusp singularities supporting finite germs which do not admit algebraically stable models. In [GR21], we generalize Favre's construction, and show that any cusps admits such dynamical data. Wahl's result has been generalized to higher dimensions by [BdFF12] to $\mathbb{Q}$-Gorenstein isolated singularities, and further generalized by [BH14, Zha17].

The case of contracting automorphisms generalizes the classification of surfaces admitting good $\mathbb{C}^\star$-actions by [OW71], and of surfaces admitting dicritical $\mathbb{C}^\star$-actions by [CMS09]. One can also give normal forms for the contracting automorphisms.

The statement (c) is available over fields of any characteristic, for algebraic singularities that are "selfsimilar", in the sense that there exists a (non-trivial) modification $\pi : X' \to (X, x_0)$ and $x_1 \in X'$ so that the formal completions of rings of regular functions of X at x_0 and of X' at x_1 are isomorphic.

The results $(b{,}c)$ have a strong connection to the study of a certain class of non-Kähler compact complex surfaces S, obtained as compactifications of the space of the orbits of a germ $f \colon (X, x_0) \circlearrowleft$ of topological degree 1. More precisely, we say that a germ $f \colon (X, x_0) \circlearrowleft$ is a *Kato germ* if it can be decomposed as $f = \pi \circ \sigma$, where $\pi \colon X_\pi \to (X, x_0)$ is a modification of (X, x_0), and $\sigma \colon (X, x_0) \to (X_\pi, x_1)$ is a local biholomorphism, with $x_1 \in \pi^{-1}(x_0)$. Automorphisms are trivially of this form by setting $\pi = \mathrm{id}$, while one can show by flattening techniques [HLT73, Hir75] that the topological degree 1 germs of Theorem B.(c) are also Kato germs (with π non an isomorphism). When the Kato germ $f = \pi \circ \sigma$ is also contracting, one can define a compact complex surface $S = S(\pi, \sigma)$ as a compactification of the orbit space of $\sigma \circ \pi$ on $X_\pi \setminus \pi^{-1}(x_0)$.

In [FR14, FFR20] we classify these surfaces, giving an alternative proof to [Kat79], which has the merit to avoid using Kodaira's classification of compact complex surfaces, and takes advantage instead of the geometry of the underlying singularities.

These themes are explored in Chapter 3.

Intersection theory on valuation spaces

One of the most fruitful strategies to study the action of a group (or semigroup) G of selfmaps of a variety X consists in finding the right functional space V where G acts, and translate the property we want to study in terms of easier-to-handle properties of this action. One can find in [FJ07, HP08, BFJ08a, Fav10, FJ11, Can11, Rug12, GR14, Xie15, BC16, GR21, DF21, Abb24] examples of this strategy at work.

In our setting, the functional space we consider is the space $\mathcal{V}_X$ of all (normalized, rank-one semi-)valuations of the singularity (X, x_0). It has been deeply studied in [FJ04] in the smooth case, and in [Fav10] in the singular case. This space is endowed naturally with a topology, and contains as a dense subset all *divisorial* valuations, which encode the exceptional primes for all resolution $\pi : X_\pi \to (X, x_0)$.

Loosely speaking, the space $\mathcal{V}_X$ ressembles a graph, modeled on the dual graph of a good resolution of (X, x_0), having branching points at any divisorial valuation (with the branches being in correspondence with the closed points of the associated exceptional primes). It can be embedded inside an infinite-dimensional vector space, consisting of *exceptional b-divisors* (in the sense of Shokurov [Sho03]), i.e., collections of exceptional divisors in any good resolution X_π of (X, x_0) satisfying a natural compatibility condition.

A selfmap $f\colon (X, x_0) \circlearrowleft$ defines naturally a continuous map $f_\bullet\colon \mathcal{V}_X \circlearrowleft$, which encodes the lifts of f to all good resolutions. For example, if ν_E is the divisorial valuation associated to an exceptional prime E of a modification $\pi\colon X_\pi \to (X, x_0)$, we can consider another modification $\varpi\colon X_\varpi \to (X, x_0)$ so that the lift $\widetilde{f} := \varpi^{-1} \circ f \circ \pi\colon X_\pi \dashrightarrow X_\varpi$ sends E onto an exceptional prime E' of X_ϖ. In this case, $f_\bullet \nu_E = \nu_{E'}$.

The proof of Theorem A, and in part of Theorem B.(c), relies on a careful description of the dynamics of $f_\bullet$, and in particular, of the structure of locally attracting fixed points (called *eigenvaluations*), and of the global attraction properties towards such eigenvaluations.

Theorem C ([GR21]). *Let $f\colon (X, x_0) \to (X, x_0)$ be a (dominant) non-invertible germ at a normal surface singularity. There exists a set $S \subset \mathcal{V}_X$ (which is either a point, a segment, or a circle) that attracts the orbit $(f_\bullet^n \nu)$ of any* divisorial *valuation $\nu \in \mathcal{V}_X^{div}$.*

One of the tools introduced in [GR21] is a distance ρ_X on $\mathcal{V}_X$, called *angular distance*. It can be described algebraically via relative Izumi constants: roughly speaking, we compare the values taken by ν and μ on any regular function $\phi\colon (X, x_0) \to \mathbb{C}$. The angular distance plays the role of the Kobayashi distance in our setting: the map $f_\bullet$ is non-expanding with respect to ρ_X.

More precisely, we show that $f_\bullet$ is a weak contraction (it strictly decreases positive distances) when f is *non-finite*: this allows to get fixed point theorems in this situation, and Theorem C with $S = \{\nu_\star\}$ a point. When f is *finite*, we apply Theorem B.(a), which simplifies greatly the geometry of $\mathcal{V}_X$.

There are three main ingredients in the study of the contraction properties of $f_\bullet$. The first ingredient is a geometrical interpretation of the angular distance, in terms of the intersection of b-divisors associated to valuations. The second ingredient is the study of the action induced by f as pullback and pushforward of b-divisors: we extend to arbitrary germs the study portrayed in [Fav10] for finite germs. The last ingredient is a positivity property of the intersection of valuations. If we denote by $\langle \nu, \mu \rangle \in (0, +\infty]$ the opposite of the intersection of the b-divisors associated to μ and ν, then the positivity property can be stated as follows.

Proposition D ([GR21]). *Let $\nu, \mu_1, \mu_2 \in \mathcal{V}_X$ be three distinct valuations. Then*

$$\langle \nu, \mu_1 \rangle \langle \nu, \mu_2 \rangle \leq \langle \nu, \nu \rangle \langle \mu_1, \mu_2 \rangle ,$$

with equality if and only if μ_1 and μ_2 belong to distinct connected components of $\mathcal{V}_X \setminus \{\nu\}$.

Proposition D provides a very powerful tool to study the birational properties of singular spaces. It is indeed one of the main tools in [GGPR19], where we study the ultrametric properties of the functional defined by $u_\lambda(\nu_1, \nu_2) = 0$ when $\nu_1 = \nu_2$, and

$$u_\lambda(\nu_1, \nu_2) := \frac{\langle \lambda, \nu_1 \rangle \cdot \langle \lambda, \nu_2 \rangle}{\langle \nu_1, \nu_2 \rangle}$$

when $\nu_1 \neq \nu_2$.

Theorem E ([GGPR19])**.** *The functional u_λ defines an (extended) ultrametric on $\mathcal{V}_X$, for either one or any $\lambda \in \mathcal{V}_X$, if and only if $\mathcal{V}_X$ is contractible.*

This statement can be reinterpreted in terms of properties of the local intersection at x_0 of branches of (X, x_0); our works generalize to general surface singularities the works of [Pło85] on $(\mathbb{C}^2, 0)$, and of [GGP18] on arborescent singularities.

We decided to split the treatment of these topics between Chapter 1 and Chapter 2: in the first chapter we present the intersection theory on valuation spaces $\mathcal{V}_X$, and Proposition D and Theorem E; in the second chapter we focus on the action induced by maps f on valuation spaces and on b-divisors, and we prove Theorem C and Theorem A.

Tangent to the identity germs and parabolic manifolds

Here we focus on the dynamics of selfmaps $f\colon (X, x_0) \circlearrowleft$ that are high order perturbations of the identity, called *tangent to the identity* germs. *Parabolic germs* (for which an iterate is tangent to the identity) have a central role in the study of dynamical systems already in dimension one, expecially when considering their deformations, when the fixed point of multiplicity h splits into h (repelling) fixed points, giving rise to a phenomenon known as *parabolic implosion*, see [Dou94, Shi00, BSU17]. Parabolic implosion, and 2-dimensional tangent to the identity germs, are central in the construction of polynomial skew-products in $\mathbb{C}^2$ having wandering domains of [ABD+16]. Parabolic germs also appear in classification problems of geometrical structures: for example, see the formal and analytic classification of real analytic surfaces in $\mathbb{C}^2$ with suitable CR-structures portrayed in [KS22].

Notice that, by [Mül87], there are infinitely many (independent) analytic vector fields tangent to any singularity (X, x_0), hence many tangent to the identity germs. While the singular setting would be interesting to develop, we remark that, at least in dimension 2, one can always lift a tangent to the identity germ $f\colon (X, x_0) \circlearrowleft$ to the minimal resolution $\pi\colon X_\pi \to (X, x_0)$, so that the lift $f_\pi\colon X_\pi \circlearrowleft$ is regular. Moreover, we can always embed (X, x_0) into an affine space $(\mathbb{C}^N, 0)$, and assume that f is the restriction of a regular germ $F\colon (\mathbb{C}^N, 0) \circlearrowleft$. For these reasons, we work in the regular setting $(X, x_0) = (\mathbb{C}^d, 0)$, and write f in local coordinates as $f(z) = z + f^{(h)}(z) + \text{h.o.t.}$, where $f^{(h)} \not\equiv 0$ is homogeneous of degree $h \geq 2$, called the *order* of f.

While the rough dynamical behavior of contracting germs is quite clear, for tangent to the identity germs the basin of attraction of f at 0 has a very rich structure to discover;

in particular, we aim at describing the basin of attraction

$$\mathcal{B}_\star = \{z \in (\mathbb{C}^d, 0) \ : \ f^n(z) \xrightarrow{\star} 0\}$$

of f at the origin (excluding the grand-orbit of 0), where $\star$ describes possible asymptotic behaviors of the orbits: they can converge tangentially to some complex direction $v \in \mathbb{P}^{d-1}_{\mathbb{C}}$, to some real direction $\zeta \in \mathbb{C}v$, or asymptotically to a (possibly formal) irreducible curve C. Even if being tangent to the identity gives restrictions to the possible asymptotic behaviors of orbits, other situations may happen; for example, orbits that can spiral around several complex directions, see [Riv98, Ron14, ABT22] and recent works by Buff and Raissy.

The basin $\mathcal{B}_\star$ can be often described as the union of preimages of so called *$\star$-parabolic manifolds*, i.e., f-invariant submanifolds Δ, so that $0 \in \partial\Delta$. This is the case for example in dimension $d = 1$, by the classical Leau-Fatou flower theorem [Lea97, Fat19]: $\mathcal{B}$ is the union of $h - 1$ basins $\mathcal{B}_\zeta$, tangent to real directions ζ.

In higher dimensions, it is not hard to show that $\mathcal{B}_v$ is empty for any $v \in \mathbb{P}^{d-1}_{\mathbb{C}}$, unless v is a *characteristic direction*: either a fixed point (*non-degenerate* case) or an indeterminacy point (*degenerate* case) for the action induced by $f^{(h)}$ on $\mathbb{P}^{d-1}_{\mathbb{C}}$. Similarly, $\mathcal{B}_C$ is empty for any irreducible curve C, unless C is f-invariant and $f|_C \neq \mathrm{id}_C$ (we call such curves *separatrices*).

The study of tangent to the identity germs $f\colon (\mathbb{C}^d, 0) \circlearrowleft$ and the construction of parabolic manifolds is intimately related to vector fields: f can be described uniquely as the time-1-flow of a (possibly formal) vector field χ of the same order as f. This vector field is denoted $\log f$, and called *infinitesimal generator* of f. Notice that χ is not necessarily saturated (i.e., we could have that the zero locus of χ has codimension 1); by considering its saturation $\hat{\chi}$ (obtained by simplifying $\chi = \sum_j \chi^j \partial_{z_j}$ by the common factor to all the coordinates χ^j), we lose the correspondence with germs given by the flow, but we retain the geometrical informations related to the induced foliation $\mathcal{F}$.

The existence of parabolic manifolds is established mainly in two cases: along *non-degenerate* characteristic directions by Hakim [Hak98], and along separatrices by [LRSV22].

The latter result is based on a reduction to so called *Ramis-Sibuya normal forms* up to weighted blow-ups, that can be seen as a local uniformization result of the infinitesimal generator χ along the separatrix C. Similarly, the former result can be seen as a construction of parabolic manifolds along (possibly transcendental) separatrices associated to non-zero eigenvalues of a reduced singularity.

In dimension 2, the singularities of χ can be resolved by [Sei68]: by studying the reduction process, Camacho and Sad prove in [CS82] the existence of separatrices for vector fields; as a consequence, Abate proves in [Aba01] the existence of parabolic curves for isolated fixed point germs. A key property here is the existence of non-degenerate characteristic directions in some resolution of χ, where we can apply Hakim's result. While the study of parabolic manifolds in dimension 2 associated to complex directions is essentially done, see e.g., [Hak98, Aba01, Mol09, Viv12, Lap16, LS18, LRRS21, LR20], new exciting phenomena have been observed recently, such as Leau-Fatou-flower type of results by [LR22], or examples by Buff-Raissy of parabolic curves and domains attached to compact leaves of the real foliation induced by $f^{(h)}$ on $\mathbb{P}^1_{\mathbb{C}}$.

Resolution of singularities of vector fields is available also in dimension 3 by [MP13], as long as we allow weighted blow-ups (along loci of singularities of the saturated vector field; we refer to these modifications as *adapted* to either χ or $f = \exp \chi$). However, separatrices do not exist in general (see [GL92]); this translates to examples of tangent to the identity germs having no parabolic manifolds attached to curves (see [AT03]). Nonetheless, the characteristic directions associated to the reduced singularities of [GL92] come necessarily from non-degenerate characteristic directions, and Hakim's result ensure the existence of parabolic curves attached to complex directions.

In [MR21], we study a rather large family of tangent to the identity isolated fixed points in $(\mathbb{C}^3, 0)$, of the form

$$f(x, y, z) = (x + yz(y - z) + P, y + x(x^2 - z^2) + Q, z + xz(y - z) + R), \qquad (*)$$

with P, Q, R of order at least 4.

The expression of $f^{(3)}$ has been inspired by examples by [Iva11] of rational maps in $\mathbb{P}^2_{\mathbb{C}}$ having no holomorphic fixed points; we deduce that f has only degenerate characteristic directions. The singularities of the infinitesimal generator χ can be resolved, obtaining a model $\pi_0 \colon X_{\pi_0} \to (\mathbb{C}^3, 0)$, which is obtained either by four standard blow-ups, or by two weighted blow-ups. These germs present a new pathological behavior.

Theorem F ([MR21])**.** *Let $f \colon (\mathbb{C}^3, 0) \circlearrowleft$ be a tangent to the identity germ of the form (*). For a Zariski-generic choice of the parameters P, Q, R, the germ f satisfies the following property:*

> *For any regular modification $\pi \colon X_\pi \to (\mathbb{C}^3, 0)$ adapted to f and dominating π_0, and for any point $p \in \pi^{-1}(0)$ in the exceptional divisor, the lift $f_\pi \colon X_\pi \circlearrowleft$ of f at p has only degenerate characteristic directions.*

The main contribution of this result, besides the example itself, is the identification of several classes of reduced singularities for the (saturated) infintesimal generator, together with conditions on the saturation divisor, that can be studied explicitly, via the techniques developed in [LRSV22], in order to describe the structure of the parabolic manifolds. Particularly interesting is the case of reduced germs having a 2-dimensional central manifold, since in this case, the available results on existence of parabolic manifolds cannot be applied. Our study is a first step towards understanding these pathological situations, possibly showing the existence of parabolic manifolds for any 3-dimensional tangent to the identity germs.

While the use of valuation techniques to the study of tangent to the identity germs is practically absent in the literature, we mention a few natural considerations. An invertible germ $f \colon (\mathbb{C}^d, 0) \circlearrowleft$ induces an action $f_\bullet \colon \mathcal{V} \circlearrowleft$, which preserves many functionals on $\mathcal{V}$ (such as skewness, log-discrepancy, multiplicity). While the set $\mathrm{Fix}(f_\bullet)$ is generically very simple, and does not give much information (for example, it does not allow to distinguish between contracting and non-contracting germs), in the tangent to the identity case $\mathrm{Fix}(f_\bullet)$ has a very rich structure (it is an infinite subcomplex of $\mathcal{V}$). It allows to identify f-invariant curves, as well as characteristic directions (though the identification is quite convoluted), and should give quite a lot of informations about the formal classification of tangent to the identity germs. Nevertheless, it remains quite hard to characterize which subcomplexes of $\mathcal{V}$ can be the set of fixed points of the action induced by a tangent to the identity germ. One can also work at the level of infinitesimal generators χ and their induced foliation $\mathcal{F}$, and consider some natural functionals induced by $\mathcal{F}$ on $\mathcal{V}$, related to the canonical bundle of $\mathcal{F}$ (see also [For98] for other approaches in this direction).

These topics will be exposed in Chapter 4, the last of this manuscript.

Other works after the PhD

The papers presented in detail in this memoir represent my primary research focus, in relation with the title of the manuscript. Here we present briefly my other results in related topics, again from my research activity after my PhD thesis. In particular, we present here a result of classification of superattracting germs in positive characteristic, the study of certain classes of non-Kähler manifolds in the setting of tropical, non-archimedean and toric geometry, and a recent preprint regarding some advances on the dynamical Manin-Mumford problem.

Classification of one-dimensional superattracting germs in positive characteristic - [Rug15]

Any superattracting germ $f\colon (\mathbb{C},0) \circlearrowleft$ is analytically conjugated to a power map $z \mapsto z^d$ for some $d \geq 2$. While this classical result extends effortlessly over any (valued) field $\mathbb{K}$ of characteristic zero, the study of normal forms in positive characteristic is more involved, because of the presence of resonances due to the Frobenius automorphism. In [Rug15], we give a complete answer to the formal and analytic classification problems of superattracting germs in dimension 1 in positive characteristic, for which only some special cases were known (see [GS11]). The formal classification is obtained by a careful study of the combinatorics regulating the conjugacy equation, while the convergence of the formal conjugacy is a fairly technical use of majorant series.

Tropical Hopf varieties - [RS17]

In a joint work with Kristin Shaw [RS17], we introduce a tropical analogue of Hopf varieties, which are the simplest case of non-Kähler varieties, and can be thought of as Kato varieties associated to a Kato datum (id, σ). The interest for these varieties comes as an investigation on what it means to be non-Kähler in a tropical and non-archimedean setting. With this in mind, we compute several invariants, such as the tropical Picard group and (p, q)-homology groups, whose corresponding objects in the complex setting allow to detect non-Kähler manifolds. Moreover, we relate our tropical construction with tropical contracting germs, and to analytic families of contracting germs (over the complex numbers), via the Berkovich analytification of a Hopf manifold (or more precisely its universal covering), defined over the field of (generalized) Laurent series (see [Pay09]).

Toric Kato manifolds - [IOPR22]

We study a special class of Kato manifolds, which we call toric Kato manifolds. They arise as compactifications for the space of orbits of suitable monomial contracting germs (more precisely, associated to matrices with columns that are either elements of the canonical basis of the euclidean space, or positive vectors). Their construction comes from toric geometry, as their universal covers are open subsets of toric algebraic varieties of non-finite type. This generalizes previous constructions of Oda [Oda78] and Tsuchihashi [Tsu87], and in complex dimension 2, retrieves the properly blown-up Inoue-Hirzebruch surfaces. We study the topological and analytical properties of toric Kato manifolds and link certain invariants to natural combinatorial data coming from the toric construction. Moreover, we produce families of flat degenerations of any toric Kato manifold, which serve as an essential tool in computing their Hodge numbers. In the last part, we investigate some Hermitian geometry properties of Kato manifolds. We give a characterization result for the existence of locally conformally Kähler metrics on any Kato manifold. Finally, we prove that no Kato manifold carries balanced metrics and that a large class of toric Kato manifolds of complex dimension ≥ 3 do not support pluriclosed metrics.

On the dynamical Manin-Mumford problem for planar endomorphisms - [DFR23]

Let X be a smooth projective variety over $\mathbb{C}$, and let $f\colon X \circlearrowleft$ be an endomorphism of it. Assume that f is *polarized*, meaning that there exists an ample line bundle L on X such that $f^*L \simeq L^{\otimes d}$ for some $d \geq 2$. The key example is that of polynomial endomorphisms on $\mathbb{P}^k_{\mathbb{C}}$ of degree d. The set $\mathrm{Preper}(f)$ of preperiodic points (i.e., points of finite orbit) of a polarized endomorphism $f\colon X \circlearrowleft$ is Zariski-dense in X (see, e.g., [Fak03, Theorem 5.1]). In particular, if $Y \subseteq X$ is a preperiodic subvariety, then $\mathrm{Preper}(f|_Y)$ is Zariski-dense in Y. The dynamical Manin-Mumford (DMM) problem studies the converse implication:

> Under which conditions on the polarized dynamical system (X, f), and on the subvariety $Y \subseteq X$, we have that Y containing a Zariski-dense subset of preperiodic points implies that Y is necessarily preperiodic?

This problem, originally proposed by Zhang, is inspired by the Manin-Mumford conjecture on torsion points on abelian varieties, solved by Raynaud [Ray83a, Ray83b]: any subvariety of an abelian variety which has a Zariski-dense subset of torsion points is the torsion translate of an abelian subvariety. As observed by Northcott, this statement can be translated as the preperiodicity of subvarieties with a Zariski-dense subset of preperiodic points by the action of $f(x) = dx$, where d is any integer ≥ 2.

Since the original conjecture [Zha95, Zha06], several step forwards and updates to the conjecture have been proposed, see, e.g., [GTZ11, GT21].

In [DFR23], we are interested in the dynamical Manin-Mumford problem for regular endomorphisms of $\mathbb{P}^2_{\mathbb{C}}$ coming from polynomial endomorphisms of $\mathbb{C}^2$. Concretely, $f\colon \mathbb{P}^2_{\mathbb{C}} \circlearrowleft$ is the homogeneization of a map $f(x, y) = \big(P(x, y), Q(x, y)\big)$, where P, Q are polynomials of degree $d \geq 2$ such that their homogeneous parts P_d and Q_d of degree d have no common factors. Note that under this assumption, f induces a rational map $f_\infty = [P_d : Q_d]$ of degree d on the line at infinity L_∞. In this setting, the DMM problem consists in showing that if a curve C has infinitely many preperiodic points, then it is itself preperiodic.

Despite the importance of this problem, very few cases are known in this setting: we mention for example the work [GNY18], that deals with product maps of the form $f(x, y) = \big(P(x), Q(y)\big)$. In their proof, the product structure of these maps is a fundamental assumption, since it allows to reduce the study to 1-dimensional dynamical phenomena.

In contrast, our recent result deals with honest 2-dimensional dynamical phenomena: we solve the DMM problem for polynomial endomorphisms as above for which the rational map f_∞ has no superattracting periodic points.

While the statement is purely algebraic, the techniques we use span between analytical methods, working either on the complex archimedean field $\mathbb{C}$ or the non-archimedean p-adic fields $\mathbb{C}_p$ (depending on context, i.e., on the multiplier of f_∞ at given fixed points at L_∞), and arithmetic ideas, related to classical concepts such heights, where we first assume that f is defined over a number field.

Présentation de mon activité de recherche

Thèmes présentés dans ce mémoire

Introduction

Les systèmes dynamiques modélisent le mouvement et le changement d'état d'un système donné dans le temps. Dans un système dynamique *discret* le temps est modélisé par les entiers. Des applications $f_t \colon X \circlearrowleft$ décrivent le changement de configuration de x à $f_t(x)$ qui se produit d'un instant $t \in \mathbb{N}$ (ou $t \in \mathbb{Z}$) au successif. Lorsque l'application $f_t =: f$ ne dépend pas de t, on dit que le système est *autonome*, et une particule démarrant en configuration x_0 à l'instant $t = 0$ sera en configuration $x_n = f^n(x_0)$ à l'instant $t = n$, où f^n est le n-ième itéré de application $f \colon X \circlearrowleft$. Les systèmes dynamiques discrets ont une contrepartie *continue*, où le temps est modélisé par les réels (ou les complexes), et l'objet étudié est le flot défini par une famille d'applications $f^t \colon X \circlearrowleft$ satisfaisant $f^{t+t'} = f^t \circ f^{t'}$ pour tout t, t'.

Le but des systèmes dynamiques et de comprendre le comportement des orbites des points dans X, et leur comportement asymptotique quand $t \to +\infty$. Souvent l'espace X des configurations est muni de structures géométriques supplémentaires, qui sont préservées par les applications agissant sur X. Selon la nature de ces structures, les objectifs et les techniques impliquées peuvent varier, et donnent des théories avec un goût très spécifique. Par exemple, lorsque X est un espace topologique, nous nous concentrons sur les propriétés asymptotiques des orbites, comme l'entropie topologique ou les systèmes topologiquement mélangeants ; lorsque nous considérons des mesures sur X, nous nous intéressons aux propriétés ergodiques du système. Les structures différentielles, analytiques ou algébriques sur X ajoutent de la rigidité aux applications que nous considérons, ce qui peut être exploité pour étudier les systèmes dynamiques correspondants.

Dans mes activités de recherche, je m'intéresse principalement à la dynamique des applications analytiques f définies sur des variétés analytiques (peut-être singulières) X, autour d'une sous-variété f-invariante Z (c'est-à-dire, $f(Z) \subseteq Z$; souvent Z est tout simplement un point fixé par f). Je travaille surtout sur le corps $\mathbb{C}$ des nombres complexes ; cependant, dans plusieurs situations telles que les déformations des systèmes dynamiques ou les spécialisations sur des sous-corps, nous sommes amenés à travailler sur des corps valués plus généraux, et en particulier sur des corps non-archimédiens.

Dans le cadre analytique, certains objets naturels (espaces, applications, champs de vecteurs, feuilletages, etc.) peuvent être vus comme des sections de faisceaux cohérents qui sont localement libres dans un ouvert de Zariski, et dégénérés le long de sous-variétés analytiques (propres), appelées *singularités* de l'objet en question. La structure de ces objets sur leur lieu régulier est plutôt simple : les points réguliers des variétés analytiques complexes ressemblent localement au germe $(\mathbb{C}^d, 0)$ de l'espace complexe à l'origine ; les applications holomorphes sont des difféomorphismes locaux en dehors de leur lieu critique ; les champs de vecteurs holomorphes dans $(\mathbb{C}^d, 0)$ sont localement équivalents à ∂_{z_1}, et les feuilletages réguliers de dimension r ressemblent localement à la distribution en plans horizontaux $(z_{r+1}, \dots, z_d) = \text{const}$, où $(z_1, \dots, z_d)$ sont des coordonnées locales en 0.

Les singularités d'une application analytique $f \colon (X, x_0) \to (Y, y_0)$ correspondent aux points dans l'ensemble critique $\mathcal{C}(f)$, où f n'est pas localement inversible. La géométrie de $\mathcal{C}(f)$ est plutôt compliquée: en éclatant à la source, on peut résoudre les singularités de $\mathcal{C}(f)$; en éclatant ultérieurement à la source et au but, on obtient des résultats de monomialisation locale (voir [Cut99, Cut17, BB23]).

Pour les systèmes dynamiques locales $f\colon (X, x_0) \circlearrowleft$ définis autour d'un point fixe $x_0 \in X$, on doit considérer des changements simultanés des coordonnées de l'espace (X, x_0), qui est à la fois la source et le but de l'application f. Cette rigidité supplémentaire des transformations autorisées rend la classification à conjugaison près très difficile à obtenir en général. La structure locale des *singularités dynamiques* (les points fixes ou périodiques d'une application f) peut être décrite soit en trouvant des *formes normales*, c'est-à-dire des expressions plus simples dans des coordonnées locales, soit en effectuant une *résolution* des singularités via des éclatements ou des transformations birationnelles plus générales; le plus souvent, on a besoin d'une combinaison des deux: une *réduction* des singularités dynamiques à transformation birationnelle près, où les singularités réduites peuvent être étudiées à travers de formes normales.

Dans l'étude des formes normales analytiques et formelles des germes $f\colon (\mathbb{C}^d, 0) \circlearrowleft$, la partie linéaire (et plus précisément, sa forme normale de Jordan) est un invariant de conjugaison qui donne une première approximation du comportement dynamique de f. Alors que les difféomorphismes analytiques génériques sont analytiquement linéarisables, des situations non génériques se présentent souvent : dans des comportements pathologiques en dynamique globale (voir par exemple la construction d'un polynôme quadratique avec ensemble de Julia de surface positive [BC12], ou la construction de domaines errants pour des endomorphismes polynomiaux [ABD$^+$16]), ou lors de l'étude des phénomènes de bifurcation dans des familles de systèmes dynamiques (voir par exemple l'implosion parabolique et semi-parabolique de [Dou94, Shi00, BSU17]); ou encore dans des applications en géométrie (voir, par exemple, [KS22]), en physique et en mécanique céleste.

Ma recherche se concentre principalement sur deux classes d'endomorphismes, où la partie linéaire ne fournit presque aucune information sur les propriétés fines de la dynamique : les endomorphismes non inversibles (par exemple, les germes super-attractifs, où la partie linéaire s'annule), et les automorphismes locaux qui sont tangents à l'identité. Dans les deux cas, un nombre infini (linéairement indépendant) d'obstructions à la réduction à des formes normales simples (polynomiales) se produit, et des formes normales explicites ne peuvent pas être trouvées. Cela suggère la richesse de l'espace des modules de ces classes de germes holomorphes à conjugaison près, et la nécessité d'une étape de réduction des singularités par des transformations biméromorphes.

Pour les germes $f\colon (\mathbb{C}^d, 0) \circlearrowleft$ obtenus comme perturbations de l'identité, ce processus se réalise via la résolution des singularités de leur *générateur infinitésimal* associé : un champ vectoriel (pas nécessairement convergent) χ dont le flot au temps 1 donne f. Dans ce cas, les singularités de f sont simplement les singularités de χ, et la résolution classique (quand elle existe) des singularités de χ et son feuilletage en courbes permet de décrire la géométrie très complexe du bassin d'attraction de f.

Dans le cas superattractif, le comportement dynamique global est assez clair : les orbites convergent exponentiellement vite à l'origine. Mais si on veut étudier des propriétés dynamiques plus fines (vitesse de convergence d'une orbite donnée, structure des courbes invariantes, etc.), la situation devient beaucoup plus complexe. Dans la recherche de modèles biméromorphes qui résolvent les singularités dynamiques données par les orbites périodiques, nous créons souvent des points d'indétermination, qui en général ne peuvent pas être éliminés. En dimension 2, l'ensemble des points d'indétermination est fini : le nouveau type de singularités dynamiques que nous souhaitons éliminer via éclatements sont les points d'indétermination périodiques. Les modèles qui n'ont pas de telles orbites sont appelés *stables algébriquement* (au sens de Fornaess et Sibony) ; la stabilité algébrique permet un bon contrôle de la dynamique, suffisant pour la construction d'objets invariants tels que mesures et courants. Pour trouver des modèles algébriquement stables, il est parfois nécessaire d'autoriser des singularités (pas trop méchantes) sur l'espace ambiant. Ce phénomène est parallèle à la nécessité de permettre des singularités quotient cyclique pour la réduction des singularités des champs de vecteurs en dimension 3, ou aux plusieurs classes de singularités apparaissant dans le Programme des Modèles Minimaux de la géométrie algébrique.

L'apparition de singularités dans les modèles algébriquement stables a motivé mon intérêt dans les systèmes dynamiques définis autour de singularités de variétés analytiques (qu'on appellera tout simplement *singularités* dans le reste de ce texte). Une autre raison est due à un phénomène très intéressant : la simple existence de systèmes dynamiques avec des propriétés spéciales force la singularité elle-même à être relativement gentille. Cela est plutôt naturel, car on s'attendrait à ce que l'existence de nombreux endomorphismes sur une singularité reflète ses symétries.

Certaines de ces données dynamiques peuvent être utilisées pour construire des variétés complexes compactes spéciales (comme des compactifications d'espaces d'orbites), qui sont non-kähleriennes. Les variétés non-kähleriennes restent assez mystérieuses, et l'interaction entre la géométrie et la dynamique permet une meilleure compréhension des deux aspects pour ces classes de variétés.

La plupart de mes contributions dans ce domaine utilisent des techniques valuatives en dynamique, une branche de recherche initiée par les travaux de Favre et Jonsson il y a environ 20 ans. Les espaces de valuations ont été intensivement étudiés depuis les travaux de Zariski, suivant son approche de résolution des singularités par uniformisation locale.

Après un lent déclin de prestige dû à l'absence de valuations dans la preuve de Hironaka de la résolution des singularités des variétés, les espaces de valuation ont fait un retour en force, à la fois grâce à leur application dans la résolution des singularités en caractéristique positive, à leur lien avec la géométrie rigide et ses structures analytiques initiée par Berkovich, et à leur utilisation dans l'étude des propriétés birationnelles des systèmes dynamiques locaux et globaux. Les espaces des valuations et les analytifiés de Berkovich sont aussi liés à la dynamique non-archimédienne, à la déformation des structures algébriques, ainsi qu'à la géométrie torique et tropicale.

Dynamique des germes non-inversibles

Comme mentionné auparavant, les formes normales des germes non-inversibles sont plus difficiles à trouver. Il existe plusieurs formes normales candidates naturelles, par exemple les applications monomiales ou les parties (quasi-)homogènes de degré minimal, mais cellesci sont trop spéciales pour modéliser même des systèmes dynamiques génériques. Voir par exemple les résultats de monomialisation (sur les coins) par Ueda et Ushiki [Ued86, Ush92], ou les résultats plus récents de Ueno (voir, par exemple, [Uen16]); ou encore la classification des germes super-attractifs conjugués à leur partie homogène de degré minimal, par [BEK12].

Comme remarqué dans [HP94], une obstruction à une liste explicite de formes normales est donnée par la structure trop compliquée de l'*ensemble critique généralisé* $C(f^\infty)$, défini comme la réunion des ensembles critiques des itérés de f. Cela nous amène à partager la tâche en deux étapes.

Étape 1: trouver des formes normales pour des familles spéciales, où ce type d'obstructions n'est pas présent (où l'ensemble critique généralisé est simple) ; parmi les chercheurs qui ont travaillé sur ce sujet, on mentionne Buff, Epstein, Favre, Hubbard, Koch, Papadopol, Ueda, Ueno, Ushiki (voir par exemple [Ued86, Ush92, HP94, Fav00, BEK12, Rug12, Rug13, Uen16, Uen19b, Uen19a]).

Étape 2: réduire tout germe non-inversible à un qui appartient à une famille étudiée dans l'*Étape* 1 par des *modifications*, c'est-à-dire, des applications birationnelles propres $\pi : X_\pi \to (X, x_0)$; ici on mentionne aussi les travaux de Diller, Favre, Gignac, Hasselblatt, Jonsson, Lin, Propp, Wulcan (voir par exemple [DF01, Fav02, HP07, FJ07, FJ11, JW11, Rug12, Lin12, LW14, GR14, GR21]).

Les deux étapes ont essentiellement été franchies en dimension 2. Une des contributions principales de mes travaux en collaboration avec William Gignac pour l'*Étape 2* est une généralisation des résultats de [FJ07] au cadre singulier, et une adaptation au cadre local des techniques développées dans [FJ11]. On peut résumer l'état de l'art dans l'énoncé suivant.

Théorème A ([FJ07, Rug12, GR14, GR21]). *Soit $f\colon (X, x_0) \circlearrowleft$ un germe (dominant) non-inversible sur une singularité normale de surface. Supposons que f n'est pas un germe fini sur une singularité cusps. Alors, pour toute modification $\pi : X_\pi \to (X, x_0)$, il en existe une autre $\pi' : X_{\pi'} \to (X, x_0)$ qui domine π (c'est-à-dire, $\pi^{-1} \circ \pi'$ est régulière), telle que le modèle $X_{\pi'}$ est* algébriquement stable *: pour tout premier exceptionnel $E \subset \pi'^{-1}(0)$, $f^n_{\pi'}(E)$ n'est pas un point d'indétermination de $f_{\pi'}$ pour $n \gg 0$.*

De plus, on peut choisir π' de façon qu'il existe $p \in X_{\pi'}$ où $f_{\pi'}$ soit rigide*: $\mathcal{C}(f^\infty_{\pi'})$ est à croisements normaux simples en p.*

La stabilité algébrique est une propriété des systèmes dynamiques algébriques, introduite par Fornaess et Sibony, voir [Sib99, Section 1.4]. Elle est cruciale pour contrôler l'action de f en cohomologie, qui permet la construction d'objets invariants, comme mesures ou courants.

Par [Fav00], nous obtenons également des formes normales pour les germes rigides *contractants* de $(\mathbb{C}^2, 0)$, où un germe est "contractant" si les valeurs propres de sa partie linéaire sont dans le disque unité. On remarque que, même en commençant avec un germe super attractif $f\colon (\mathbb{C}^2, 0) \circlearrowleft$, la réduction aux germes rigides de [FJ07, Rug12] ne donne pas nécessairement des germes rigides contractants : on pourrait obtenir des germes rigides dont la partie linéaire a des valeurs propres $(0, \lambda)$ avec λ un nombre complexe quelconque. Le cas non contractant est très subtil: il présente des phénomènes de Stokes lorsque λ

est une racine de l'unité, qui empêchent la convergence de la conjugaison formelle (mais permet la convergence le long de secteurs); lorsque $|\lambda| > 1$ on a des propriétés de sommabilité encore plus faibles. Comme contributions de mon travail à *Étape 1*, on peut trouver dans [Rug12] des formes normales formelles et des commentaires sur la sommabilité des applications de conjugaison pour les germes rigides non contractants en dimension 2. Dans [Rug13], il y a une généralisation partielle du résultat de Favre en dimensions supérieures, où nous mettons en évidence de nouveaux phénomènes de résonance pour les germes rigides, qui font intervenir la partie linéaire et la partie monomiale du germe rigide.

Les thèmes autour du Théorème A seront développés dans le Chapître 2.

Singularités supportant des systèmes dynamiques spéciaux

Dans Théorème A on peut remarquer l'exception remarquable des germes finis sur des singularités cusp. Cela représente une instance d'un phénomène plus large, qui relie la géométrie d'une singularité au type de systèmes dynamiques qu'elle peut supporter. Il est relativement simple de montrer que toute singularité (X, x_0) admets des endomorphismes non-finis et de grand degré topologique, ainsi que des automorphismes d'ordre infini qui sont des perturbations de grand ordre de l'identité, par [Mül87]. Mais quand on impose des propriétés dynamiques spéciales de l'endomorphisme $f\colon (X, x_0) \circlearrowleft$, on trouve que la singularité (X, x_0) doit elle aussi être spéciale. Dans le théorème suivant, on retrouve trois énoncés dans le cas de singularités normales de surfaces.

Théorème B. *Une singularité normale de surface (X, x_0) admet :*

(a) [Wah90] un endomorphisme fini non-trivial si et seulement si elle est log-canonique.

(b) [FR14] un automorphisme contractant si et seulement si elle est quasi-homogène.

(c) [FFR20] un endomorphisme non-fini de degré topologique 1 si et seulement si elle est sandwich.

Dans le cas d'endomorphismes finis, Favre donne une preuve alternative du théorème de Wahl, et montre l'existence de singularités cusps avec des germes finis qui n'admettent pas de modèles algébriquement stables. Dans [GR21], on généralise la construction due à Favre, en montrant que toute singularité cusp admet une telle donnée dynamique. Le résultat de Wahl a été généralisé en dimension supérieure par [BdFF12] pour les singularités isolées $\mathbb{Q}$-Gorenstein, et d'autres généralisations ont été données par [BH14, Zha17].

Le cas d'automorphismes contractants généralise la classification des surfaces avec une bonne action $\mathbb{C}^\star$ par [OW71], et dec elles avec une action $\mathbb{C}^\star$ dicritique par [CMS09]. Dans ce cas, on peut exhiber des formes normales pour les automorphismes contractants.

L'énoncé (c) est valable sur des corps de n'importe quelle caractéristique, pour des singularités algébriques "autosimilaires", dans le sens qu'il existe une modification $\pi : X' \to (X, x_0)$ (non triviale) et $x_1 \in X'$ tels que le complété formel de l'anneau des fonctions régulières de X en x_0 et de X' en x_1 sont isomorphes.

Les résultats (b,c) ont un lien très étroit à l'étude de certaines classes de surfaces complexes compactes non-kahleriennes S, obtenues comme compactifications de l'espace des orbites d'un germe $f\colon (X,x_0)\circlearrowleft$ de degré topologique égal à 1. Plus précisément, un germe $f\colon (X,x_0)\circlearrowleft$ est dit *de Kato* s'il peut être factorisé en $f=\pi\circ\sigma$, où $\pi\colon X_\pi\to(X,x_0)$ est une modification de (X,x_0), et $\sigma\colon (X,x_0)\to(X_\pi,x_1)$ est un biholomorphisme local, pour un certain $x_1\in\pi^{-1}(x_0)$. Automorphismes sont évidemment de cette forme, en désignant $\pi=\mathrm{id}$; on peut montrer par des techniques d'aplatissement [HLT73, Hir75] que les germes de degré topologique 1 dans Théorème B.(c) sont des germes de Kato (avec π qui n'est pas un isomorphisme). Quand le germe de Kato $f=\pi\circ\sigma$ est en plus contractant, on peut construir une surface complexe compacte $S=S(\pi,\sigma)$ en tant que compactification de l'espace des orbites de $\sigma\circ\pi$ sur $X_\pi\setminus\pi^{-1}(x_0)$.

En [FR14, FFR20], nous donnons une classification de ces surfaces, en obtenant ainsi une preuve alternative de [Kat79], qui a le mérite de ne pas utiliser la classification de Kodaira des surfaces complexes compactes, mais elle profite plutôt de la géométrie des singularités sous-jacentes.

Ces thèmes seront explorés dans le Chapître 3.

Théorie de l'intersection sur les espaces de valuation

L'une des stratégies les plus fructueuses pour étudier l'action d'un groupe (ou semi-groupe) G d'endomorphismes d'une variété X consiste en trouver le bon espace fonctionnel V sur lequel G agit, et en traduire la propriété qu'on veut étudier en termes de propriétés plus faciles à gérer de cette action. On peut trouver dans [FJ07, HP08, BFJ08a, Fav10, FJ11, Can11, Rug12, GR14, Xie15, BC16, GR21, DF21, Abb24] des exemples de cette stratégie à l'œuvre.

Dans notre contexte, l'espace fonctionnel que nous considérons est l'espace $\mathcal{V}_X$ de toutes les (semi-)valuations (normalisées) de rang 1 de la singularité (X,x_0). Il a été étudié en profondeur dans [FJ04] dans le cas lisse, et dans [Fav10] dans le cas singulier. Cet espace est naturellement muni d'une topologie, et contient comme sous-ensemble dense toutes les valuations *divisorielles*, qui codent les diviseurs exceptionnels pour toute résolution $\pi : X_\pi\to(X,x_0)$.

L'espace $\mathcal{V}_X$ ressemble à un graphe, modélisé sur le graphe dual d'une bonne résolution de (X,x_0), ayant des points de branchement à toute valuation divisorielle (les branches correspondant aux points fermés des diviseurs exceptionnels associés). Il peut être immergé dans un espace vectoriel de dimension infinie, constitué de *b-diviseurs exceptionnels* (au sens de Shokurov [Sho03]), c'est-à-dire des collections de diviseurs exceptionnels dans toute bonne résolution X_π de (X,x_0) et satisfaisant une condition de compatibilité naturelle.

Un germe d'endomorphisme $f\colon (X,x_0)\circlearrowleft$ définit naturellement une application continue $f_\bullet\colon \mathcal{V}_X\circlearrowleft$, qui code les relevés de f dans toute résolution birationnelle. Par exemple, soit ν_E une valuation divisorielle associée à un premier exceptionnel E d'une modification $\pi\colon X_\pi\to(X,x_0)$. On peut trouver une autre modification $\varpi\colon X_\varpi\to(X,x_0)$ telle que le relèvement $\widetilde{f}:=\varpi^{-1}\circ f\circ\pi\colon X_\pi\dashrightarrow X_\varpi$ envoie E sur (le point générique d') un premier exceptionnel E' de X_ϖ. Dans ce cas, $f_\bullet\nu_E=\nu_{E'}$.

La preuve du Théorème A, et en partie du Théorème B.(c), repose sur une description minutieuse de la dynamique de $f_\bullet$, et en particulier, sur la description de la structure des points fixes localement attractifs (appelés *valuations propres*), et des propriétés d'attraction globale vers de telles valuations propres.

Théorème C ([GR21])**.** *Soit $f\colon (X,x_0) \to (X,x_0)$ un (germe dominant) non-inversible dans une singularité de surface normale. Il existe un ensemble $S \subset \mathcal{V}X$ (qui est soit un point, un segment, ou un cercle) qui attire l'orbite $(f_\bullet^n \nu)$ de toute valuation* divisoriale *$\nu \in \mathcal{V}_X^{div}$.*

Un des outils introduits dans [GR21] est une distance ρ_X sur $\mathcal{V}_X$, appelée *distance angulaire*. Elle peut être décrite algébriquement via les constantes relatives d'Izumi : en gros, nous comparons les valeurs prises par ν et μ sur toute fonction régulière $\phi\colon (X,x_0) \to \mathbb{C}$. La distance angulaire joue le rôle de la distance de Kobayashi dans notre cadre : l'application $f_\bullet$ est non-expansive par rapport à ρ_X.

Plus précisément, nous montrons que $f_\bullet$ est une contraction faible (elle diminue strictement les distances positives) lorsque f est *non-finie* : cela permet d'obtenir des théorèmes de point fixe dans cette situation, et le Théorème C avec $S = \nu_\star$ un point. Lorsque f est *finie*, nous appliquons le Théorème B.(a), ce qui simplifie grandement la géométrie de $\mathcal{V}_X$.

Il y a trois ingrédients principaux pour l'étude des propriétés de contraction de $f_\bullet$.

Le premier ingrédient est une interprétation géométrique de la distance angulaire, en termes d'intersection des b-diviseurs associés aux valuations.

Le deuxième ingrédient est l'étude de l'action induite par f en tant que tirée-en-arrière et poussée-en-avant des *b*-diviseurs : nous étendons aux germes arbitraires l'étude présentée dans [Fav10] pour les germes finis.

Le dernier ingrédient est une propriété de positivité de l'intersection des valuations. Si nous désignons par $\langle \nu, \mu \rangle \in (0, +\infty]$ l'opposé de l'intersection des *b*-diviseurs associés à μ et ν, alors la propriété de positivité peut être formulée comme suit.

Proposition D ([GR21])**.** *Soient $\nu, \mu_1, \mu_2 \in \mathcal{V}_X$ trois valuations distinctes. Alors*

$$\langle \nu, \mu_1 \rangle \langle \nu, \mu_2 \rangle \leq \langle \nu, \nu \rangle \langle \mu_1, \mu_2 \rangle ,$$

avec égalité si et seulement si μ_1 et μ_2 appartiennent à des composantes connexes distinctes de $\mathcal{V}_X \setminus \nu$.

Proposition D constitue un outil très puissant pour étudier les propriétés birationnelles des espaces singuliers. En effet, c'est l'un des principaux outils dans [GGPR19], où nous étudions les propriétés ultramétriques de la fonction définie par $u_\lambda(\nu_1, \nu_2) = 0$ lorsque $\nu_1 = \nu_2$, et

$$u_\lambda(\nu_1, \nu_2) := \frac{\langle \lambda, \nu_1 \rangle \cdot \langle \lambda, \nu_2 \rangle}{\langle \nu_1, \nu_2 \rangle}$$

lorsque $\nu_1 \neq \nu_2$.

Théorème E ([GGPR19])**.** *La fonctionnelle u_λ définit une ultramétrique (étendue) sur $\mathcal{V}_X$, pour un λ donné ou pour tout $\lambda \in \mathcal{V}_X$, si et seulement si $\mathcal{V}_X$ est contractile.*

Cette affirmation peut être réinterprétée en termes de propriétés de l'intersection locale en x_0 des branches de (X, x_0); nos travaux généralisent au cas d'une singularité normale ceux de [Pło85] sur $(\mathbb{C}^2, 0)$, et de [GGP18] sur les singularités arborescentes.

Nous avons décidé de séparer le traitement de ces sujets entre les Chapîtres 1 et 2 : dans le premier chapitre, nous présentons la théorie de l'intersection sur les espaces de valuation $\mathcal{V}_X$, ainsi que Proposition D et Théorème E ; dans le deuxième chapitre, nous nous concentrons sur l'action induite par les applications f sur les espaces de valuation et sur les b-diviseurs, et prouvons Théorème C et Théorème A.

Germes tangents à l'identité et variétés paraboliques

On étudie la dynamique des automorphismes $f\colon (X, x_0) \circlearrowleft$ qui sont des perturbations de grand ordre de l'identité, appelés germes *tangents à l'identité*. Les *germes paraboliques* (dont un itéré est tangent à l'identité) jouent un rôle central dans l'étude des systèmes dynamiques déjà en dimension 1, surtout lorsqu'on considère leurs déformations: le point fixe de multiplicité h se déforme en h points fixes (répulsifs), donnant lieu à un phénomène connu sous le nom d'*implosion parabolique*, voir [Dou94, Shi00, BSU17]. L'implosion parabolique et les germes tangents à l'identité en deux dimensions jouent un rôle central dans la construction d'endomorphismes polynomiaux de $\mathbb{C}^2$ ayant des domaines errants, voir [ABD+16]. Les germes paraboliques apparaissent également dans les problèmes de classification des structures géométriques : par exemple, voir la classification formelle et analytique des surfaces analytiques réelles dans $\mathbb{C}^2$ ayant des structures CR adéquates, de [KS22].

Remarquez que, d'après [Mül87], il existe une infinité (indépendante) de champs vectoriels analytiques tangents à n'importe quelle singularité (X, x_0), donc de nombreux germes tangents à l'identité. Bien que le cadre singulier serait intéressant à développer, nous notons qu'au moins en dimension 2, on peut toujours relever un germe tangent à l'identité $f\colon (X, x_0) \circlearrowleft$ à la résolution minimale $\pi\colon X_\pi \to (X, x_0)$, de sorte que le relevé $f_\pi\colon X_\pi \circlearrowleft$ soit régulier. De plus, on peut toujours plonger (X, x_0) dans un espace affine $(\mathbb{C}^N, 0)$, et supposer que f est la restriction d'un germe régulier $F\colon (\mathbb{C}^N, 0) \circlearrowleft$. Pour ces raisons, nous travaillons dans le cadre régulier $(X, x_0) = (\mathbb{C}^d, 0)$, et écrivons f dans des coordonnées locales comme $f(z) = z + f^{(h)}(z) + \text{h.o.t.}$, où $f^{(h)} \not\equiv 0$ est homogène de degré $h \geq 2$, appelé l'*ordre* de f.

Alors que le comportement dynamique approximatif des germes contractants est assez clair, pour les germes tangents à l'identité le bassin d'attraction de f en 0 a une structure très riche à découvrir ; en particulier, nous visons à décrire les propriétés du bassin d'attraction. En particulier, on veut décrire le bassin d'attraction

$$\mathcal{B}_\star = \{z \in (\mathbb{C}^d, 0) \;:\; f^n(z) \xrightarrow{\star} 0\}$$

de f à l'origine (en excluant la grand-orbite de 0), où $\star$ décrit le comportement asymptotique des orbites : elles peuvent converger tangentiellement le long d'une direction complexe $v \in \mathbb{P}^{d-1}_{\mathbb{C}}$, d'une direction réelle $\zeta \in \mathbb{C}v$, ou d'une courbe irréductible (éventuellement formelle) C. Bien que le fait d'être tangent à l'identité impose des restrictions au comportement asymptotique des orbites, on peut avoir d'autres comportements possibles; par exemple des orbites peuvent spiraler autour de plusieurs directions complexes, voir [Riv98, Ron14, ABT22] et les travaux récents par Buff et Raissy.

Le bassin $\mathcal{B}_\star$ peut souvent être décrit comme l'union des préimages des *variétés $\star$-paraboliques*, c'est-à-dire des sous-variétés invariantes Δ pour lesquelles $0 \in \partial\Delta$. C'est le cas, par exemple, en dimension $d = 1$, par le théorème classique de la fleur de Leau-Fatou [Lea97, Fat19] : $\mathcal{B}$ est l'union de $h - 1$ bassins $\mathcal{B}_\zeta$, tangents aux directions réelles ζ.

En dimensions supérieures, il n'est pas difficile de montrer que $\mathcal{B}_v$ est vide pour tout $v \in \mathbb{PC}^{d-1}$, sauf si v est une *direction caractéristique* : soit un point fixe (*cas non dégénéré*) soit un point d'indétermination (*cas dégénéré*) pour l'action induite par $f^{(h)}$ sur $\mathbb{P}^{d-1}_{\mathbb{C}}$. De même, $\mathcal{B}_C$ est vide pour toute courbe irréductible C, sauf si C est f-invariante et $f|_C \neq \mathrm{id}_C$ (nous appelons de telles courbes *séparatrices*).

L'étude des germes tangents à l'identité $f \colon (\mathbb{C}^d, 0) \circlearrowleft$ et la construction des variétés paraboliques sont intimement liées aux champs de vecteurs : f peut être décrit de manière unique comme le flot au temps 1 d'un champ de vecteurs (pas forcement convergent) χ,du même ordre que f. Ce champs de vecteurs χ est souvent noté $\log f$, et appelé *générateur infinitésimal* de f. Remarquez que χ n'est pas nécessairement saturé (i.e., le lieu de zéros de χ pourrait avoir codimension 1); en considérant sa saturation $\hat{\chi}$ (qui simplifie $\chi = \sum_j \chi^j \partial_{z_j}$ par le facteur commun à toutes les coordonnées χ^j), nous perdons la correspondance avec les germes donnés par le flot, mais nous conservons les informations géométriques liées à la feuilletage induit $\mathcal{F}$.

L'existence de variétés paraboliques est obtenue dans deux cas : le long les directions caractéristiques *non-dégénérées* par [Hak98], et le long de séparatrices par [LRSV22].

Le dernier résultat est basé sur une réduction à éclatement à poids près à des formes normales, appelées *formes normales de Ramis-Sibuya*; cela peut être vu comme un résultat d'uniformisation locale du générateur infinitésimal χ le long de la séparatrice C. De même, le premier résultat peut être considéré comme une construction de variétés paraboliques le long de séparatrices (en générale transcendants) associées aux valeurs propres non nulles d'une singularité réduite.

En dimension 2, les singularités de χ peuvent être résolues par [Sei68] : en étudiant le processus de réduction, Camacho et Sad prouvent dans [CS82] l'existence de séparatrices pour les champs de vecteurs ; en conséquence, Abate prouve dans [Aba01] l'existence de courbes paraboliques pour les germes de points fixes isolés. Une propriété clé ici est l'existence de directions caractéristiques non dégénérées dans certaines résolutions de χ, où l'on peut appliquer le résultat de Hakim.

Bien que l'étude des variétés paraboliques en dimension 2 associées à des directions complexes soit essentiellement achevée, voir par exemple [Hak98, Aba01, Mol09, Viv12, Lap16, LS18, LRRS21, LR20], de nouveaux phénomènes passionnants ont été observés récemment, comme des résultats de type fleur de Leau-Fatou par [LR22], ou des exemples de courbes et de domaines paraboliques attachés à des feuilles compactes du feuilletage réel induit par $f^{(h)}$ sur $\mathbb{P}^1_{\mathbb{C}}$ par Buff-Raissy.

La résolution des singularités des champs de vecteurs est également disponible en dimension 3 par [MP13], si on permet des éclatements à poids (le long de lieux de singularités du champ de vecteurs saturé ; nous appelons ces modifications *adaptées* à χ, ou à $f = \exp \chi$). Cependant, les séparatrices n'existent pas en général (voir [GL92]) ; cela se traduit par des exemples de germes tangents à l'identité n'ayant pas de variétés paraboliques attachées à des courbes (voir [AT03]). Néanmoins, les directions caractéristiques associées aux singularités réduites de [GL92] proviennent nécessairement de directions caractéristiques non dégénérées, et le résultat de Hakim assure l'existence de courbes paraboliques attachées à des directions complexes.

Dans [MR21], nous étudions une famille assez grande de germes tangents à l'identité de $(\mathbb{C}^3, 0)$ avec 0 point fixe isolé, de la forme

$$f(x, y, z) = (x + yz(y - z) + P, y + x(x^2 - z^2) + Q, z + xz(y - z) + R), \qquad (*)$$

avec P, Q, R d'ordre au moins 4. L'expression de $f^{(3)}$ a été inspirée par des exemples de [Iva11] de fonctions rationnelles dans $\mathbb{P}^2_{\mathbb{C}}$ n'ayant pas de points fixes holomorphes ; on déduit que f n'a que des directions caractéristiques dégénérées. Les singularités du générateur infinitésimal χ peuvent être résolues, en obtenant un modèle $\pi_0 \colon X_{\pi_0} \to (\mathbb{C}^3, 0)$, composé de 4 éclatements standard, ou bien de 2 éclatements à poids. Ces germes présentent un nouveau comportement pathologique.

Théorème F ([MR21]). *Soit $f \colon (\mathbb{C}^3, 0) \circlearrowleft$ un germes tangent à l'identité de la forme (*). Pour un choix Zariski-générique des paramètres P, Q, R, le germes f satisfait la propriété suivante :*

> *Pour toute modification régulière $\pi \colon X_\pi \to (\mathbb{C}^3, 0)$ adaptée à f et dominant π_0, et pour tout point $p \in \pi^{-1}(0)$ dans le diviseur exceptionnel, le relèvement $f_\pi \colon X_\pi \circlearrowleft$ de f en p a uniquement des directions caractéristiques dégénérées.*

La contribution principale de ce résultat, en plus que pour l'exemple lui-même, est l'identification de plusieurs classes de singularités réduites pour le générateur infinitésimal (saturé), ainsi que des conditions sur le diviseur de saturation, qui peuvent être étudiées explicitement, via les techniques développées dans [LRSV22], afin de décrire la structure des variétés paraboliques. Particulièrement intéressant est le cas des germes réduits ayant une variété centrale de dimension 2, car dans ce cas, les résultats disponibles sur l'existence de variétés paraboliques ne peuvent pas être appliqués. Notre étude est une première étape vers la compréhension de ces situations pathologiques, montrant éventuellement l'existence de variétés paraboliques pour tous les germes tangents à l'identité en dimension 3.

Bien que l'utilisation des techniques d'évaluation pour l'étude des germes tangents à l'identité soit pratiquement absente dans la littérature, nous mentionnons quelques considérations naturelles. Un germe inversible $f \colon (\mathbb{C}^d, 0) \circlearrowleft$ induit une action $f_\bullet \colon \mathcal{V} \circlearrowleft$, qui préserve de nombreuses fonctionnelles sur $\mathcal{V}$ (telles que le défaut, la log-discrepance, la multiplicité). Alors que l'ensemble $\mathrm{Fix}(f_\bullet)$ est génériquement très simple et ne fournit pas beaucoup d'informations (par exemple, il ne permet pas de distinguer entre les germes contractants et non contractants), dans le cas tangent à l'identité $\mathrm{Fix}(f_\bullet)$ possède une structure très riche (il est un sous-complexe infini de $\mathcal{V}$). Il permet d'identifier les courbes invariantes par f, ainsi que les directions caractéristiques (bien que l'identification soit assez complexe), et devrait fournir beaucoup d'informations sur la classification formelle des germes tangents à l'identité. Néanmoins, il reste assez difficile de caractériser quels sous-complexes de $\mathcal{V}$ peuvent être obtenus comme l'ensemble des points fixes de l'action induite par un germe tangent à l'identité. On peut également travailler au niveau des générateurs infinitésimaux χ et de leur feuilletage induit $\mathcal{F}$, et considérer certaines fonctionnelles naturelles induites par $\mathcal{F}$ sur $\mathcal{V}$, liées au fibré canonique de $\mathcal{F}$ (voir aussi [For98] pour d'autres approches dans cette direction).

Ces sujets seront traités dans le Chapître 4, le dernier de ce manuscrit.

Autres travaux après le doctorat

Les articles présentés en détail dans ce mémoire représentent mon axe principal de recherche, en relation avec le titre du manuscrit. Ici, nous présentons brièvement d'autres résultats dans des domaines proches, toujours obtenus après mes travaux de thèse. En particulier, nous présentons ici un résultat de classification des germes super-attractifs en caractéristique positive, l'étude de certaines classes de variétés non-kählériennes dans le cadre de la géométrie tropicale, non-archimédienne et torique, ainsi qu'une prépublication récente concernant des avancées sur le problème de Manin-Mumford dynamique.

Classification des germes super-attractifs unidimensionnels en caractéristique positive - [Rug15]

Tout germe superattractif $f\colon (\mathbb{C},0)\circlearrowleft$ est analytiquement conjugué à une puissance $z \mapsto z^d$ pour un certain $d \geq 2$. Alors que ce résultat classique s'étend facilement sur n'importe quel corps (valué) $\mathbb{K}$ de caractéristique nulle, l'étude des formes normales en caractéristique positive est plus complexe, en raison de la présence de résonances dues à l'automorphisme de Frobenius. Dans [Rug15], nous donnons une réponse complète aux problèmes de classification formelle et analytique des germes superattractifs en dimension 1 en caractéristique positive, pour lesquels seuls certains cas particuliers étaient connus (voir [GS11]). La classification formelle est obtenue par une étude minutieuse de la combinatoire provenant de l'équation de conjugaison, tandis que la convergence de la conjugaison formelle est obtenue par une utilisation assez technique des séries majorantes.

Variétés de Hopf tropicales - [RS17]

Dans un travail en collaboration avec Kristin Shaw [RS17], nous introduisons un analogue tropical des variétés de Hopf, qui sont le cas le plus simple de variétés non-Kähleriennes, et peuvent être considérées comme des variétés de Kato associées à un donnée de Kato (id, σ). L'intérêt pour ces variétés réside dans une investigation sur ce que signifie être non-Kählérien dans un cadre tropical et non-archimédien. Dans cette optique, nous calculons plusieurs invariants, tels que le groupe de Picard tropical et les groupes d'homologie (p, q), dont les objets correspondants dans le cadre complexe permettent de détecter les variétés non-Kählériennes. De plus, nous relions notre construction tropicale avec des germes contractants tropicaux, et avec des familles analytiques de germes contractants (sur les nombres complexes), via l'analytifié de Berkovich d'une variété de Hopf (ou plutot de son revêtement universel) définie sur un corps de séries de Laurent généralisées (voir [Pay09]).

Variétés de Kato toriques - [IOPR22]

Nous étudions une classe spéciale de variétés de Kato, que nous appelons les variétés de Kato toriques. Elles apparaissent comme des compactifications de l'espace des orbites de germes contractants monomiaux (plus précisément, associés à des matrices dont les colonnes sont soit des éléments de la base canonique, soit des vecteurs positifs). Leur construction vient de la géométrie torique, et en fait leurs revêtements universels sont des ouverts de variétés algébriques toriques de type non-fini. Cela généralise les constructions précédentes d'Oda [Oda78] et de Tsuchihashi [Tsu87], et en dimension complexe 2, on retrouve les surfaces d'Inoue-Hirzebruch (éclatées). Nous étudions les propriétés topologiques et analytiques des variétés de Kato toriques et relions certains invariants à des données combinatoires naturelles provenant de la construction torique. De plus, nous produisons des familles de dégénérescences plates de n'importe quelle variété Kato torique, qui servent d'outil essentiel pour calculer leurs nombres de Hodge. Dans la dernière partie, nous étudions certains propriétés des variétés de Kato du point de vue de la géométrie hermitienne. Nous donnons un résultat de caractérisation pour l'existence de métriques localement conformément kählériennes sur n'importe quelle variété de Kato. Enfin, nous prouvons qu'aucune variété de Kato ne porte de métriques équilibrées et qu'une grande classe de variétés de Kato toriques de dimension complexe ≥ 3 ne supportent pas de métriques pluricloses.

Sur le problème de Manin-Mumford dynamique pour les endomorphismes du plan - [DFR23]

Soit X une variété projective lisse sur $\mathbb{C}$, et soit $f\colon X \circlearrowleft$ un endomorphisme de celle-ci. Supposons que f soit *polarisé*, ce qui signifie qu'il existe un fibré en droites ample L sur X tel que $f^*L \simeq L^{\otimes d}$ pour un certain $d \geq 2$. L'exemple clé est celui des endomorphismes polynomiaux sur $\mathbb{P}^k_{\mathbb{C}}$ de degré d.

L'ensemble $\mathrm{Preper}(f)$ des points prépériodiques d'un endomorphisme polarisé $f\colon X \circlearrowleft$ est Zariski-dense dans X (voir, par exemple, [Fak03, Théorème 5.1]). En particulier, si $Y \subseteq X$ est une sous-variété prépériodique, alors $\mathrm{Preper}(f|_Y)$ est Zariski-dense dans Y.

Le problème dynamique de Manin-Mumford (DMM) étudie l'implication inverse :

> Sous quelles conditions sur le système dynamique polarisé (X, f), et sur la sous-variété $Y \subseteq X$, avons-nous que si Y contient un sous-ensemble Zariski-dense de points prépériodiques, alors Y est nécessairement prépériodique ?

Ce problème, initialement proposé par Zhang, s'inspire de la conjecture de Manin-Mumford sur les points de torsion des variétés abéliennes, résolue par Raynaud [Ray83a, Ray83b]: toute sous-variété d'une variété abélienne qui possède un sous-ensemble Zariski-dense de points de torsion est la translation de torsion d'une sous-variété abélienne. Comme l'a observé Northcott, cette assertion peut être traduite comme la prépériodicité des sous-variétés avec un sous-ensemble Zariski-dense de points prépériodiques par l'action de $f(x) = dx$, où d est un entier quelconque ≥ 2.

Depuis la conjecture initiale [Zha95, Zha06], plusieurs avancées et mises à jour de la conjecture ont été proposées, voir [GTZ11, GT21].

Dans [DFR23], nous nous intéressons au problème de Manin-Mumford dynamique pour les endomorphismes réguliers de $\mathbb{P}^2_{\mathbb{C}}$ provenant des endomorphismes polynomiaux de $\mathbb{C}^2$. Concrètement, $f\colon \mathbb{P}^2_{\mathbb{C}} \circlearrowleft$ est l'homogénéisation d'une application $f(x,y) = \big(P(x,y), Q(x,y)\big)$, où P, Q sont des polynômes de degré $d \geq 2$ tels que leurs parties homogènes P_d et Q_d de degré d n'ont pas de facteurs communs. Sous cette hypothèse, f induit une application rationnelle $f_\infty = [P_d : Q_d]$ de degré d sur la droite à l'infini L_∞. Dans ce cadre, le problème de DMM consiste à montrer qu'une courbe C ayant une infinité de points prépériodiques est elle-même prépériodique.

Malgré l'importance de ce problème, très peu de cas sont connus dans ce cadre : nous mentionnons par exemple le travail [GNY18], qui traite le cas des applications produit de la forme $f(x,y) = \big(P(x), Q(y)\big)$. Dans leur preuve, la structure produit de ces applications est une propriété fondamentale, car elle permet de réduire l'étude à des phénomènes dynamiques 1-dimensionnels.

Notre résultat récent concerne des situations dynamiques vraiment 2-dimensionnels : nous résolvons le problème de DMM pour les endomorphismes polynomiaux ci-dessus pour lesquels l'application rationnelle f_∞ n'a pas de points périodiques superattractifs.

Bien que l'énoncé soit purement algébrique, les techniques que nous utilisons comprennent des méthodes analytiques, en travaillant (selon le contexte, i.e., le multiplicateur de f_∞ sur le point fixe sous exploration) à la fois sur le corps archimédien complexe $\mathbb{C}$, ou bien sur les corps p-adiques non archimédiens $\mathbb{C}_p$; ainsi que des idées arithmétiques, liées au concept classique de hauteurs, où nous supposons d'abord que f est défini sur un corps de nombres.

Notations

1.1.1

$\mathrm{Sing}(X)$	singularities of a complex manifold X.
$\mathcal{O}_{X,x_0}$	ring of regular functions of the singularity (X, x_0).
$\eta_\pi^{\pi'}$	the natural birational map $\eta_\pi^{\pi'} = \pi^{-1} \circ \pi' \colon X_{\pi'} \dashrightarrow X_\pi$.
$\mathcal{E}(\pi)_{\mathbb{K}}$	the $\mathbb{K}$-vector space of exceptional $\mathbb{K}$-divisors of π ($\mathbb{K} = \mathbb{R}$ omitted).
Γ_π	the dual graph of π.
Γ_π^*	the vertices of the dual graph of π.
$(R_X, \mathfrak{m}_X)$	the local ring obtained as the formal completion of $\mathcal{O}_{X,x_0}$.

1.1.2

$\hat{\mathcal{V}}_X$	the space of centered semivaluations (X omitted when $X = \mathbb{C}^2$).
triv_{x_0}	the trivial valuation at x_0.
$\hat{\mathcal{V}}_X^*$	the space of finte semivaluations (X omitted when $X = \mathbb{C}^2$).

1.1.3

ord_E	order of vanishing along E.
div_E	a non-normalized divisorial valuation.
$\hat{\mathcal{V}}_X^{\mathrm{div}}$	set of divisorial valuations.
$\hat{\mathcal{V}}_X^{\mathrm{qm}}$	set of quasimonomial valuations.
int_C	a non-normalized curve semivaluation.

1.1.4

$\mathcal{V}_X^\bullet$	set of normalizable semivaluations.
$\mathcal{V}_X^{\mathfrak{a}}$	set of semivaluations normalized by $\nu(\mathfrak{a}) = 1$.
$\mathcal{V}_X$	set of semivaluations normalized by $\nu(\mathfrak{m}_X) = 1$.
$\nu^\bullet$	class of semivaluation in $\mathcal{V}_X^\bullet$.
$\nu^{\mathfrak{a}}$	representative of ν in $\mathcal{V}_X^{\mathfrak{a}}$.
ν_E	normalized representative of div_E in $\mathcal{V}_X$.
ν_C	normalized representative of int_C in $\mathcal{V}_X$.
$\leq$	natural partial order on $\hat{\mathcal{V}}_X$.
$\leq^{\mathfrak{a}}$	partial order induced by $\mathfrak{a}$ on $\mathcal{V}_X^{\mathfrak{a}}$ and $\mathcal{V}_X^\bullet$.
$U_{\leq}^{\mathfrak{a}}(\mu^\bullet)$	weakly closed subset of $\mathcal{V}_X^\bullet$.
$U^{\mathfrak{a}}(\mu^\bullet)$	weakly open subset of $\mathcal{V}_X^\bullet$.

1.1.5

cen_π	reduction map (center in X_π).
$U_\pi(p)$	weakly open set $\mathrm{cen}_\pi^{-1}(p)$.

1.1.6

$\check{E}$	dual element of E with respect to the basis $(E_1, \ldots, E_n)$ of $\mathcal{E}(\pi)$.

1.1.7

$Z_\pi(\mathfrak{a})$	dual divisor in $\mathcal{E}(\pi)_{\mathbb{R}}$ associated to the $\mathfrak{m}_X$-primary ideal $\mathfrak{a}$.
$b_E^{\mathfrak{a}}$	generalized multiplicity of the exceptional prime E with respect of the $\mathfrak{m}_X$-primary ideal $\mathfrak{a}$.
$Z_\pi(\nu)$	dual divisor in $\mathcal{E}(\pi)_{\mathbb{R}}$ associated to the semivaluation $\nu \in \hat{\mathcal{V}}_X^*$.

1.1.8	
b-$\mathcal{E}(X)$	Weil exceptional b-divisors of (X, x_0).
c-$\mathcal{E}(X)$	Cartier exceptional b-divisors of (X, x_0).
b-$\mathrm{Nef}(X)$	nef b-divisors of (X, x_0).
$Z(\mathfrak{a})$	b-divisors associated to the $\mathfrak{m}_X$-primary ideal $\mathfrak{a}$.
$Z(\nu)$	b-divisors associated to the finite semivaluation ν.
1.1.9	
Γ_π	realized dual graph in $\mathcal{E}(\pi)_\mathbb{R}$.
ev_π	the evaluation map $\nu \mapsto Z_\pi(\nu)$.
emb_π	the continuous embedding $\mathrm{emb}_\pi \colon \Gamma_\pi \to \mathcal{V}_X$.
$\mathcal{S}_\pi$	embedded dual graph associated to the good resolution π in $\mathcal{V}_X$.
$\varprojlim \Gamma_\pi$	universal dual graph.
ev	the natural isomorphism $\mathrm{ev}\colon \mathcal{V}_X \to \varprojlim \Gamma_\pi$.
r_π	retraction map.
$\mathcal{T}_\nu \mathcal{V}_X$	tangent space of $\mathcal{V}_X$ at $\nu \in \mathcal{V}_X$.
$\mathcal{S}_X$	skeleton of X.
emb	natural continuous bijection $\mathrm{emb}\colon \varinjlim \Gamma_\pi \to \mathcal{V}_X^{\mathrm{qm}}$.
1.2.1	
$Z \cdot W$	intersection of b-divisors.
$\langle Z, W \rangle$	opposite of intersection of b-divisors.
$\langle \nu, \mu \rangle$	intersection of semivaluations.
$\langle \nu, \mathfrak{a} \rangle$	intersection of a semivaluation with an ideal.
$\mathcal{V}_X^Z$	set of semivaluations normalized by $-Z \cdot Z(\nu) = 1$.
$\hat{\mathcal{V}}_X^{Z,\infty}$	finite semivaluations such that $-Z \cdot Z(\nu) = +\infty$.
1.2.3	
$\mathrm{Div}(\omega)$	divisor associated to a 2-form.
K_π	relative canonical divisor of X_π.
$A(E_i)$	log-discrepancy of the exceptional prime E_i.
$\delta(E_i)$	degree of E_j in $\mathit{\Gamma}_\pi$.
K_X	canonical b-divisor of (X, x_0).
$\mathbf{1}_\pi$	reduced exceptional divisor of π.
$\mathbf{1}_X$	reduced exceptional b-divisor of (X, x_0).
$K_X^{\log}$	log-canonical b-divisor of (X, x_0).
1.2.4	
$A_X(\nu)$	log-discrepancy of a semi-valuation $\nu \in \hat{\mathcal{V}}_X^*$.
$\hat{\phi}$	generalized Puiseux series.
$\hat{\beta}$	second parameter of Puiseux pair associated to a valuation.
$\hat{\phi}_j$	coefficient of generalized Puiseux series ($j \geq 1$).
$\hat{\beta}_j$	exponent of generalized Puiseux series ($j \geq 1$).
$\hat{\beta}_\infty$	sup of $\hat{\beta}_j$.
1.2.5	
$\mathrm{lct}(X)$	log-canonical threshold of (X, x_0).
1.2.7	
$\beta(\nu\vert\mu)$	relative skewness of ν over μ.
$\alpha_Z(\nu)$	skewness of ν with respect to the b-divisor Z.
$\alpha(\nu)$	skewness of ν (homogeneous quadratic).
$\hat{\mathcal{V}}_X^\alpha$	centered semivaluations of finite skewness.
1.2.8	
$\rho_X(\nu, \mu)$	angular distance on $\mathcal{V}_X^\bullet$.

1.3.1	
$\mathcal{B}_X$	set of branches on (X, x_0).
u	candidate ultrametric distance on $\mathcal{B}$ associated to multiplicity.
u_L	candidate ultrametric distance on $\mathcal{B}_X$ associated to a branch L.
1.3.2	
A_π	strict transform of A in X_π.
A_π^{exc}	exceptional transform of A in X_π.
1.3.3	
u_λ	candidate ultrametric distance on $\mathcal{V}_X$.
2.1.1	
$\mathcal{C}_f^b$	branches contracted by f.
$\mathcal{C}_f$	semivaluations contracted by f.
$f_\bullet$	action induced by f on normalized semivaluations.
$c_{\mathfrak{a}_X}^{\mathfrak{a}_Y}(f, \nu)$	attraction rate of f along ν with respect to the given normalizations.
$\mathrm{Ind}\,(f_\pi)$	indeterminacy set of f_π.
2.1.2	
$k_{D \xrightarrow{f} D'}$	ramification index along D.
$e_{D \xrightarrow{f} D'}$	tolopogical degree restricted to D.
$\mathrm{d}f_\bullet$	tangent map induced by $f_\bullet$ on $\mathcal{T}_\nu \mathcal{V}_X$.
$c_\alpha(f, \nu_C)$	attraction rate speed of f along the contracted curve valuation ν_C with respect to α.
$c_A(f, \nu_C)$	attraction rate speed of f along the contracted curve valuation ν_C with respect to A.
2.1.3	
$m(f, \nu)$	multiplicity of f along ν.
$\mathcal{D}(\pi)$	non-exceptional divisors on the model π.
b-$\mathcal{D}(X)$	Weil non-exceptional b-divisors.
c-$\mathcal{D}(X)$	Cartier non-exceptional b-divisors.
$\mathrm{pr}_{\mathcal{E}}$	projection to the exceptional part of a b-divisor.
$\hat{Z}(\nu_C)$	Cartier non-exceptional b-divisor associated to a curve semivaluation ν_C.
2.1.5	
R_f	Jacobian divisor of a map f.
2.1.6	
$\hat{e}_Z$	functional on $\hat{\mathcal{V}}_X^*$ of intersection with the b-divisor Z.
e_Z	functional on $\mathcal{V}_X$ of intersection with the b-divisor Z.
$\mathcal{S}(Z)$	finite subgraph of $\mathcal{V}_X$ outside of which e_Z is locally constant.
$\mathcal{S}_X(V)$	skeleton in $\mathcal{V}_X$ generated by V.
$r_X(V)$	natural retraction to $\mathcal{S}_X(V)$.
$\widehat{\Gamma}_\pi$	cone over Γ_π.
$\widehat{\mathrm{ev}}_\pi$	lift of the evaluation map to $\hat{\mathcal{V}}_X^*$.
$\widehat{\mathrm{emb}}_\pi$	lift of the embedding to.
$\widehat{\mathcal{S}}_\pi$	cone over $\mathcal{S}_\pi$.
$\hat{r}_\pi$	retraction to $\widehat{\mathcal{S}}_\pi$.
$\widehat{\mathcal{S}}_X(V)$	cone over $\mathcal{S}_X(V)$.
$\hat{r}_X(V)$	retraction to $\widehat{\mathcal{S}}_X(V)$.
$\mathcal{R}(\mathfrak{a})$	set of rees valuations associated to the ideal $\mathfrak{a}$.
$\mathcal{S}_f$	critical skeleton associated to f.

2.2.2

$\mathcal{V}_X^\alpha$	normalized valuations of finite skewness.

2.3.1

$\mathrm{pr}_{\mathcal{E},\pi}(Z)$	π-incarnation of $\mathrm{pr}_{\mathcal{E}}(Z)$ for $Z \in b\text{-}\mathcal{D}(X)$.

2.3.3

$\gamma(f,\nu)$	Poincaré series of the attraction rates of f along ν.
$c_\infty(f)$	first dynamical degree of f.

3.1.1

$L_\varphi^\varepsilon(X,x_0)$	link of a singularity (X,x_0).
X_φ^ε	standard open neighborhood of a singularity (X,x_0).
$\overline{X_\varphi^\varepsilon}$	standard closed neighborhood of a singularity (X,x_0).
$S_\varphi^\varepsilon(X)$	shell of a singularity (X,x_0).
$X_\varphi^{\varepsilon^-,\varepsilon^+}$	standard open neighborhood of the shell $S_\varphi^\varepsilon(X)$ with $\varepsilon^- < \varepsilon < \varepsilon^+$.

3.1.2

$S(\pi,\sigma)$	Kato surface associated to the Kato datum (π,σ).

4.1.2

$f^n(z) \xrightarrow{\star} 0$	convergence of an orbit to 0 asymptotically along $\star$.

4.1.3

$\mathcal{B}_\star(f)$	$\star$-basin of attraction of a tangent to the identity germ $f\colon (\mathbb{C}^d,0) \circlearrowleft$ at 0.

4.2.1

$\log f$	infinitesimal generator of f.
$\exp\chi$	time-1 flow of χ.

Chapter 1

Surface singularities and valuation spaces

Introduction

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This chapter is devoted to my contributions to the study of valuation spaces $\mathcal{V}_X$ associated to normal surface singularities (X, x_0). Valuations have been around since the works of Dedeking and Weber [DW82], under the name of *places*: they can be described as maps ν from the ring of (non-zero) regular functions on (X, x_0) with values on a totally ordered abelian group, satisfying some additional properties, and up to ordered bijections on the value groups.

The set $\mathcal{ZR}(X, x_0)$ of all valuations associated to a singularity (X, x_0) appears in the litterature since the works of Ostrowski [Ost35, Chapter 12 p.392], under the name of *absolute Riemann surface*. In [Zar39], Zariski classifies valuations of algebraic surfaces in order to give a proof of resolution of surface singularities based on valuation spaces and local uniformization. Later in [Zar40], the space $\mathcal{ZR}(X, x_0)$ (referred to as the *Riemann manifold* of (X, x_0)) is endowed with a natural topology, which makes it a compact (but not Hausdorff) topological space . The now called *Zariski-Riemann space* $\mathcal{ZR}(X, x_0)$ is one of the fundamental tools for Zariski's approach to resolution of singularities via local uniformization; geometrically, it can also be described as the projective limit of the total spaces X_π for any possible good resolution $\pi\colon X_\pi \to (X, x_0)$.

Valuation spaces lost their central role in algebraic geometry and singularity theory because of the celebrated Hironaka's [Hir64] resolution of singularities in characteristic zero, which makes no use of valuations. However, the lack of success to the attempts to adapt Hironaka's proof to positive characteristics generated a revival of valuation spaces. Moreover, some variants of the Zariski-Riemann space, and in particular the space of valuations $\mathcal{V}_X$ we consider in this memoir, showed several connections with non-archimedean geometry in the sense of Berkovich (see [Ber90, Jon12]) and the related connections with tropical geometry (see [Pay09]) and deformations of metrics and endomorphisms via hybrid spaces (see, e.g., [BJ17, Fav20]); we mention also the recent applications on k-stability sparked by the works of Li [Li15], and of course the applications in dynamical systems, some of which will be mentioned in Chapter 2.

The valuative space $\mathcal{V}_X$ (which technically is the set of normalized semi-valuations of rank 1 on (X, x_0)) can be described via a process of completion (in the sense of metric spaces) of the set $\mathcal{V}_X^{\text{div}}$ of *divisorial valuations*, which are defined by the order of vanishing along an exceptional prime $E \subseteq \pi^{-1}(x_0)$, where $\pi\colon X_\pi \to (X, x_0)$ is any good resolution. As a topological space, $\mathcal{V}_X$ is compact and Hausdorff, and in fact it is the largest Hausdorff quotient of $\mathcal{ZR}(X, x_0)$ when working with surfaces, by [dF19]. Allthough it does not admit a Berkovich analytic structure globally, it does locally, as showed by [Fan18]: with this additional structure $\mathcal{V}_X$ is often referred to as the *non-archimedean link* of (X, x_0), referring to the fact that it can be seen of a non-archimedean degeneration of the classical link of a singularity.

Valuations $\nu \in \mathcal{V}_X$ can be also described in terms of their associated b-divisors (in the sense of Shokurov [Sho03]), see, e.g., [BdFF12]. This allows to transport intersection theory techniques to the space of valuations. Our major contribution in [GR21] to this topic consists in a positivity property of such intersection on valuation spaces $\mathcal{V}_X$ associated to surface singularities (X, x_0), see Theorem 1.2.15. After reviewing some background in Section 1.1, we introduce b-divisors and develop the intersection theory on valuations in Section 1.2. We also introduced a distance ρ_X on $\mathcal{V}_X$, called *angular distance*, that will play a central role in the contents of Chapter 2.

The positivity property of intersection proved itself a powerful tool to study geometrical properties related to valuations. In Section 1.3, we present the results obtained in [GGPR19] on the ultrametric properties of a functional u_λ related to the study of the local intersection of branches on a surface singularity (X, x_0).

1.1 Singularities and valuation spaces

1.1.1 Surface singularities and resolution of singularities

A complex analytic space is a second countable Hausdorff topological space X that is locally shaped as the zero-locus $Z(\phi)$ of a holomophic map $\phi\colon \Omega \subseteq \mathbb{C}^n \to \mathbb{C}^s$.

Holomorphic maps $f\colon X \to Y$ between complex analytic spaces are continuous maps that are locally the restriction to $X \underset{\text{loc}}{=} Z(\phi)$ of a holomorphic map $\Omega \subseteq \mathbb{C}^n \to \mathbb{C}^m$ sending $Z(\phi)$ to $Z(\psi) \underset{\text{loc}}{=} Y$.

Complex analytic spaces together with holomorphic maps form a category, whose isomorphisms are also called holomorphic equivalences.

Definition 1.1.1. Let X be a complex analytic space. A point $x_0 \in X$ is a *regular* point of X if there exists a neighborhood U of x_0 in X, which is holomorphically equivalent to an open neighborhood of the origin in $\mathbb{C}^d$. A point of X which is not regular is called *singular*.

If all the points of X are regular, then X is a complex manifold. The subset $\mathrm{Sing}(X)$ of singular points of X is a closed complex analytic subset of X strictly included in X (see e.g. [dJP00, Section 6.3]).

We will be interested in local properties of complex analytic spaces. As such, we will be considering germs (X, x_0) at some point $x_0 \in X$ of a complex analytic space X. We call such a datum a *(germ of) singularity* (even if x_0 is actually a regular point of X). As we fix a chart centered at x_0, we may always assume that $X = Z(\phi)$ is the zero-locus of a holomorphic map $\phi\colon \mathbb{C}^n \to \mathbb{C}^s$, and $x_0 = 0$.

Unless otherwise specified, germs of singularities we consider are always assumed to be irreducible. In this case the regular part $X \setminus \mathrm{Sing}(X)$ is a complex manifold of a given dimension d, that is called the *dimension* of (X, x_0). Then a *surface singularity* is the germ of an irreducible singularity of dimension 2. We will often assume that all our spaces are also *normal*. We recall that a singularity is normal when it is irreducible and its local ring $\mathcal{O}_{X,x_0}$ is integrally closed (in its field of fractions). Equivalently, any bounded function defined on $X \setminus Z$, where $Z \subsetneq (X, x_0)$ is a proper analytic subspace, extends through the singularity. Normality implies that the singular locus has codimension at least 2, hence normal surface singularities are isolated. A complex analytic space is normal if it is normal at any of its points.

To better understand the geometry of a singularity, we are led to relate them to regular varieties, by means of either resolutions, or deformations. We will focus on the first aspect, and recall some definitions related to birational geometry.

Definition 1.1.2. Let (X, x_0) be a surface singularity. A *modification* of (X, x_0) is a proper holomorphic map $\pi\colon X_\pi \to (X, x_0)$, where X_π is a normal complex surface, and π is a biholomorphism over $X \setminus \{x_0\}$.

A modification is a *resolution* of (X, x_0) if in addition X_π is regular. Such a resolution is *good* if the exceptional locus $\pi^{-1}(x_0)$ is a divisor whose support has simple normal crossings.

By results due to Zariski [Zar39] in dimension 2, and Hironaka [Hir64] in higher dimensions, resolutions and good resolutions always exist (in characteristic zero). For surfaces there exist unique *minimal* resolutions and good resolutions (see [Lau71, Chapter 5]), where minimality is considered with respect to the partial order $\leq$ on modifications given by $\pi \leq \pi'$ if $\eta_\pi^{\pi'} := \pi^{-1} \circ \pi'\colon X_{\pi'} \to X_\pi$ is regular.

A modification $\pi\colon X_\pi \to (X, x_0)$, or its total space X_π, is sometimes called a *(birational) model* of (X, x_0). To study their properties, we are interested in the set of divisors supported

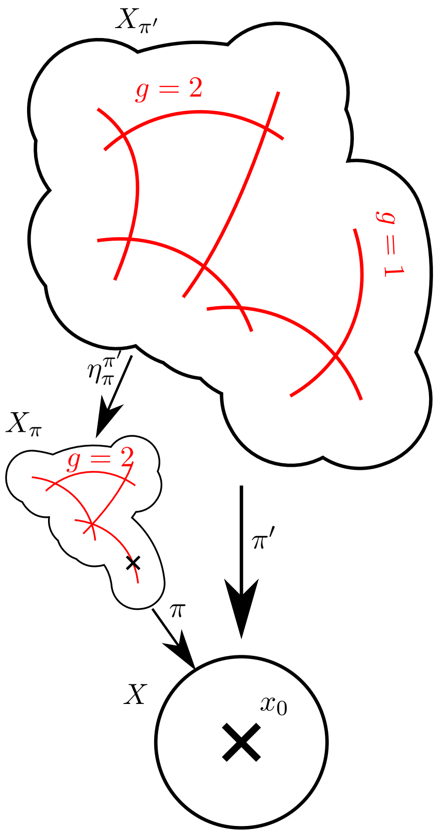

Figure 1.1: A good resolution dominating a modification of a singularity.

on the *exceptional divisor* $\pi^{-1}(x_0)$. To this end, we denote by $\mathcal{E}(\pi)_R$ the R-module of formal sums of the form

$$\sum_{E \in \Gamma_\pi^*} a_E E$$

where Γ_π^* denotes the set of *exceptional primes* of π, i.e., the irreducible components of the exceptional divisor $\pi^{-1}(x_0)$. The coefficients a_E vary in $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and we will omit the index R when $R = \mathbb{R}$.

When π is a good resolution, we may associate to π the *dual graph* of its exceptional divisor, that we denote by Γ_π. It is a (unoriented) simplicial graph, whose set of vertices is given by Γ_π^*. We then declare that two exceptional primes $E \neq F$ are connected by an edge for any point in $E \cap F$.

We denote by $(R_{X,x_0}, \mathfrak{m}_{X,x_0})$ the local ring obtained as the formal completion of $\mathcal{O}_{X,x_0}$; to simplify notations, we will often write $(R_X, \mathfrak{m}_X)$ when the point x_0 is clear from context.

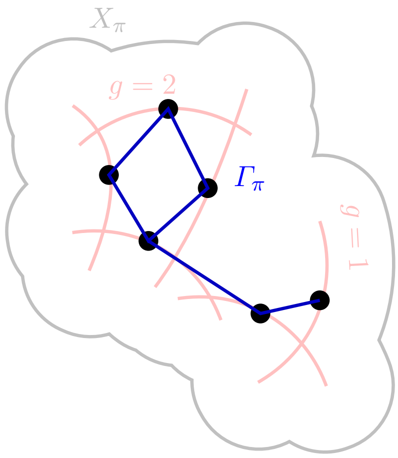

Figure 1.2: Dual graph associated to a good resolution.

Given a $\mathfrak{m}_X$-primary ideal $\mathfrak{a} \subset R_X$ (that is, a proper ideal containing some power of $\mathfrak{m}_X$), one can find good resolutions of (X, x_0) which resolve $\mathfrak{a}$ in the following sense.

Definition 1.1.3. If $\mathfrak{a} \subset R_X$ is an $\mathfrak{m}_X$-primary ideal, then a *log resolution* of $\mathfrak{a}$ is a good resolution π of (X, x_0) such that the ideal sheaf $\pi^*\mathfrak{a}$ is locally principal.

Again, log resolutions always exist: one can first perform the normalized blow-up of the ideal $\mathfrak{a}$, and then resolve the singularities one obtains in order to get a good resolution.

1.1.2 Valuation spaces of normal surface singularities

Let (X, x_0) be a normal surface singularity. In this section we will introduce and discuss the basic structure of an associated space $\mathcal{V}_X$ of (semi-)valuations. In the regular case $(X, x_0) = (\mathbb{C}^2, 0)$, the space $\mathcal{V}_{\mathbb{C}^2} = \mathcal{V}$ was introduced and described in detail by Favre-Jonsson, who called it the *valuative tree* [FJ04, FJ07, Jon12]. In the singular setting, the spaces $\mathcal{V}_X$ are analyzed in [Fav10, GR21], and they have appeared in a somewhat different vein in the works [Fan14, Thu07, dF19]. Our aim in this section is to give a fairly detailed treatment of the anatomy of $\mathcal{V}_X$, in the spirit of [Jon12, §7]. We recall that R_X denotes the completed local ring $\hat{\mathcal{O}}_{X,x_0}$, and $\mathfrak{m}_X$ is its unique maximal ideal.

Definition 1.1.4. A *rank one semivaluation* on R_X is a function $\nu\colon R_X \to \mathbb{R} \cup \{+\infty\}$ satisfying $\nu(\phi\psi) = \nu(\phi) + \nu(\psi)$ and $\nu(\phi + \psi) \geq \min\{\nu(\phi), \nu(\psi)\}$ for all $\phi, \psi \in R_X$, and $\nu(\lambda) = 0$ for any $\lambda \in \mathbb{C}^*$. Such a semivaluation is said to be *centered* if in addition $\nu(R_X) \geq 0$, and $\nu(\mathfrak{m}_X) > 0$, where for any ideal $\mathfrak{a} \subseteq R_X$, we set $\nu(\mathfrak{a}) := \min\{\nu(\phi) \mid \phi \in \mathfrak{a}\}$.
The collection of all rank one centered semivaluations on R_X will be denoted $\hat{\mathcal{V}}_X$.

Remark 1.1.5. Spaces of semivaluations appear often in non-archimedean geometry as analytifications of algebro-geometric objects defined over non-archimedean fields. Specifically, in Berkovich's formalism of non-archimedean analytic geometry (see [Ber90]), the fundamental spaces are sets of rank one semivaluations. In the case when X is a complex algebraic surface, the set $\hat{\mathcal{V}}_X$ can be interpreted within this formalism as follows: if we equip $\mathbb{C}$ with the trivial absolute value and consider the analytification X^{an} of X, then $\hat{\mathcal{V}}_X$ is the open subset of X^{an} consisting of all points whose reduction in X is the point x_0. We will not make any explicit use of this interpretation, so familiarity with Berkovich's theory is not essential in what follows.

The space $\hat{\mathcal{V}}_X$ is naturally endowed with a topology (called *weak topology*), which is the initial topology with respect to the family of evaluation maps $\mathrm{ev}_\phi : \nu \mapsto \nu(\phi)$, where ϕ varies among elements (or equivalently, ideals) of R_X.

Among centered semivaluations, a special role is played by the *trivial valuation* triv_{x_0}, which is the unique valuation satisfying $\mathrm{triv}_{x_0}(\mathfrak{m}_X) = +\infty$. Any other centered valuation $\nu \in \hat{\mathcal{V}}_X^* := \hat{\mathcal{V}}_X \setminus \{\mathrm{triv}_{x_0}\}$ is called *finite*.

1.1.3 Classification of valuations

In his foundational work on resolution of singularities in dimension two [Zar39], Zariski classified finite semivaluations $\nu \in \hat{\mathcal{V}}_X^*$ into four types according to certain associated algebraic invariants of ν (the *rational rank* and *transcendence degree*), and then characterized these types geometrically. While we will not care too much about the algebraic invariants here, the geometric characterizations will be crucial. In the terminology of Favre-Jonsson, any $\nu \in \hat{\mathcal{V}}_X^*$ is either a divisorial valuation, an irrational valuation, a curve semivaluation or an infinitely singular valuation. In the language of Berkovich spaces, they correspond to points of type II, III, I and IV respectively.

A *divisorial* valuation $\nu \in \hat{\mathcal{V}}_X^*$ is a valuation which is proportional to the order of vanishing along some exceptional prime divisor in some modification of (X, x_0). Explicitly, ν is divisorial if there exists some modification $\pi \colon X_\pi \to (X, x_0)$, some exceptional prime divisor E of X_π, and some constant $\lambda > 0$ such that $\nu(\phi) = \lambda \mathrm{ord}_E(\phi \circ \pi)$ for all $\phi \in R_X$. In this case we say that ν is *realized by* π. We denote by div_E the divisorial valuation obtained as above with $\lambda = 1$. If π' is a modification dominating π, and if F is the strict transform of E in $X_{\pi'}$, then $\mathrm{div}_E(\phi) = \mathrm{ord}_E(\phi \circ \pi) = \mathrm{ord}_F(\phi \circ \pi') = \mathrm{div}_F(\phi)$. Hence the divisorial valuation does not depend on the model π where the exceptional prime appears; in particular we may as well assume that π is a good resolution of (X, x_0). The set of divisorial valuations will be denoted as $\hat{\mathcal{V}}_X^{\mathrm{div}}$.

Divisorial valuations belong to a wider class of valuations, called *quasimonomial*. Let π be any good resolution of (X, x_0), and let E, F be two exceptional primes of π intersecting at a point $p \in X_\pi$. There then exist local holomorphic coordinates (z, w) of X_π around p for which E and F have defining equations $\{z = 0\}$ and $\{w = 0\}$ respectively. For any real $r, s \geq 0$ not both zero, we consider the *monomial* valuation at p with *weights* (r, s) with respect to the coordinates (z, w), defined on $\hat{\mathcal{O}}_{X_\pi, p}$ as

$$\mu_{r,s}\Big(\sum_{i,j} a_{ij} z^i w^j\Big) := \min\{ir + js : a_{ij} \neq 0\}.$$

We obtain a valuation $\nu_{r,s} \in \hat{\mathcal{V}}_X^*$ as the pushforward of $\mu_{r,s}$ via π: $\nu_{r,s}(\phi) := \pi_*\mu_{r,s}(\phi) = \mu_{r,s}(\phi \circ \pi)$. When $(r,s) = (1,0)$, we recover $\nu_{1,0} = \mathrm{div}_E$, and similarly $\nu_{0,1} = \mathrm{div}_F$. More generally, whenever $\frac{s}{r} \in \mathbb{Q}$, we obtained a divisorial valuation (associated to an exceptional prime obtained after a toric blow-up above p). When $\frac{s}{r} \in \mathbb{R}_{>0} \setminus \mathbb{Q}$, we say that $\nu_{r,s}$ is *irrational*.

Quasimonomial valuations (i.e., divisorial and irrational valuations together) are also called *Abhyankar* valuations, referring to their properties in terms of their algebraic invariants (the sum of the rational rank and transcendence degree are maximal for these valuations). We denote by $\hat{\mathcal{V}}_X^{\mathrm{qm}}$ their set.

A *curve semivaluation* is a semivaluation associated to an irreducible formal curve germ (C, x_0) in X passing through x_0. Any such formal curve germ can be uniformized: there exists a formal parameterization $h\colon (\mathbb{C}, 0) \to (C, x_0)$. Explicitly, a curve semivaluation associated to C is any semivaluation of the form $\nu(\phi) = \lambda \mathrm{ord}_0(\phi \circ h)$ for $\lambda > 0$. The case $\lambda = 1$ will be denoted by int_C. Observe that curve semivaluations are *not* valuations, since their kernel $\nu^{-1}(+\infty)$ is the prime ideal $\mathfrak{p}$ of R_X corresponding to the curve germ (C, x_0) in (X, x_0).

Valuations that do not fall under one of the previous three classes are called *infinitely singular*. They are characterized either in terms of algebraic invariants, or in terms of their value group (not being finitely generated over $\mathbb{Z}$). They can be though as valuations associated to "curves of infinite multiplicity", interpretation that inspired their name.

For more details on the classification of valuations in dimension 2, we refer to [Zar39, FJ04].

1.1.4 Normalizations

As one can notice from the classification above, the space $\hat{\mathcal{V}}_X$ is naturally a cone over triv_{x_0}, in the sense that for any $\lambda \in \mathbb{R}_{>0}$ and any finite semivaluation $\nu \in \hat{\mathcal{V}}_X^*$, we have a ray of (pairwise distinct) semivaluations $\lambda\nu$, so that $\lambda\nu \to \mathrm{triv}_{x_0}$ whenever $\lambda \to +\infty$.

It is natural to consider semivaluations up to multiples: we denote by $\mathcal{V}_X^\bullet$ the quotient of $\hat{\mathcal{V}}_X^*$ by the equivalence relation $\nu \sim \lambda\nu$ for any $\lambda \in \mathbb{R}_{>0}$. We call any equivalence class a *normalizable semivaluation*. This name hints to the fact that it is often useful to fix a representant of a given class, i.e., consider sections of the natural projection $\mathrm{pr}\colon \hat{\mathcal{V}}_X^* \to \mathcal{V}_X^\bullet$, called *normalizations*.

Definition 1.1.6. Let $\mathfrak{a}$ be an ideal in R_X. The set of *normalized valuations* with respect to $\mathfrak{a}$ is

$$\mathcal{V}_X^{\mathfrak{a}} = \{\nu \in \hat{\mathcal{V}}_X^* \ : \ \nu(\mathfrak{a}) = 1\}.$$

Given a valuation $\nu \in \hat{\mathcal{V}}_X^*$, we denote by $\nu^\bullet$ its equivalence class in $\mathcal{V}_X^\bullet$, and by $\nu^{\mathfrak{a}}$ the unique valuation (when it exists) in $\mathcal{V}_X^{\mathfrak{a}}$ representing $\nu^\bullet$.

The most common choices are:

- The normalization with respect to the maximal ideal $\mathfrak{m}_X$. In this case, we will simply write $\mathcal{V}_X := \mathcal{V}_X^{\mathfrak{m}_X}$. We also set $\nu_E = \mathrm{div}_E^{\mathfrak{m}_X}$ and $\nu_C = \mathrm{int}_C^{\mathfrak{m}_X}$.

- The normalization with respect to an irreducible element $\phi \in R_X$, which corresponds to taking $\mathfrak{a} = \phi R_X$. In this case, we write $\mathcal{V}_X^\phi := \mathcal{V}_X^{\phi R_X}$.

Notice that in the latter case, not all normalizable valuations $\nu^\bullet \in \mathcal{V}_X^\bullet$ admit a representant in $\mathcal{V}_X^\phi$: we miss exactly the class of the curve semivaluation int_C with $C = \{\phi = 0\}$. In general, any normalizable valuation is represented in $\mathcal{V}_X^\mathfrak{a}$ if and only if the ideal $\mathfrak{a}$ is $\mathfrak{m}_X$-primary.

The weak topology on $\hat{\mathcal{V}}_X$ induces a subspace topology on $\mathcal{V}_X^\mathfrak{a}$ for any ideal $\mathfrak{a}$, and a quotient topology on $\mathcal{V}_X^\bullet$. When $\mathfrak{a}$ is $\mathfrak{m}_X$-primary, the spaces $\mathcal{V}_X$ and $\mathcal{V}_X^\mathfrak{a}$ are homeomorphic, via the the restriction of the natural projection $\mathrm{pr}\colon \hat{\mathcal{V}}_X^* \to \mathcal{V}_X^\bullet$ to $\mathcal{V}_X^\mathfrak{a}$. Allthought this topology is not metrizable, and in fact not separable and not first countable, in some sense it is determined by the behavior of sequences: $\hat{\mathcal{V}}_X$ is Frechet-Uryson (topological closure is detected by sequences), and sequentially compact (the two conditions together are referred as *angelic*, see [Poi13]). It is also compact Hausdorff, path connected, and locally contractible.

Related to the weak topology is a natural partial order $\leq$ on $\hat{\mathcal{V}}_X$, defined by $\nu \leq \mu$ if and only if $\nu(\phi) \leq \mu(\phi)$ for any $\phi \in R_X$. This induces a partial order $\leq^\mathfrak{a}$ on $\mathcal{V}_X^\mathfrak{a}$, which in turns induces a partial order $\leq^\mathfrak{a}$ on $\mathcal{V}_X^\bullet$, by setting $\nu^\bullet \leq^\mathfrak{a} \mu^\bullet$ if and only if $\mu(\mathfrak{a})\nu(\phi) \leq \nu(\mathfrak{a})\mu(\phi)$ for any ν, μ representing $\nu^\bullet$ and $\mu^\bullet$ respectively. Notice that the partial order $\leq^\mathfrak{a}$ defined on $\mathcal{V}_X^\bullet$ depends on $\mathfrak{a}$. Given a $\mathfrak{m}_X$-primary ideal $\mathfrak{a}$, and $\mu^\bullet \in \mathcal{V}_X^\bullet$, the set

$$U_\geq^\mathfrak{a}(\mu^\bullet) := \{\nu^\bullet \in \mathcal{V}_X^\bullet \mid \nu^\bullet \geq^\mathfrak{a} \mu^\bullet\}$$

is weakly closed, since we can write it as

$$U_\geq^\mathfrak{a}(\mu^\bullet) = \bigcap_\psi \Big\{\nu \in \hat{\mathcal{V}}_X^* \mid \mu(\mathfrak{a}) \operatorname{ev}_\psi(\nu) \geq \mu(\psi) \operatorname{ev}_\mathfrak{a}(\nu)\Big\},$$

where μ is any representant of $\mu^\bullet$, and ψ varies among all irreducible elements of R_X. It is a (non-trivial) consequence of the fact that (X, x_0) has dimension 2, that

$$U^\mathfrak{a}(\mu^\bullet) := \{\nu^\bullet \in \mathcal{V}_X^\bullet \mid \nu^\bullet >^\mathfrak{a} \mu^\bullet\} = U_\geq^\mathfrak{a}(\mu^\bullet) \setminus \{\mu^\bullet\}$$

is weakly-open.

These opens generate the weak topology (as long as we let $\mathfrak{a}$ vary).

1.1.5 Center of a valuation and reduction map

Let $\nu \in \hat{\mathcal{V}}_X^*$, and let $\pi\colon X_\pi \to (X, x_0)$ be a modification of (X, x_0). By the valuative criterion of properness, ν has a unique *center* in X_π, that is, a unique scheme-theoretic point $\xi \in X_\pi$ with the property that ν takes nonnegative values on the local ring $\mathcal{O}_{X_\pi,\xi}$ and strictly positive values exactly on its maximal ideal $\mathfrak{m}_\xi$. We will write this $\xi = \mathrm{cen}_\pi(\nu)$. Note, the map $\mathrm{cen}_\pi\colon \hat{\mathcal{V}}_X^* \to X_\pi$ is typically called the *reduction map* in the context of non-archimedean analytic geometry; it is anti-continuous in the sense that the inverse image of a Zariski closed set is weakly open.

Since $\mathrm{cen}_\pi(\lambda\nu) = \mathrm{cen}_\pi(\nu)$ for any $\lambda \in \mathbb{R}_{>0}$, the reduction map is well defined on $\mathcal{V}_X^\bullet$. The weakly open sets $U_\pi(p) = \mathrm{cen}_\pi^{-1}(p)$ (seen as subsets of either $\hat{\mathcal{V}}_X^*$, $\mathcal{V}_X^\bullet$ or $\mathcal{V}_X^\mathfrak{a}$), where p is a closed point in $\pi^{-1}(x_0)$, play an important role for the study of lifts of holomorphic maps between normal surface singularities (see Proposition 2.1.3). These open sets generate the weak topology (as long as π varies among all modifications, and not only resolutions).

If π' is a modification dominating π, then one easily sees that $\mathrm{cen}_\pi = \eta_\pi^{\pi'} \circ \mathrm{cen}_{\pi'}$. In particular the center $\mathrm{cen}_\pi(\nu)$ of a centered valuation $\nu \in \hat{\mathcal{V}}_X$ is always an infinitely near (scheme theoretic) point $\xi \in \pi^{-1}(x_0)$.

Centers allow to transport valuations from one model to the other. Suppose that $\nu \in \hat{\mathcal{V}}_X^*$ has a closed point $p = \mathrm{cen}_\pi(\nu)$ as center in some model $\pi\colon X_\pi \to (X, x_0)$. Then we can write $\nu = \pi_*\mu = \mu(\,\cdot \circ \pi)$, where $\mu \in \hat{\mathcal{V}}_{X_\pi}^*$ is a finite valuation at (X_π, p).

The situation at non-closed points is simpler (at least for surfaces), since in that case $\xi = \mathrm{cen}_\pi(\nu) = E \in \Gamma_\pi^*$ is an exceptional prime, and ν is a multiple of the divisorial valuation div_E.

1.1.6 The intersection theory of good resolutions

Let $\pi\colon X_\pi \to (X, x_0)$ be a good resolution. We now turn our attention to the intersection theory of X_π.

In particular, we want to study (relative) positivity properties for exceptional divisors, namely the notions of relative ampleness and nefness (see e.g. [Laz04, §1.7]). The relative ampleness of a divisor $D \in \mathcal{E}(\pi)$ is defined in terms of projective embeddings, but can be characterized numerically by having $D \cdot E > 0$ for each of the exceptional primes E of π, a characterization which extends to exceptional $\mathbb{R}$-divisors [Fel08, Theorem B]. Similarly, an $\mathbb{R}$-divisor $D \in \mathcal{E}(\pi)_\mathbb{R}$ is *relatively nef* if $D \cdot E \geq 0$ for each $E \in \Gamma_\pi^*$.

The intersection product on X_π naturally extends to $\mathbb{R}$-divisors, and induces a symmetric bilinear form on $\mathcal{E}(\pi)_\mathbb{R}$, that we call *intersection form*. The key property of the intersection form is that it is negative definite (see [Zar39, Gra62]), which allows to describe precisely the cone of relatively nef divisors. In fact, the intersection form induces a canonical isomorphism between $\mathcal{E}(\pi)_\mathbb{R}$ and its dual $\mathcal{E}(\pi)_\mathbb{R}^\vee$. Let $\mathcal{B} = (E_1, \ldots, E_n)$ be the exceptional primes of π, that form a $\mathbb{Z}$-basis of $\mathcal{E}(\pi)$. By construction, the matrix M representing the intersection form has negative entries on the diagonal, and non-negative entries outside. We will denote by $\mathcal{B}^\vee = (\check{E}_1, \ldots, \check{E}_n) \in \mathcal{E}(\pi)_\mathbb{Q}$ the dual basis to $(E_1, \ldots, E_n)$. Concretely, $\check{E}_i$ is the unique $\mathbb{Q}$-divisor with the property that $\check{E}_i \cdot E_j = \delta_{ij}$ for each j. In particular, the divisors $\check{E}_i$ are *relatively nef*.

An easy consequence of the projection formula tells us that if π' is a good resolution dominating π and if E' is the strict transform in $X_{\pi'}$ of an exceptional prime E of π, then $\eta_\pi^{\pi'*}\check{E} = \check{E}'$.

By direct computation, one can easily check that the coordinates of $\check{E}_i$ with respect to $\mathcal{B}$ are exactly the entries on the i-th row of M^{-1}. As a consequence, the intersection form is represented by M^{-1} in the basis $\mathcal{B}^\vee$ (considered up to the canonical identification described above). In other terms,

$$\check{E}_i = \sum_j (\check{E}_i \cdot \check{E}_j) E_j.$$

We state here an easy but important linear algebra lemma, that will allow to draw further consequences on the positivity of $\check{E}_i$.

Lemma 1.1.7. *Let V be a finite dimensional vector space endowed with a negative definite symmetric bilinear form (that we denote as the intersection product). Suppose that $E_1, \dots, E_n$ is a basis of V with the property that $(E_i \cdot E_j) \geq 0$ for each $i \neq j$. Let $\check{E}_1, \dots, \check{E}_n \in V$ be the corresponding dual basis with respect to the bilinear form. Then the following statements hold.*

- *For all i and j, one has $(\check{E}_i \cdot \check{E}_j) \leq 0$.*
- *Let Γ be the graph with vertices $E_1, \dots, E_n$ and an edge connecting two vertices E_i and E_j if and only if $(E_i \cdot E_j) > 0$. Then $(\check{E}_i \cdot \check{E}_j) < 0$ if and only if the vertices E_i and E_j belong to the same connected component in Γ.*

The proof is a consequence of the Gram-Schmidt orthogonalization algorithm applied to the opposite of the bilinear form (which is an inner product), see [GR21, Lemma 1.8].

Since the dual graph of a good resolution of a normal surface singularity is always connected, we deduce that the dual divisors $\check{E}_i$ are strictly anti-effective.

Evidently, an exceptional $\mathbb{R}$-divisor D is relatively nef (resp. relatively ample) if and only if $D = \sum_i \lambda_i \check{E}_i$ with $\lambda_i \geq 0$ (resp. > 0). Therefore $\mathrm{Nef}(\pi)$ is a strict convex polyhedral cone, generated by $\check{E}_i$, and $\mathrm{Amp}(\pi)$ is its (nonempty) interior (and conversely, $\mathrm{Nef}(\pi)$ is the closure $\overline{\mathrm{Amp}(\pi)}$). In particular, one can easily construct bases of $\mathcal{E}(\pi)_\mathbb{R}$ consisting of integral relatively ample divisors, which after scaling can even be taken to be relatively very ample. As an immediate consequence, one also obtains the following.

Corollary 1.1.8. *If D_1 and D_2 are nonzero relatively nef divisors, then $D_1 \cdot D_2 < 0$. In particular, nonzero relatively nef divisors are strictly anti-effective.*

Without further mention, with nef and ample we will always mean the relative versions with respect to the given good resolution π.

1.1.7 Ideals, valuations and divisors

In order to exploit local intersection theory, we associate to any $\mathfrak{m}_X$-primary ideal and any valuation a divisor, encoding their numerical properties.

We start with a $\mathfrak{m}_X$-primary ideal $\mathfrak{a}$, and take any log resolution $\pi \colon X_\pi \to (X, x_0)$. This means that the sheaf of ideals $\pi^*\mathfrak{a}$ is locally principal on X_π:

$$\pi^*\mathfrak{a} = \mathcal{O}_{X_\pi}(Z_\pi(\mathfrak{a}))$$

for a suitable divisor $Z_\pi(\mathfrak{a})$, of the form

$$Z_\pi(\mathfrak{a}) = -\sum_{E \in \Gamma_\pi^*} b_E^{\mathfrak{a}} E, \tag{1.1}$$

with $b_E^{\mathfrak{a}} = \mathrm{ord}_E(\pi^*\mathfrak{a}) \in \mathbb{N}^*$. Notice that if $\pi' \colon X_{\pi'} \to (X, x_0)$ is another log resolution of $\mathfrak{a}$ dominating π, then $(\eta_\pi^{\pi'})^* Z_\pi(\mathfrak{a}) = Z_{\pi'}(\mathfrak{a})$.

The divisor $Z_\pi(\mathfrak{a})$ is clearly anti-effective, and in fact it is relatively nef, a direct consequence of the definition of a log-resolution.

Remark 1.1.9. Notice that if π is a good resolution of (X, x_0) and $Z \in \mathcal{E}(\pi)$ is a divisor for which the line bundle $\mathcal{O}_{X_\pi}(Z)$ is relatively base point free, then π will be a log resolution of the ideal $\mathfrak{a} := \pi_* \mathcal{O}_{X_\pi}(Z)$, and $Z = Z_\pi(\mathfrak{a})$. Since any relatively very ample divisor Z is so that $\mathcal{O}_{X_\pi}(Z)$ is relatively base point free, we deduce that every relatively very ample divisor $Z \in \mathcal{E}(\pi)$ is of the form $Z = Z_\pi(\mathfrak{a})$ for some $\mathfrak{m}_X$-primary ideal $\mathfrak{a}$.

Since it is always possible to find a basis of $\mathcal{E}(\pi)_\mathbb{R}$ consisting of relatively very ample divisors, we deduce that we can always find $\mathfrak{m}_X$-primary ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ for which π is a log resolution and for which the divisors $Z_\pi(\mathfrak{a}_i)$ form a basis of $\mathcal{E}(\pi)_\mathbb{R}$.

Let now $\nu \in \hat{\mathcal{V}}_X^*$ be a valuation, and let $\pi\colon X_\pi \to (X, x_0)$ be a good resolution of (X, x_0). If $\xi := \mathrm{cen}_\pi(\nu) = E \in \Gamma_\pi^*$, then ν is the divisorial valuation $\lambda \mathrm{div}_E$ for some $\lambda > 0$, and we can evaluate ν on $\mathcal{E}(\pi)$ by setting $\nu(D) = \lambda a_E$, where $D = \sum_{F \in \Gamma_\pi^*} a_F F$. In the general case, we can still evaluate ν on $\mathcal{E}(\pi)$, by setting $\nu(D) := \mu(\psi)$, where $\nu = \pi_* \mu$ with $\mu \in \hat{\mathcal{V}}_{X_\pi, \xi}^*$ and $\psi \in \mathcal{O}_{X_\pi, \xi}$ is a defining equation of $D \in \mathcal{E}(\pi)$ around ξ.

The evaluation $\nu\colon \mathcal{E}(\pi) \to \mathbb{R}$ is $\mathbb{Z}$-linear, and hence extends linearly to an evaluation map $\nu\colon \mathcal{E}(\pi)_\mathbb{R} \to \mathbb{R}$ on $\mathbb{R}$-divisors. Since the intersection product on $\mathcal{E}(\pi)_\mathbb{R}$ is non-degenerate (it is negative-definite), we have a natural identification of $\big(\mathcal{E}(\pi)_\mathbb{R}\big)^\vee$ with $\mathcal{E}(\pi)_\mathbb{R}$, and we conclude the following.

Proposition 1.1.10. *For each $\nu \in \hat{\mathcal{V}}_X^*$ and each good resolution $\pi\colon X_\pi \to (X, x_0)$, there is a unique $\mathbb{R}$-divisor $Z_\pi(\nu) \in \mathcal{E}(\pi)_\mathbb{R}$ such that $\nu(D) = Z_\pi(\nu) \cdot D$ for each $D \in \mathcal{E}(\pi)_\mathbb{R}$.*

When $\nu = \mathrm{div}_E$ for some $E \in \Gamma_\pi^*$, then $Z_\pi(\mathrm{div}_E)$ is simply the element $\check{E}$ of the dual basis $(\check{E})_{E \in \Gamma_\pi^*}$ of the basis Γ_π^* of $\mathcal{E}(\pi)$.

More generally, if the center $\xi = \mathrm{cen}_\pi(\nu)$ lies within exactly one exceptional prime E of π, then $Z_\pi(\nu) = \lambda \check{E}$ for some $\lambda > 0$. Otherwise, ξ must be the intersection point of two exceptional primes, say E and F, in which case $Z_\pi(\nu) = r\check{E} + s\check{F}$ for some $r, s > 0$. Either way, we see that $Z_\pi(\nu)$ is relatively nef (hence anti-effective by Corollary 1.1.8).

If $\pi'\colon X_{\pi'} \to (X, x_0)$ is another good resolution dominating π, then we have again $(\eta_\pi^{\pi'})^* Z_\pi(\nu_E) = Z_{\pi'}(\nu_E)$. This property does not hold for valuations whose center $\xi = \mathrm{cen}_\pi(\nu)$ is a closed point, but we still have $(\eta_\pi^{\pi'})_* Z_{\pi'}(\nu_E) = Z_\pi(\nu_E)$.

Notice that the collection of all $Z_\pi(\nu)$ where π varies among good resolutions of (X, x_0) contains all the informations about the given valuation ν. In fact, one can evaluate semi-valuations on ideals using intersections in the following way.

Proposition 1.1.11. *Let $\nu \in \hat{\mathcal{V}}_X^*$ and let $\mathfrak{a} \subset R_X$ be an $\mathfrak{m}_X$-primary ideal. Let $\pi\colon X_\pi \to (X, x_0)$ be a log resolution of $\mathfrak{a}$. Then $\nu(\mathfrak{a}) = -Z_\pi(\nu) \cdot Z_\pi(\mathfrak{a})$.*

Proof. Let ξ be the center of ν in X_π, so that $\nu = \pi_* \mu$ with $\mu \in \hat{\mathcal{V}}_{X_\pi, \xi}^*$. By construction, the value of ν on $\mathfrak{a} \subset R_X$ agrees with the value of μ on the ideal $(\pi^* \mathfrak{a})_\xi \subset \mathcal{O}_{X_\pi, \xi}$. Because π is a log resolution of $\mathfrak{a}$, the ideal sheaf $\pi^* \mathfrak{a}$ is the ideal sheaf of the divisor $-Z_\pi(\mathfrak{a})$, and thus in particular $\mu((\pi^* \mathfrak{a})_\xi) = \nu(-Z_\pi(\mathfrak{a})) = -Z_\pi(\nu) \cdot Z_\pi(\mathfrak{a})$, completing the proof. $\square$

1.1.8 B-divisors and dual graphs

In the previous section, we attached to both $\mathfrak{m}_X$-primary ideals $\mathfrak{a}$ and finite valuations ν a family of $(Z_\pi)_\pi$ of $\mathbb{R}$-divisors, where $Z_\pi \in \mathcal{E}(\pi)_\mathbb{R}$. These divisors satisfy a compatibility property; namely, $(\eta_\pi^{\pi'})_* Z_{\pi'} = Z_\pi$ for any π' dominating π. The models π vary among "suitable" good resolutions (log resolutions of $\mathfrak{a}$ in the ideal case, plain good resolutions in the valuation case), but one can extend the definition of Z_π for any modifications π by setting $Z_\pi := (\eta_\pi^{\pi'})_* Z_{\pi'}$ for any π' suitable good resolution dominating π (they always exist). The compatibility condition ensures that the definition is well posed. The collection of all these divisors $Z = (Z_\pi)_\pi$ is a *b-divisor* in the sense of Shokurov [Sho03] (see also [BFJ08a, Fav10]).

Definition 1.1.12. Let (X, x_0) be a normal surface singularity. A *(Weil) (exceptional) b-divisor* Z on (X, x_0) is a collection $Z = (Z_\pi)_\pi$ of Weil $\mathbb{R}$-divisors $Z_\pi \in \mathcal{E}(\pi)_\mathbb{R}$ for any modification $\pi \colon X_\pi \to (X, x_0)$, satisfying the relation $(\eta_\pi^{\pi'})_* Z_{\pi'} = Z_\pi$ for all modifications π, π' with π' dominating π. The divisor Z_π is called the *incarnation* of Z in the model π.

If moreover there exists a modification π so that $Z_{\pi'} = (\eta_\pi^{\pi'})^* Z_\pi$ for all modifications π' dominating π, we say that Z is a *Cartier* b-divisor, *determined* by the model π.

We denote by b-$\mathcal{E}(X)$ the set of Weil exceptional b-divisors, and by c-$\mathcal{E}(X)$ its subset of *Cartier* b-divisors. Finally, we say that a b-divisor $Z = (Z_\pi)_\pi$ is *nef* if Z_π is nef for all modifications π, and denote by b-$\mathrm{Nef}(X)$ the set of nef b-divisors.

In other terms, the space b-$\mathcal{E}(X)$ of b-divisors is the projective limit of the $\mathbb{R}$-vector spaces $\mathcal{E}(\pi)_\mathbb{R}$, with respect to the inverse system given by all modifications, ordered by dominance. In particular, the space of b-divisors inherits a structure of $\mathbb{R}$-vector space, as well as a topology, when we endow $\mathcal{E}(\pi)_\mathbb{R} \cong \mathbb{R}^{\#\Gamma_\pi^*}$ with its euclidean topology. Being the directed poset of good resolutions not countable (unless the base field is), the topology on b-$\mathcal{E}(X)$ is not sequential. It can be described as the topology of simple convergence: a net $(Z^\alpha)_\alpha$ converges to Z if and only if the sequence Z_π^α converges to Z_π in $\mathcal{E}(\pi)_\mathbb{R}$ for any good resolution π.

Remark 1.1.13. Often in our examples, a Weil b-divisor $Z = (Z_\pi)_\pi$ will be constructed by giving a sequence $\pi_0 < \pi_1 < \pi_2 < \ldots$ of increasing models (ordered as usual by dominance), a sequence $Z_n \in \mathcal{E}(\pi_n)_\mathbb{R}$ of divisors satisfying $(\eta_{\pi_n}^{\pi_m})_* Z_m = Z_n$ for any $m > n$, and setting $Z_\pi = (\eta_{\pi_n}^\pi)^* Z_n$, where n is the largest integer so that $\pi \geq \pi_n$, if such a n exists, and extended by projection to all models.

In this case, we will say that Z is determined by the models $(\pi_n)_n$.

In the previous section, we have seen that to any $\mathfrak{m}_X$-primary ideal $\mathfrak{a}$ we can associate an exceptional nef Cartier b-divisor $Z(\mathfrak{a}) = (Z_\pi(\mathfrak{a}))_\pi$. Similarly, to any finite valuation $\nu \in \hat{\mathcal{V}}_X^*$ we can associate an exceptional nef Weil v-divisor $Z(\nu) = (Z_\pi(\nu))_\pi$, which is Cartier if and only if ν is divisorial. In particular, if div_E is a divisorial valuation associated to an exceptional prime $E \in \Gamma_\pi^*$, then $Z(\mathrm{div}_E)$ is the Cartier b-divisor determined by $\check{E}$ in the model X_π.

1.1.9 Universal dual graph

The construction of b-divisors can be used to introduce valuation spaces in a more geometric way, via projective limits of dual graphs.

This is achieved by realizing the dual graph of a given good resolution as an embedded graph inside $\mathcal{E}(\pi)_\mathbb{R}$. To relate dual graphs to valuations via b-divisors, it is convenient to fix a normalization. We consider the one with respect to the maximal ideal $\mathfrak{m}_X$ (we could work with any $\mathfrak{m}_X$-primary ideal $\mathfrak{a}$ instead). We recall that given a divisorial valuation div_E associated to an exceptional prime $E \in \mathit{\Gamma}_\pi^*$, we denote by $b_E = \mathrm{div}_E(\mathfrak{m}_X) \geq 1$ the normalizing factor, called *generic multiplicity* of the valuation div_E.

Remark 1.1.14. For technical reasons, we will ask that any good resolution π has the additional property that any two distinct exceptional primes $E, F \in \mathit{\Gamma}_\pi^*$ intersect at at most one point. In fact, if E and F intersect at $N \geq 2$ points, it suffices to blow-up all points these points, but for one. In particular, we lose the uniqueness of the minimal good resolution, but in exchange we get that an edge between two exceptional primes is uniquely determined by the endpoints.

Let now $\pi \colon X_\pi \to (X, x_0)$ be a good resolution of (X, x_0). We recall that the dual graph $\mathit{\Gamma}_\pi$ of π has $\mathit{\Gamma}_\pi^*$ as set of vertices, and an edge between two exceptional primes $E \neq F$ if and only if they intersect.

Definition 1.1.15. The *(realized) dual graph* Γ_π of π is the subset of $\mathcal{E}(\pi)_\mathbb{R}$ consisting of the divisors $\frac{\check{E}}{b_E}$ for each of the exceptional primes E, as well as the straight line segment connecting $\frac{\check{E}}{b_E}$ and $\frac{\check{F}}{b_F}$ if E and F intersect in X_π.

The additional condition on good resolutions ensures that Γ_π is a realization of $\mathit{\Gamma}_\pi$ inside $\mathcal{E}(\pi)_\mathbb{R}$.

Proposition 1.1.16. *Let $\pi \colon X_\pi \to (X, x_0)$ be a log-resolution of $\mathfrak{m}_X$. The map $\mathrm{ev}_\pi \colon \mathcal{V}_X \to \mathcal{E}(\pi)_\mathbb{R}$ which takes $\nu \in \mathcal{V}_X$ to its associated divisor $Z_\pi(\nu)$ is continuous and has Γ_π as its image. Moreover, there is a continuous embedding $\mathrm{emb}_\pi \colon \Gamma_\pi \to \mathcal{V}_X$ such that $\mathrm{ev}_\pi \circ \mathrm{emb}_\pi$ is the identity on Γ_π.*

The notation ev_π and emb_π is chosen to agree with that of [Jon12], where ev_π is called *evaluation*.

Proof. To see that ev_π is continuous is suffices to check that $Z_\pi(\nu) \cdot Z$ varies continuously with ν for all divisors Z in a basis $Z_1, \ldots, Z_n \in \mathcal{E}(\pi)_\mathbb{R}$. By Remark 1.1.9, we may choose such a basis consisting of divisors of the form $Z_i = Z_\pi(\mathfrak{a}_i)$ for $\mathfrak{m}_X$-primary ideals $\mathfrak{a}_i \subset R_X$. Applying Proposition 1.1.11, we have $Z_\pi(\nu) \cdot Z_i = -\nu(\mathfrak{a}_i)$, which varies continuously in ν by the definition of the weak topology. Therefore ev_π is continuous.

Let $\nu \in \mathcal{V}_X$ and let $\xi = \mathrm{cen}_\pi(\nu)$. Assume first that ξ lies within a unique exceptional prime E of π. In this case we have seen $Z_\pi(\nu) = \lambda \check{E}$ for some $\lambda > 0$. However, because ν is normalized, we must have $1 = \nu(\mathfrak{m}) = -Z_\pi(\nu) \cdot Z_\pi(\mathfrak{m}) = \lambda b_E$, proving that $Z_\pi(\nu) = b_E^{-1} \check{E} \in \Gamma_\pi$. Assume next that ξ is the intersection point of two exceptional primes E and F of π. Then $Z_\pi(\nu) = r\check{E} + s\check{F}$ for some $r, s > 0$ satisfying the normalization condition $1 = \nu(\mathfrak{m}) = -Z_\pi(\nu) \cdot Z_\pi(\mathfrak{m}) = rb_E + sb_F$. This says exactly that $Z_\pi(\nu)$ lies on the straight line segment between $b_E^{-1}\check{E}$ and $b_F^{-1}\check{F}$, and thus lies within Γ_π. To complete the proof, we need only construct the embedding emb_π. This is done as follows. First, the vertex points

$b_E^{-1}\check{E}$ in Γ_π are mapped by emb_π to the normalized divisorial valuation $b_E^{-1}\mathrm{div}_E$. Then, the edge points $r\check{E}+s\check{F}$ are mapped by emb_π to the monomial valuation $\nu_{r,s}$ at the point $E\cap F$. It is trivial to check that emb_π has the desired properties. $\square$

Let now $\pi\colon X_\pi \to (X,x_0)$ be any log resolution of $\mathfrak{m}_X$, and set $\pi' = \pi\circ\eta$, where η is the blow-up of a point $p\in\pi^{-1}(x_0)$, with exceptional divisor $\eta^{-1}(p) =: G$. The fact that π is a log resolution guarantees that $\pi^*\mathfrak{m}_X$ has no base points on $\pi^{-1}(x_0)$, and

$$b_G = \sum_{E\ni p} b_E.$$

Moreover, by a direct computation, we get that

$$\check{G} = \sum_{E\ni p} \eta^*\check{E} - G. \tag{1.2}$$

As a consequence, the natural projection $\eta_*\colon \mathcal{E}(\pi')_\mathbb{R} \to \mathcal{E}(\pi)_\mathbb{R}$ induces a surjective map $\eta_*\colon \Gamma_{\pi'} \to \Gamma_\pi$.

An analogous statement holds when considering any two log-resolutions $\pi\leq\pi'$ of $\mathfrak{m}_X$, since we can decompose $\eta_\pi^{\pi'}$ as a composition of point blow-ups. We deduce that the family $(\eta_\pi^{\pi'})_{\pi\leq\pi'}$ form an inverse system, of which we can take the projective limit $\varprojlim \Gamma_\pi$, sometimes called the *universal dual graph* associated to (X,x_0). By construction $\varprojlim \Gamma_\pi$ naturally embeds in the set of (exceptional) $\mathbb{R}$-b-divisors b-$\mathcal{E}(X)$. Moreover, by the universal property of the inverse limit, the maps ev_π induce a continuous map $\mathrm{ev}\colon \mathcal{V}_X \to \varprojlim \Gamma_\pi$. The structure of $\mathcal{V}_X$ is then given by the following theorem.

Theorem 1.1.17. *The map* $\mathrm{ev}\colon \mathcal{V}_X \to \varprojlim \Gamma_\pi$ *is a homeomorphism.*

It is a matter of unravelling definitions to notice that the map ev sends $\nu\in\mathcal{V}_X$ to its associated b-divisor $Z(\nu)$.

The proof of this theorem can be found in [Jon12, Theorem 7.9] for $X=\mathbb{C}^2$, but in fact the argument works equally well in our singular setting. As a consequence of the theorem, the map $r_\pi := \mathrm{emb}_\pi\circ\mathrm{ev}_\pi$ is a retraction of $\mathcal{V}_X$ onto the *embedded dual graph* $\mathcal{S}_\pi := \mathrm{emb}_\pi(\Gamma_\pi)\subset\mathcal{V}_X$. More difficult to see is that in fact $\mathcal{V}_X$ *deformation retracts* onto $\mathcal{S}_\pi$ (see [Ber90, Thu07]) and thus in particular has the homotopy type of the finite graph $\mathcal{S}_\pi\cong\Gamma_\pi$.

Theorem 1.1.17 allows to better understand the structure of $\mathcal{V}_X$ as a $\mathbb{R}$-tree (in the smooth case), or as a graph of real trees over the simplicial graph Γ_π associated to any good resolution of (X,x_0) (see [GGPR19, Section 2.7] for further details). In particular, $\mathcal{V}_X$ is locally shaped as an $\mathbb{R}$-tree; the branching points at $\nu\in\mathcal{V}_X$, defined as the local connected components of $\mathcal{V}_X\setminus\{\nu\}$ at ν, are called the *tangent vectors* at ν, and the associated tangent space is denoted by $\mathcal{T}_\nu\mathcal{V}_X$. This space consists of one point when ν is a curve semivaluation or infinitely singular, and of two points when ν in irrational; when $\nu=\nu_E$ is a divisorial valuation, then $\mathcal{T}_{\nu_E}\mathcal{V}_X$ is in bijection with the (closed) points of E.

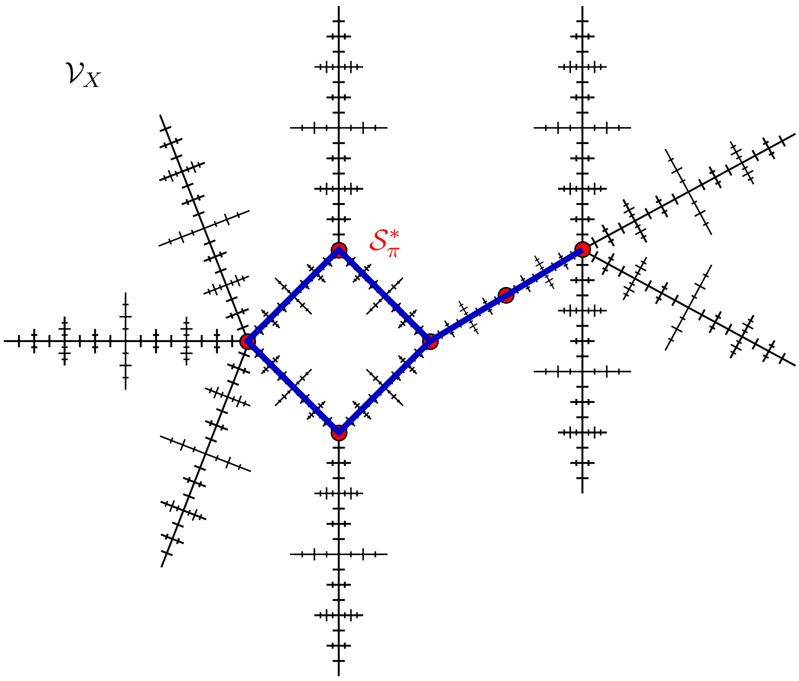

Figure 1.3: The valuation space $\mathcal{V}_X$ and the skeleta associated to the minimal resolution π of (X, x_0).

Remark 1.1.18. If π_0 is a minimal good resolution of (X, x_0) dominated by π, then, using the fact that π is obtained from π_0 by a composition of point blowups, one sees that Γ_π deformation retracts onto a subset homeomorphic to the dual graph of π_0, and thus in fact $\mathcal{V}_X$ has the homotopy type of the dual graph of any minimal good resolution. Note, any two minimal good resolutions of (X, x_0) have homeomorphic dual graphs, an easy consequence of [Lau71, Theorem 5.12]. Moreover, minimal good resolutions are obtained from the unique good resolution in the sense of Laufer by a sequence of blowups of *satellite points*, i.e., points in the intersection of two exceptional primes (see Remark 1.1.14). This operation preserves the embedded dual graph, and we deduce that the image through emb_π of the dual graph Γ_{π_0} of any minimal good resolution π_0 does not depend on π. We denote such image as $\mathcal{S}_X$, and refer to it as the *skeleton* of $\mathcal{V}_X$.

For any log resolution π of $\mathfrak{m}_X$, we make Γ_π into a *metric* graph by specifying the lengths of its edges: if E and F are distinct intersecting exceptional primes of π, the length of the edge from $b_E^{-1}\check{E}$ to $b_F^{-1}\check{F}$ is set to be $\frac{1}{b_E b_F}$. The metric has the very useful property that if π' is a good resolution dominating π, then the continuous map $\mathrm{ev}_{\pi'} \circ \mathrm{emb}_\pi$ is an isometric embedding $i_{\pi'\pi} \colon \Gamma_\pi \to \Gamma_{\pi'}$. One can prove this easily when π' is obtained from π by a point blowup; the general case then follows by induction. Also by induction one sees that the inclusions $i_{\pi'\pi}$ are compatible in that $\mathrm{emb}_\pi = \mathrm{emb}_{\pi'} \circ i_{\pi'\pi}$. It therefore makes sense to speak of a direct limit $\varinjlim \Gamma_\pi$ and an induced continuous map $\mathrm{emb}\colon \varinjlim \Gamma_\pi \to \mathcal{V}_X$. Note that by definition a valuation $\nu \in \mathcal{V}_X$ is quasimonomial if and only if it lies in the image $\mathcal{S}_\pi$ of emb_π for some π, so the image of $\mathrm{emb}\colon \varinjlim \Gamma_\pi \to \mathcal{V}_X$ is precisely the set $\mathcal{V}_X{}^{\mathrm{qm}}$ of quasimonomial valuations. It follows from this and Theorem 1.1.17 that quasimonomial valuations are dense in $\mathcal{V}_X$. In fact, one can show that all four types of semivaluations are dense in $\mathcal{V}_X$.

Theorem 1.1.19. *The map* $\mathrm{emb}\colon \varinjlim \Gamma_\pi \to \mathcal{V}_X^{qm}$ *is a continuous bijection, but is not a homeomorphism.*

Of course, the map emb cannot be a homeomorphism, since $\varinjlim \Gamma_\pi$ is by construction a metric space, whereas the weak topology is not metrizable. However we may push forward the metric through emb and obtain a new, strictly stronger topology on the space of quasimonomial valuations. We denote by d this metric on $\mathcal{V}_X{}^{\mathrm{qm}}$, and we call its induced topology the *strong topology*.

1.2 Intersection theory on valuation spaces

1.2.1 Intersection of b-divisors and valuations

We have seen how, given any model $\pi\colon X_\pi \to (X, x_0)$, we have an intersection form defined on $\mathcal{E}(\pi)_\mathbb{R}$, which is negative definite. These intersection forms are compatible to one another, in the sense of the projection formula: if π' is another model dominating π, and $\eta = \eta_\pi^{\pi'}\colon X_{\pi'} \to X_\pi$ is the natural projection, then

$$E' \cdot \eta^* D = \eta_* E' \cdot D$$

for any $E' \in \mathcal{E}(\pi')_\mathbb{R}$ and $D \in \mathcal{E}(\pi)_\mathbb{R}$. We want to use this compatibility to extend these intersection forms to an intersection form on *b*-divisors. The main application we have in mind is to be able to develop an intersection theory on $\mathfrak{m}_X$-primary ideals, and on semivaluations.

Let now $Z = (Z_\pi)_\pi$ and $W = (W_\pi)_\pi$ be two exceptional b-divisors on (X, x_0). For each model $\pi\colon X_\pi \to (X, x_0)$, we can consider the intersection of their incarnations $Z_\pi \cdot W_\pi$. If we take a model $\pi'\colon X_{\pi'} \to (X, x_0)$ dominating π, then the compatibility condition described above gives

$$Z_\pi \cdot W_\pi = \eta_* Z_{\pi'} \cdot W_\pi = Z_{\pi'} \cdot \eta^* W_\pi.$$

In particular, if W is a Cartier b-divisor determined by the model π, then we can define the intersection $Z \cdot W$ as the constant value $Z_{\pi'} \cdot W_{\pi'} = Z_\pi \cdot W_\pi$ for any model π' dominating π.

When W is not Cartier, the sequence (or more precisely, the net) $(Z_\pi \cdot W_\pi)_\pi$ is not definitely constant, and in general does not converge to any value, as the following example shows.

Example 1.2.1. Over $(X, x_0) = (\mathbb{C}^2, 0)$, we consider the sequence of point blow-ups $\eta_n \colon X_{n+1} \to (X_n, p_n)$ defined recursively as follows. We start by blowing up the origin $p_0 = 0 \in \mathbb{C}^2$, and set $E_0 := \eta_0^{-1}(p_0) \subset X_1$ its exceptional divisor. Then, suppose we have defined η_{n-1}. Then we pick a point $p_n \in E_{n-1}$ which is free (for example, the intersection with the strict transform of the curve $\{y = 0\}$), let η_n be the blow-up of p_n, and set $E_n := \eta_n^{-1}(p_n)$.

For any $n \in \mathbb{N}^*$, we set $\pi_n := \eta_0 \circ \cdots \circ \eta_{n-1} \colon X_n \to (\mathbb{C}^2, 0)$. By abuse of notation, we denote by E_k the exceptional prime given by the strict transform of $E_k \subset X_{k+1}$ in X_n for any $n > k$. Consider

$$Z_{\pi_n} = \sum_{k=0}^{n-1} (k+1) E_k \qquad \text{and} \qquad W_{\pi_n} = \sum_{k=0}^{n-1} (-1)^k E_k.$$

Then $(Z_{\pi_n})_n$ and $(W_{\pi_n})_n$ induce Weil b-divisors Z and W. A direct computation shows $Z_{\pi_n} \cdot W_{\pi_n} = (-1)^n$.

This example illustrates how, in order to define a meaningful notion of intersection (and self-intersection) on b-divisors, we need to impose either a control on the growth of the coefficients of the incarnations, or some monotonicity on the intersections.

The first approach starts form the remark that the pairing $b\text{-}\mathcal{E}(X) \times c\text{-}\mathcal{E}(X) \to \mathbb{R}$ defined above induces an intersection form on $c\text{-}\mathcal{E}(X)$. We can extend this intersection form to the completion of $c\text{-}\mathcal{E}(X)$ inside $b\text{-}\mathcal{E}(X)$ (with respect to the intersection form itself). This maximal space where we can define the intersection form is called the space of L^2 b-divisors. Roughly speaking, it is made by b-divisors of the form $\sum_E \lambda_E Z(E)$ with $\sum_E \lambda_E^2 < +\infty$, where $Z(E)$ is the Cartier b-divisor determined by the exceptional prime E on the minimal model π_E where E appears (its existence is a consequence of the existence of minimal models in dimension 2). This space is quite useful when studying the action induced by a germ on b-divisors (for further details in the global setting, see, e.g., [BFJ08a]).

The second approach will allow us to define the intersection on the space $b\text{-}\mathrm{Nef}(X)$ of nef b-divisors, which contains the case of b-divisors associated to $\mathfrak{m}_X$-primary ideals and valuations.

We have seen that any nef divisor is automatically anti-effective. In the case of nef b-divisors, we can deduce the following property.

Lemma 1.2.2. *Let $Z = (Z_\pi)_\pi \in b\text{-}\mathrm{Nef}(X)$ be a nef b-divisor. Let $\pi \colon X_\pi \to (X, x_0)$ be any resolution, and let π' be another resolution dominating π. Then $Z_{\pi'} - \eta^* Z_\pi$ is antieffective, where $\eta = \eta_\pi^{\pi'}$.*

Proof. Up to factorizing η into composition of point blow-ups, we may assume that η is the blow-up of a point $p \in \pi^{-1}(x_0)$. Set $D = Z_{\pi'} - \eta^* Z_\pi$. The statement corresponds to proving that $D \cdot \check{E}' \leq 0$ for any $E' \in \Gamma_{\pi'}^*$. If E' is the strict transform by η of an exceptional prime $E \in \Gamma_\pi^*$, then we have

$$D \cdot \check{E}' = Z_{\pi'} \cdot \eta^* \check{E} - \eta^* Z_\pi \cdot \eta^* \check{E} = \eta_* Z_{\pi'} \cdot \check{E} - Z_\pi \cdot \check{E} = 0,$$

since $\eta_* Z_{\pi'} = Z_\pi$ by definition of b-divisor.

It remains to check the case of $G = \eta^{-1}(p)$. In this case, by applying Equation (1.2), we get

$$D \cdot \check{G} = (Z_{\pi'} - \eta^* Z_\pi) \cdot \left(\sum_{E \ni p} \eta^* \check{E}' - G \right) = -Z_{\pi'} \cdot G \leq 0,$$

where we used here that $Z_{\pi'}$ is nef. □

The next definition allows to restate Lemma 1.2.2 by saying that any nef b-divisor is decreasing.

Definition 1.2.3. Let $Z = (Z_\pi)_\pi \in b\text{-}\mathcal{E}(X)$ be a b-divisor. We say that Z is *increasing* (resp., *decreasing*) if for any $\pi' \geq \pi$ models, we have that $Z_{\pi'} - \eta^* Z_\pi$ is effective (resp., antieffective), where $\eta = \eta_\pi^{\pi'}$.

As a direct consequence, if Z, W are two b-divisors, with Z *decreasing* and W *nef* b-divisor, then we have that

$$Z_{\pi'} \cdot W_{\pi'} = (\eta^* Z_\pi + D) \cdot W_{\pi'} = \eta^* Z_\pi \cdot W_{\pi'} + D \cdot W_{\pi'} = Z_\pi \cdot W_\pi + D \cdot W_{\pi'} \leq Z_\pi \cdot W_\pi,$$

where we used the fact that $D := Z_{\pi'} - \eta^* Z_\pi$ is antieffective by definition, and $W_{\pi'}$ is nef.

Definition 1.2.4. Let Z, W are two b-divisors, with Z *decreasing* and W *nef* b-divisor. We set

$$Z \cdot W := \inf_\pi Z_\pi \cdot W_\pi \in [-\infty, +\infty),$$

where π varies among all resolutions of (X, x_0).

Notice that if Z is increasing, then $-Z$ is decreasing, and we can define analogously the intersection $Z \cdot W$ by replacing the inf by the sup.

Notice that if also Z is nef, then $Z \cdot W \leq 0$, with equality if and only if either $Z = 0$ or $W = 0$. Moreover, in general this intersection could take the value $-\infty$. In order to work with positive values instead, we set $\langle Z, W \rangle := -Z \cdot W$ for any nef b-divisors $Z, W \in b\text{-}\mathrm{Nef}(X)$.

By applying this definition to the b-divisors associated to finite semivaluations, we get the notion of intersection of semivaluations.

Definition 1.2.5. Let $\nu, \mu \in \hat{\mathcal{V}}_X^*$ be two finite semivaluations. Their intersection $\langle \nu, \mu \rangle$ is defined as

$$\langle \nu, \mu \rangle := -Z(\nu) \cdot Z(\mu) \in (0, +\infty].$$

Remark 1.2.6. Similarly, if $\mathfrak{a}$ is a $\mathfrak{m}_X$-primary ideal, we can set

$$\langle \nu, \mathfrak{a} \rangle := -Z(\nu) \cdot Z(\mathfrak{a}) \in (0, +\infty).$$

By Proposition 1.1.11, we simply have that $\langle \nu, \mathfrak{a} \rangle = \nu(\mathfrak{a})$.

Remark 1.2.7. We can use any nef b-divisor $Z \in b\text{-Nef}(X)$ to normalize semivaluations, by setting

$$\mathcal{V}_X^Z := \{\nu \in \hat{\mathcal{V}}_X^* \mid -Z \cdot Z(\nu) = 1\}.$$

Any $\nu \in \hat{\mathcal{V}}_X^*$ so that $-Z \cdot Z(\nu) < +\infty$ has a unique equivalent semivaluation in $\mathcal{V}_X Z$. Notice that the set $\hat{\mathcal{V}}_X^{Z,\infty} := \{\nu \in \hat{\mathcal{V}}_X^* \mid -Z \cdot Z(\nu) = +\infty\}$ might be empty, for example when Z is Cartier.

This extends the normalizations by ideals $\mathfrak{a}$ introduced in Definition 1.1.6. Another class of important normalizations is given by the choice $Z = Z(\mu)$ for any semi-valuation $\mu \in \hat{\mathcal{V}}_X^*$. In this case $\hat{\mathcal{V}}_X^{Z,\infty}$ is either empty, or given by $\{\mu^\bullet\}$, depending on whether $\langle \mu, \mu \rangle$ is finite or not.

1.2.2 Continuity properties of the intersection

Proposition 1.2.8. *The intersection form $b\text{-Nef}(X) \times b\text{-Nef}(X) \to [-\infty, 0]$ which associates to two nef b-divisors (Z, W) their intersection $Z \cdot W$ is upper semi-continuous. In particular, the map $\hat{\mathcal{V}}_X^* \times \hat{\mathcal{V}}_X^* \to (0, +\infty]$ defined by $(\nu, \mu) \mapsto \langle \nu, \mu \rangle$ is lower semi-continuous.*

Proof. Denote by Φ the intersection form on $b\text{-Nef}(X)$, by Φ_π the intersection form on $\mathcal{E}(\pi)_\mathbb{R}$. If we denote by $\mathrm{pr}_\pi \colon b\text{-}\mathcal{E}(X) \to \mathcal{E}(\pi)_\mathbb{R}$ the natural projection $\mathrm{pr}_\pi(Z) = Z_\pi$, then we have that $\Phi_\pi \circ (\mathrm{pr}_\pi \times \mathrm{pr}_\pi)(Z, W) = Z_\pi \cdot W_\pi$ is continuous, and $\Phi = \inf_\pi \Phi_\pi \circ (\mathrm{pr}_\pi \times \mathrm{pr}_\pi)$ is upper semi-continuous. □

Notice that these maps are not continuous, not even on finite semivaluations.

Example 1.2.9. Let $\pi_0 \colon X_{\pi_0} \to (\mathbb{C}^2, 0)$ be the blow-up of the origin. Consider a sequence $(p_n)_{n \in \mathbb{N}^*}$ of points in $E_0 = \pi_0^{-1}(0)$, with $p_n \neq p_m$ for any $n \neq m$, and set $\nu_n = (\pi_0)_* \mathrm{ord}_{p_n}$. The b-divisors $Z_n := Z(\nu_n)$ are all nef and Cartier, and they converge in $b\text{-}\mathcal{E}(X)$ to $Z_0 = Z(\mathrm{div}_{E_0})$. But we have that $Z_n \cdot Z_n = -2 \neq -1 = Z_0 \cdot Z_0$.

The map is not even continuous when we fix one of the two b-divisors, see Example 1.2.13 for an example involving log-discrepancies.

We conclude this section by showing that, in the case of (distinct normalized) valuations, we can always compute their intersection in a model (and we do not need to work on all models).

Proposition 1.2.10. *Let (X, x_0) be a normal surface singularity, and $\nu, \mu \in \hat{\mathcal{V}}_X^*$ be two semivaluations. If ν and μ have different centers in X_π for a good resolution $\pi \colon X_\pi \to (X, x_0)$, then*

$$Z(\nu) \cdot Z(\nu) = Z_\pi(\nu) \cdot Z_\pi(\nu).$$

Proof. Let $\xi = \mathrm{cen}_\pi(\nu)$ and $\zeta = \mathrm{cen}_\pi(\mu)$ be the centers of ν and μ. If $\xi = E \in \Gamma_\pi^*$ is an exceptional prime of π, then $\nu = \lambda \mathrm{div}_E$ for some $\lambda > 0$, and $Z(\nu)$ is a Cartier b-divisor determined in π, from which we deduce the statement. A similar statement holds if we switch the roles of ν and μ, hence we may assume that ξ and ζ are (distinct) closed points. Let $\pi' \geq \pi$ be any good resolution dominating π. Then $\eta = \eta_\pi^{\pi'}$ can be factorized as $\eta_2 \circ \eta_1$, where $\eta_1 \colon X_{\pi_1} \to X_\pi$ is a sequence of point blow-ups above ξ, and $\eta_2 \colon X_{\pi'} \to X_{\pi_1}$

is a sequence of point blow-ups not above ξ. By contruction, η_1 is an isomorphism over a neighborhod of ζ, and we have that $Z_{[}(\pi)_1]\mu = \eta_1^* Z_{[}(\pi)_1]\mu$. We deduce by the projection formula that

$$Z_{[}(\pi)_1]\nu \cdot Z_{[}(\pi)_1]\mu = Z_{[}(\pi)_1]\nu \cdot \eta_1^* Z_\pi(\mu) = (\eta_1)_* Z_{[}(\pi)_1]\nu \cdot Z_\pi(\mu) = Z_\pi(\nu) \cdot Z_\pi(\mu).$$

The same argument used for η_2, switching the roles of ν and μ, allow us to conclude that

$$Z_{\pi'}(\nu) \cdot Z_{\pi'}(\mu) = Z_{\pi_1}(\nu) \cdot Z_{\pi_1}(\mu) = Z_\pi(\nu) \cdot Z_\pi(\mu).$$

The statement follows. □

1.2.3 Log-canonical b-divisor

There are several ways to measure "how singular" a singularity is. One can think for example at the multiplicity of (X, x_0), or at the Hilbert-Samuel polynomial (see, e.g., [Kol07, Section 2.8]). For our purposes, we will consider a functional A defined on the valuation space, and called the *log-discrepancy*. The log-discrepancy measures the positivity properties of the canonical bundle, and its avatars in different birational models.

In particular, we will be interested in the minimum it takes on normalized valuations, called *log-canonical threshold*. Among several classes of singularities, we will distinguish *Kamawata log-terminal (klt)* singularities, for which the log-canonical threshold is strictly positive, and *log-canonical (lc)* singularities, for which the log-canonical threshold is non-negative.

In this section we review the classification of such singularities in the case of surfaces. As usual, (X, x_0) denotes a normal surface singularity and $(R_X, \mathfrak{m}_X)$ denotes the completed local ring $\hat{\mathcal{O}}_{X,x_0}$. We also fix a nontrivial holomorphic 2-form ω on (X, x_0); the vanishing of ω defines a Weil divisor on X that we denote by $\operatorname{Div}(\omega)$.

Given a good resolution $\pi\colon X_\pi \to (X, x_0)$, the *relative canonical divisor* of π is the divisor $K_\pi \in \mathcal{E}(\pi)_\mathbb{Q}$ defined by the equality $\operatorname{Div}(\pi^*\omega) = \pi^* \operatorname{Div}(\omega) + K_\pi$, where here $\pi^* \operatorname{Div}(\omega)$ refers to the Mumford pull-back of $\operatorname{Div}(\omega)$, see [Mat02, p. 195]. The relative canonical divisor K_π does not depend on the choice of ω.

In order to study the positivity propertis of the relative canonical divisor, we label the exceptional primes of the good resolution π as $E_1, \dots, E_n$, write

$$K_\pi = \sum_i (A_i - 1) E_i.$$

The number $A_i = A(E_i)$ is known as the *log-discrepancy* of the exceptional prime E_i. We denote by $A = (A_i)_i$ the vector of log-discrepancies of the exceptional primes in Γ_π^*.

The Mumford pull-back $\pi^* \operatorname{Div}(\omega)$ is numerically trivial on $\mathcal{E}(\pi)$ by definition, and the adjunction formula for X_π gives

$$2g(E) - 2 = K_E = (K_\pi + E) \cdot E. \tag{1.3}$$

for every $E \in \Gamma_\pi^*$, where here $g(E)$ denotes the genus of E, and K_E denotes the canonical bundle of E. By applying Equation (1.3) to E_j for all $j = 1, \dots, n$, we get

$$2g(E_j) - 2 = \sum_i A_i E_i \cdot E_j - \sum_{i \neq j} E_i \cdot E_j = (MA)_j - \delta(E_j),$$

where $\delta(E_j)$ is the degree of E_j in the graph Γ_π, and M is the intersection matrix on $\mathcal{E}(\pi)$. Hence, we can compute the vector of log-discrepancies by the formula

$$A = M^{-1}(2g - 2 - \delta),$$

where $2g - 2 - \delta$ denotes the vector of entries $2g(E_j) - 2 - \delta(E_j)$.

Let $\pi' \colon X_{\pi'} \to (X, x_0)$ be a good resolution dominating π, and set $\eta = \eta_\pi^{\pi'}$. We have

$$\begin{aligned} K_{\pi'} &= \operatorname{Div}(\pi'^* \omega) - \pi'^* \operatorname{Div}(\omega) \\ &= \operatorname{Div}(\eta^* \pi^* \omega) - \eta^* \operatorname{Div}(\pi^* \omega) + \eta^* \operatorname{Div}(\pi^* \omega) - \eta^* \pi^* \operatorname{Div}(\omega) \\ &= K_\eta + \eta^* K_\pi, \end{aligned} \tag{1.4}$$

where K_η is the relative canonical divisor of $\eta \colon X_{\pi'} \to X_\pi$.

Following [BdFF12], we put all together the collection of relative canonical divisors, obtaining a (Weil) b-divisor $K_X = (K_\pi)_\pi$, called the *canonical b-divisor* of (X, x_0). We also denote by $\mathbb{1}_X = (\mathbb{1}_\pi)_\pi$ the *b*-divisor whose incarnation in any model π is given by the reduced exceptional divisor $\mathbb{1}_\pi = \sum_{E \in \Gamma_\pi^*} E$.

Definition 1.2.11. Let (X, x_0) be a normal surface singularity. The *b*-divisor $K_X^{\log} := K_X + \mathbb{1}_X$ is called the *log-canonical b-divisor* of (X, x_0).

1.2.4 Log-discrepancy

Assume now that $\pi \colon X_\pi \to (X, x_0)$ is a good resolution. Let $p \in \pi^{-1}(x_0)$ be a point in the exceptional divisor of π, $\eta \colon X_{\pi'} \to X_\pi$ be the blow-up of p, and $\pi' = \pi \circ \eta$. By applying Equation (1.4) in this situation, we get

$$K_{\pi'} + \mathbb{1}_{\pi'} = K_\eta + \eta^* K_\pi + \mathbb{1}_{\pi'} = \eta^*(K_\pi + \mathbb{1}_\pi) + \begin{cases} G & \text{if } p \text{ is free,} \\ 0 & \text{if } p \text{ is satellite,} \end{cases}$$

where $G = \eta^{-1}(p)$.

From this computation, we deduce that the log-canonical b-divisor is increasing, and we can give the following definition.

Definition 1.2.12. Let $\nu \in \hat{\mathcal{V}}_X^*$ be a finite semi-valuation. Its *log-discrepancy* is defined as

$$A_X(\nu) := K_X^{\log} \cdot Z(\nu) \in \mathbb{R} \cup \{+\infty\}.$$

We will simply write $A_X = A$ unless we need to specify the singularity we are working on. Notice that if $\nu = \mathrm{div}_E$, then $A(\mathrm{div}_E) = 1 + \mathrm{ord}_E(K_\pi) = A(E)$ is the coefficient in front of E of $K_\pi + E$, for any model π that contains E as exceptional prime. The computation above also shows that if E and F are exceptional primes of a good resolution π intersecting in a point p, and if $\nu_{r,s}$ is the monomial valuation at p with weights r and s, then $A(\nu_{r,s}) = rA(E) + sA(F)$.

Being defined as the supremum of continuous functions, the map $\nu \mapsto A(\nu)$ is lower semi-continuous on $\hat{\mathcal{V}}_X^*$.

In order to show that this map is not continuous, we give here another description of valuations on $(X, x_0) = (\mathbb{C}^2, 0)$, that is better adapted to work with log-discrepancy (see [FJ04, Chapter 4] for more details). On $(\mathbb{C}^2, 0)$, we fix some coordinates (x, y), and consider space $\mathcal{V}x$ of normalized valuations with respect to x. Any such valuation can be described by a pair $(\hat{\phi}, \hat{\beta})$, where $\hat{\phi}$ is a generalized Puiseux series:

$$\hat{\phi} = \sum_{j=1}^{N} \hat{\phi}_j x^{\hat{\beta}_j},$$

where $N \in \mathbb{N} \cup +\infty$, $\hat{\phi}_j \in \mathbb{C}^*$, $\hat{\beta}_j$ is an increasing sequence in $\mathbb{Q}_{>0}$, and $(0, +\infty] \ni \hat{\beta}_\infty \geq \sup_j \hat{\beta}_j$, with equality when $N = +\infty$.

There is a natural surjective map (of metrized $\mathbb{R}$-trees) between the space of these pairs, and the space $\mathcal{V}^x$ of x-normalized valuations, that sends the parameter $1 + \hat{\beta}$ to the log-discrepancy A. To give an intuition to this surjection, let C be an irreducible curve in $(\mathbb{C}^2, 0)$, not coinciding with $\{x = 0\}$, and let $\big(x, \hat{\phi}(x)\big)$ be a Puiseux parametrization of C. Then the pair $(\hat{\phi}, +\infty)$ is sent to the curve valuation int_C^x. The four families of valuations correspond to the following situations:

- divisorial valuations are the ones with $N < +\infty$ and $\hat{\beta} \in \mathbb{Q}_{>0}$;
- irrational valuations are the ones with $N < +\infty$ and $\hat{\beta} \in \mathbb{R}_{>0} \setminus \mathbb{Q}$;
- curve semivaluations are the ones with $\hat{\beta}_j$ having bounded denominators and $\hat{\beta} = +\infty$;
- infinitely singular valuations are the ones with $N = +\infty$ and $\hat{\beta}_j$ having unbounded denominators.

We can now to describe the discontinuity of log-discrepancy.

Example 1.2.13. Take now $\hat{\beta}_j$ to be an increasing sequence of positive rational numbers, converging towards a finite vaue $\hat{\beta}$. For example, $\hat{\beta}_j = 1 - 2^{-j} \to 1 = \hat{\beta}$. For any $n \in \mathbb{N}^* \cup \{+\infty\}$, set

$$\hat{\phi}^n := \sum_{j=1}^{n} x^{\hat{\beta}_j}.$$

Then the sequence of curve semivaluations ν_n associated to $(\hat{\phi}^n, +\infty)$ converges towards the infintely singular valuation ν_∞ associated to $(\hat{\phi}^\infty, \hat{\beta})$, but $A(\nu_n) = +\infty$ for any $n \in \mathbb{N}^*$, while $A(\nu_\infty) = 1 + \hat{\beta} = 2$.

For other details on the construction of A as well as some other properties, we refer to [Fav10, BdFFU15, Jon12]. The log-discrepancy $A(\nu)$ is also called the *thinness* of ν, see [FJ04, FJ07, Fav10, BFJ08b] and the *weight* of ν, see [MN15, NX16, BN16].

1.2.5 Log-canonical threshold, klt and lc singularities

Since the log-canonical b-divisor is increasing above any good resolution π, we infer that $A(\nu)$ realizes its inf in some monomial valuations $\nu \in \mathcal{S}_\pi$ on π. This minimum is denoted by $\mathrm{lct}(X)$, and called the *log-canonical threshold* of (X, x_0).

Notice that $\mathrm{lct}(X)$ depends on the normalization chosen on $\mathcal{V}_X$, but its sign does not. A singularity (X, x_0) is called *kamawata log-terminal (klt)* if $\mathrm{lct}(X) > 0$, and *log-canonical (lc)* if $\mathrm{lct}(X) \geq 0$.

These classes of singularities appear very naturally when studying minimal models in the sense of birational geometry, and are in some sense mild singularities.

For the case of surfaces, one can classify these singularities (see, e.g., [KM98, Section 4.1]): klt singularities are exactly quotient singularities, while lc singularities that are not klt are either cusp singularities (the exceptional divisor if its minimal good resolution is a cycle of rational curves), or simple elliptic singularities (the exceptional divisor if its minimal good resolution is an irreducible elliptic curve), or quotients of these.

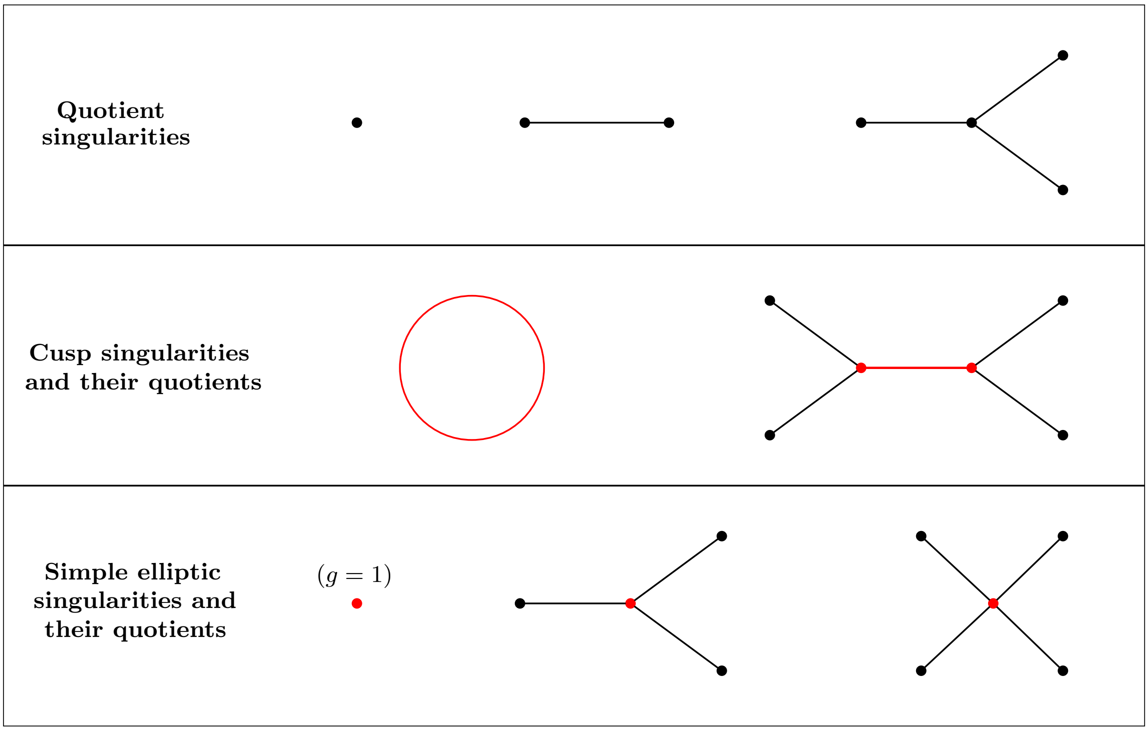

Figure 1.4: The possible skeleta $\mathcal{S}_X \subset \mathcal{V}_X$ of lc singularities (X, x_0). Here dots are drawn only at forks and endpoints of $\mathcal{S}_X$, and a point in $\mathcal{S}_X$ is red if and only if $A(\nu) = 0$.

1.2.6 Positivity properties of intersection of valuations

We show here a fundamental property of the intersection of b-divisors on surface singularities, proved in [GR21], and illustrate some of its applications later on. In order to state this property, we need a preliminary definition.

Definition 1.2.14. Let (X, x_0) be a normal surface singularity, and let $\nu, \mu_1, \mu_2 \in \mathcal{V}_X$ be three normalized semivaluations. We say that ν *disconnects* μ_1 *and* μ_2, or that ν *disconnects the pair* (μ_1, μ_2), if either $\nu = \mu_1$, $\nu = \mu_2$, or μ_1 and μ_2 belong to different connected components of $\mathcal{V}_X \setminus \{\nu\}$.

Notice that we can consider $\mathcal{V}_X$ endowed indifferently with either the weak or the strong topology, since the connected components of $\mathcal{V}_X \setminus \{\nu\}$ are the same for the two topologies.

Theorem 1.2.15. *Let (X, x_0) be a normal surface singularity. Let $\nu, \mu_1, \mu_2 \in \hat{\mathcal{V}}_X^*$ be finite semivaluations. Then*

$$\langle \nu, \mu_1 \rangle \langle \nu, \mu_2 \rangle \leq \langle \nu, \nu \rangle \langle \mu_1, \mu_2 \rangle \,, \tag{1.5}$$

with equality if and only if $\nu^\bullet$ disconnects $\mu_1^\bullet$ and $\mu_2^\bullet$ in $\mathcal{V}_X^\bullet$.

Proof. Being Equation (1.5) homogeneous of degree 2 on ν, and of degree 1 on μ_1 and μ_2, we can work in $\mathcal{V}_X$ instead of $\hat{\mathcal{V}}_X^*$.

If $\nu = \mu_1$ or $\nu = \mu_2$, then Equation (1.5) clearly holds as an equality. If $\mu := \mu_1 = \mu_2 \neq \nu$, consider a model π where the centers of μ and ν are different, and by Proposition 1.2.10 and Cauchy-Schwarz inequality we get

$$\langle \nu, \mu \rangle^2 = \big(- Z_\pi(\nu) \cdot Z_\pi(\mu) \big)^2 < \big(- Z_\pi(\nu) \cdot Z_\pi(\nu) \big) \big(- Z_\pi(\mu) \cdot Z_\pi(\mu) \big) \leq \langle \nu, \nu \rangle \langle \mu, \mu \rangle \,.$$

Assume now that ν, μ_1, μ_2 are all distinct. Again by Proposition 1.2.10, we can reduce to the situation where ν, μ_1 and μ_2 are divisorial valuations associated to distinct exceptional primes E, F_1, F_2 in a suitable good resolution π. In this situation, the statement becomes

$$(\check{E} \cdot \check{F}_1)(\check{E} \cdot \check{F}_2) \leq (\check{E} \cdot \check{E})(\check{F}_1 \cdot \check{F}_2), \tag{1.6}$$

with equality if and only if E disconnects F_1 and F_2 inside Γ_π.

Let $D_1, \ldots, D_{n-2}$ be the exceptional primes of π that are not equal to E or F_1, and set

$$Q = \check{F}_1 - \frac{\check{E} \cdot \check{F}_1}{\check{E} \cdot \check{E}} \check{E}.$$

Notice that Equation (1.6) is equivalent to $Q \cdot \check{F}_2 \leq 0$.

Consider the basis $D_1, \ldots, D_{n-2}, F_1, -\check{E}$ of $\mathcal{E}(\pi)_\mathbb{R}$, which for simplicity we denote by $u_1, \ldots, u_n$. This basis has the property that $u_i \cdot u_j \geq 0$ for all $i \neq j$ (while $u_i \cdot u_i < 0$ for all i). Moreover we have $Q \cdot u_i = 0$ for all $i \neq n-1$, and $Q \cdot u_{n-1} = 1$.

If we write $Q = a_1 u_1 + \cdots + a_n u_n$, then

$$\begin{pmatrix} u_1 \cdot u_1 & \cdots & u_n \cdot u_1 \\ \vdots & \ddots & \vdots \\ u_1 \cdot u_n & \cdots & u_n \cdot u_n \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} Q \cdot u_1 \\ \vdots \\ Q \cdot u_n \end{pmatrix}. \tag{1.7}$$

By solving this linear system, we get $a_i = (u_i^* \cdot u_{n-1}^*)$. By Lemma 1.1.7 applied to the basis $u_1, \ldots, u_n$, the matrix $(u_i^* \cdot u_j^*)_{ij}$ has non-positive entries. We conclude that $a_i \leq 0$ for each i. Because each of the divisors u_i are effective, it follows that $Q = \sum a_i u_i$ is anti-effective. In particular, $Q \cdot \check{F}_2 \leq 0$, as desired.

To determine when we have an equality, we need to be more careful. First notice that $a_n = 0$. In fact $u_j \cdot u_n = 0$ for all $j = 1, \ldots, n-1$, while $u_n \cdot u_n = \check{E} \cdot \check{E} < 0$. We deduce that $a_n = 0$ from the last line of Equation (1.7), and the fact that $Q \cdot u_n = 0$.

If we consider the first $n-1$ equations of the linear system Equation (1.7) (for the variables a_i, $i = 1, \ldots, n-1$) the associated matrix U_E is the intersection matrix of $\overline{\pi^{-1}(x_0) \setminus E}$. By Lemma 1.1.7, $a_i = (u_i^* \cdot u_{n-1}^*) < 0$ for $i = 1, \ldots, n-1$ if and only if u_i and F belong to the same connected component of $\overline{\pi^{-1}(x_0) \setminus E}$. We conclude by noticing that $Q \cdot \check{F}_2 = a_i$ if $F_2 = u_i$.

□

Remark 1.2.16. Being Cartier nef b-divisors a positive linear combination of b-divisors associated to divisorial valuations, and being Equation (1.6) linear on μ_1 and μ_2, we can replace these valuations in Equation (1.6) with any Cartier nef b-divisors, for example b-divisors associated to $\mathfrak{m}_X$-primary ideals. By continuity arguments, we can also drop the Cartier hypothesis. In general, we cannot replace ν by nef b-divisors.

1.2.7 Skewness and relative skewness

It turns out that the intersection of two valuations has an algebraic interpretation, in terms of (relative) Izumi constants, also known as (relative) skewness.

Definition 1.2.17. Let $\nu, \mu \in \hat{\mathcal{V}}_X^*$ be two finite semivaluations. Their *relative skewness* is defined as

$$\beta(\nu|\mu) := \sup_{\mathfrak{a}} \frac{\nu(\mathfrak{a})}{\mu(\mathfrak{a})},$$

where $\mathfrak{a}$ varies among all $\mathfrak{m}_X$-primary ideals.

Proposition 1.2.18. *For any finite semivaluations $\nu, \mu \in \hat{\mathcal{V}}_X^*$, we have*

$$\beta(\nu|\mu) = \frac{\langle \nu, \nu \rangle}{\langle \nu, \mu \rangle}.$$

Proof. From Theorem 1.2.15, we have $\langle \nu, \mu \rangle \langle \nu, \mathfrak{a} \rangle \leq \langle \nu, \nu \rangle \langle \mu, \mathfrak{a} \rangle$, from which we infer

$$\frac{\nu(\mathfrak{a})}{\mu(\mathfrak{a})} = \frac{\langle \nu, \mathfrak{a} \rangle}{\langle \mu, \mathfrak{a} \rangle} \leq \frac{\langle \nu, \nu \rangle}{\langle \nu, \mu \rangle},$$

and $\beta(\nu|\mu) \leq \dfrac{\langle \nu, \nu \rangle}{\langle \nu, \mu \rangle}$ by taking the sup.

Let now $\pi \colon X_\pi \to (X, x_0)$ be a good resolution. Since relatively ample divisors are dense in the nef cone of π, there exist relatively ample divisors D_n with $D_n \to Z_\pi(\nu)$ as $n \to +\infty$.

For each n, pick $k_n \in \mathbb{N}$ large enough so that $k_n D_n$ is an integral relatively very ample divisor, and hence $k_n D_n = Z_\pi(\mathfrak{a}_n)$ for some $\mathfrak{m}_X$-primary ideal $\mathfrak{a}_n$. Then

$$\frac{\nu(\mathfrak{a}_n)}{\mu(\mathfrak{a}_n)} = \frac{Z_\pi(\nu) \cdot D_n}{Z_\pi(\mu) \cdot D_n} \to \frac{Z_\pi(\nu) \cdot Z_\pi(\nu)}{Z_\pi(\mu) \cdot Z_\pi(\nu)}.$$

Hence $\beta(\nu|\mu) \geq \frac{Z_\pi(\nu) \cdot Z_\pi(\nu)}{Z_\pi(\mu) \cdot Z_\pi(\nu)}$, and we conclude by taking the sup over all good resolutions π (notice that the denominator is constant for all models dominating one where the centers of μ and ν are distinct). □

Instead of comparing two valuations to one another, we can compare the value that a valuation ν takes on a $\mathfrak{m}_X$-primary ideal with a given positive functional on $\mathfrak{m}_X$-primary ideals.

Definition 1.2.19. Let $\nu \in \hat{\mathcal{V}}_X^*$ be a finite semivaluation. Its *skewness* with respect to a non-trivial effective b-divisor Z is defined as

$$\alpha_Z(\nu) = \sup_{\mathfrak{a}} \frac{\nu(\mathfrak{a})}{Z \cdot Z(\mathfrak{a})}.$$

Similarly to what we did in Proposition 1.2.18, we can show that

$$\alpha_Z(\nu) = \frac{\langle \nu, \nu \rangle}{Z \cdot Z(\nu)}.$$

Typical choices for the b-divisor Z are $-Z(\mathfrak{m}_X)$, or $Z = -Z(\mathrm{int}_C)$ for some irreducible curve C.

We will also set $\alpha(\nu) = \langle \nu, \nu \rangle$, and denote by $\hat{\mathcal{V}}_X^\alpha$ the set of centered semivaluations with $\alpha(\nu) < +\infty$. This set contains $\hat{\mathcal{V}}_X^{\mathrm{qm}}$ the set of quasimonomial valuations.

1.2.8 Angular distance

The metric d described in Section 1.1.9 has proved useful a number of times in previous works for studying the geometry of valuation spaces and developing potential theory on them, see [FJ04, FJ07, FJ05a, FJ05b, FJ11, BFJ14, BN16]. For us, however, another metric, which we will call the *angular metric*, shall prove to be of greater use in our dynamical setting.

Definition 1.2.20. The *angular distance* $\rho(\nu, \mu)$ between two semivaluations $\nu, \mu \in \mathcal{V}_X^\bullet$ is given by

$$\rho_X(\nu, \mu) := \log[\beta(\nu \mid \mu)\beta(\mu \mid \nu)] = \log \frac{\langle \nu, \nu \rangle \langle \mu, \mu \rangle}{\langle \nu, \mu \rangle^2} \in [0, +\infty]. \tag{1.8}$$

We will often omit the index X if the singularity (X, x_0) is clear from context. The term angular distance alludes to the fact that $\rho(\nu, \mu)$ gives a measure of the angle between ν and μ with respect to the intersection of valuations, which is in fact given by $\arccos(e^{-\frac{1}{2}\rho(\nu,\mu)})$.

Proposition 1.2.21. *The angular distance ρ gives a metric on the set of valuations of finite skewness in $\mathcal{V}_X$.*

Proof. If $\nu, \mu \in \mathcal{V}_X$ are both valuations of finite skewness, then $\rho(\nu, \mu)$ is finite. Clearly ρ is symmetric, and positive definite. To show the triangle inequality, let $\gamma \in \mathcal{V}_X$ be another semivaluation of finite skewness. Then by Proposition 1.2.18 we get

$$\begin{aligned}
\beta(\nu \mid \mu)\beta(\mu \mid \nu) &= \sup_{\mathfrak{a}} \left(\frac{\nu(\mathfrak{a})\gamma(\mathfrak{a})}{\mu(\mathfrak{a})\gamma(\mathfrak{a})}\right) \times \sup_{\mathfrak{a}} \left(\frac{\mu(\mathfrak{a})\gamma(\mathfrak{a})}{\nu(\mathfrak{a})\gamma(\mathfrak{a})}\right) \\
&\leq \sup_{\mathfrak{a}} \left(\frac{\nu(\mathfrak{a})}{\gamma(\mathfrak{a})}\right) \times \sup_{\mathfrak{a}} \left(\frac{\gamma(\mathfrak{a})}{\mu(\mathfrak{a})}\right) \times \sup_{\mathfrak{a}} \left(\frac{\mu(\mathfrak{a})}{\gamma(\mathfrak{a})}\right) \times \sup_{\mathfrak{a}} \left(\frac{\gamma(\mathfrak{a})}{\nu(\mathfrak{a})}\right) \\
&= \beta(\nu \mid \gamma)\beta(\gamma \mid \mu)\beta(\mu \mid \gamma)\beta(\gamma \mid \nu).
\end{aligned}$$

Upon taking logarithms, this gives $\rho(\nu, \mu) \leq \rho(\nu, \gamma) + \rho(\gamma, \mu)$, as desired. $\square$

1.3 Applications to ultrametric distances

1.3.1 Ultrametrics on sets of branches

Let (X, x_0) be a normal surface singularity. A *branch* on it is an irreducible germ of formal curve on (X, x_0). Its set, denoted by $\mathcal{B}_X$ (we omit the index X in the smooth case), is endowed with an inner product, the local intersection product at x_0. Note that on arbitrary normal surface singularities the intersection numbers are defined in the sense of Mumford [Mum61], and take rational (possibly non-integral) positive values.

In [Pło85], Płoski studied the intersection of branches in the smooth case $(X, x_0) = (\mathbb{C}^2, 0)$, opportunely renormalized. His result can be stated in terms of ultrametric properties of the functional u, defined on $\mathcal{B} \times \mathcal{B}$ by $u(A, B) = \frac{m(A)m(B)}{A \cdot B}$:

$$u(A, B) \leq \max\{u(A, C), u(C, B)\},$$

with equality whenever $u(A, C) \neq u(C, B)$. This result has been then generalized by [GGP18] to *arborescent* singularities.

Definition 1.3.1. A normal surface singularity (X, x_0) is called *arborescent* if its skeleton $\mathcal{S}_X$ is a tree.

Equivalently, arborescent singularities are characterized by the existence of a modification π so that the dual graph Γ_π is a tree, or that this property holds for all good modifications, or again that the valuation space $\mathcal{V}_X$ is contractible. In particular, rational singularities are arborescent, but, for arborescent singularities, the exceptional primes may have arbitrary genus.

In their work, given a fixed branch $L \in \mathcal{B}_X$, they consider the functional u_L defined on pair of branches in $\mathcal{B}_X \setminus \{L\}$ by

$$u_L(A, B) = \frac{(L \cdot A)(L \cdot B)}{A \cdot B},$$

where we set $u_L(A,B) = 0$ when $A = B$. Notice that in the smooth case, $u(A,B) = u_L(A,B)$ for any generic branch L of multiplicity 1 (it suffices that L is transversal to A and B). One can then recover Płoski's result by noticing that it suffices to check the ultrametric property on finite sets of branches.

The map u_L takes values in $[0,+\infty)$, and vanishes if and only if $A = B$, since $L \cdot A > 0$ for any $A \in \mathcal{B}_X$. We may extend u_L on the whole space of branches $\mathcal{B}_X$ by allowing the value $+\infty$, when either $L = A$ or $L = B$, but still setting $u_L(L,L) = 0$. Without further notice, we will consider this extension of u_L with values in $[0,+\infty]$, and consider metrics as extended metrics (i.e., with values in $[0,+\infty]$).

In [GGPR19], we complete this picture, by showing that the ultrametric character of the functional u_L completely characterizes arborescent singularities.

Theorem 1.3.2. *A normal surface singularity (X,x_0) is arborescent if and only if there exists $L \in \mathcal{B}_X$ for which u_L is an ultrametric, if and only if u_L are ultrametrics for any choice of $L \in \mathcal{B}_X$.*

1.3.2 Intersection of branches and of valuations

In order to extend the statement of Theorem 1.3.2 to the valuation setting, and relate it to intersection theory of valuations and b-divisors, we need to clarify the links between these objects.

Let $A, B \in \mathcal{B}_X$ be two branches. Their intersection in the sense of Mumford is defined as

$$A \cdot B := \pi^* A \cdot \pi^* B,$$

where $\pi\colon X_\pi \to (X,x_0)$ is any good resolution, and π^* denotes the *total transform* operator. Notice that the right-hand side does not depend on the resolution chosen because of the projection formula.

Given a branch $A \in \mathcal{B}_X$, and a good modification $\pi\colon X_\pi \to (X,x_0)$, we can decompose the total transform $\pi^* A$ into the sum

$$\pi^* A = A_\pi + A_\pi^{\mathrm{exc}},$$

where A_π is the *strict transform* of A, defined as the irreducible (non-exceptional) divisor satisfying $\pi(A_\pi) = A$, and $A_\pi^{\mathrm{exc}} \in \mathcal{E}(\pi)_{\mathbb{Q}}$ is the *exceptional transform* of A. This decomposition allows us to establish a connection between the intersection of branches, and the intersection of valuations.

Proposition 1.3.3. *Let $A, B \in \mathcal{B}_X$ be two branches on (X,x_0), and let $\pi\colon X_\pi \to (X,x_0)$ be any good resolution. Then*

$$A_\pi^{\mathrm{exc}} = -Z_\pi(\mathrm{int}_A) \quad \textit{and} \quad A \cdot B = \langle \mathrm{int}_A, \mathrm{int}_B \rangle .$$

Proof. Let $D \in \mathcal{E}(\pi)$ be any π-exceptional divisor. By definition, $Z_\pi(\mathrm{int}_A)$ is the unique exceptional divisor satisfying

$$Z_\pi(\mathrm{int}_A) \cdot D = \mathrm{int}_A(D).$$

Write now $\pi^*A = A_\pi + A_\pi^{\mathrm{exc}}$. By applying the projection formula, we get

$$0 = \pi^*A \cdot D = A_\pi \cdot D + A_\pi^{\mathrm{exc}} \cdot D = \mathrm{int}_A(D) + A_\pi^{\mathrm{exc}} \cdot D.$$

Being D arbitrary, we infer that $A_\pi^{\mathrm{exc}} = -Z_\pi(\mathrm{int}_A)$.

Consider now the branches A and B, that we assume distinct (if $A = B$ both computation of intersections give $+\infty$, and we are done).

Let $\pi\colon X_\pi \to (X, x_0)$ be an embedded resolution of $A + B$. In particular, the strict transforms A_π and B_π of A and B do not intersect, and int_A and int_B have distinct centers in X_π.

In this case, we have

$$A \cdot B = \pi^*A \cdot \pi^*B = \big(A_\pi - Z_\pi(\mathrm{int}_A)\big) \cdot \pi^*B = A_\pi \cdot \pi^*B,$$

where the last equality comes by the projection formula. But then we have

$$A_\pi \cdot \pi^*B = A_\pi \cdot \big(B_\pi - Z_\pi(\mathrm{int}_B)\big) = -A_\pi \cdot Z_\pi(\mathrm{int}_B),$$

where we used the fact that A_π and B_π do not intersect. Finally, we have

$$-A_\pi \cdot Z_\pi(\mathrm{int}_B) = -\mathrm{int}_A\big(Z_\pi(\mathrm{int}_B)\big) = -Z_\pi(\mathrm{int}_A) \cdot Z_\pi(\mathrm{int}_B) = \langle \mathrm{int}_A, \mathrm{int}_B \rangle\,,$$

where the last equality is a consequence of Proposition 1.2.10. □

1.3.3 Ultrametrics on valuation spaces

Proposition 1.3.3 allows us to extend our setting from branches to valuations.

Definition 1.3.4. Let X be a normal surface singularity, and let $\lambda \in \hat{\mathcal{V}}_X^*$ be any semivaluation. Let $\nu_1, \nu_2 \in \hat{\mathcal{V}}_X^*$ be any finite semivaluations on X. We set:

$$\boxed{u_\lambda(\nu_1, \nu_2)} := \begin{cases} \dfrac{\langle \lambda, \nu_1 \rangle \cdot \langle \lambda, \nu_2 \rangle}{\langle \nu_1, \nu_2 \rangle} & \text{if } \nu_1^\bullet \neq \nu_2^\bullet, \\ 0 & \text{if } \nu_1^\bullet = \nu_2^\bullet. \end{cases} \tag{1.9}$$

Recall that $\nu_1^\bullet = \nu_2^\bullet$ if and only if ν_1 and ν_2 are proportional. Usually, we fix a normalization, for example the one given by the maximal ideal, and consider u_λ as acting of pairs of normalized semivaluations in $\mathcal{V}_X$. In this case, it is often convenient to pick $\lambda \in \mathcal{V}_X$ as well. Since the expression in Equation (1.9) is homogeneous of degree 2 on λ, this renormalization changes u_λ by a positive multiplicative constant. Notice also that Equation (1.9) is homogeneous of degree 0 on both ν_1 and ν_2, hence it induces a map, also denoted by u_λ on pairs of normalizable semivaluations in $\mathcal{V}_X^\bullet$.

In what follows, we will implicitly work with a fixed normalization, for example the one given by the maximal ideal.

If L is a branch, and $\lambda = \mathrm{int}_L$ is the associated curve semivaluation, then $u_L(A, B) = u_\lambda(\mathrm{int}_A, \mathrm{int}_B) = u_\lambda(\nu_A, \nu_B)$. Hence u_L corresponds to the restriction of u_λ on curve semivaluations (under the natural identification of a branch A with its corresponding semivaluation ν_A).

Remark 1.3.5. Since $\langle \nu_1, \nu_2 \rangle < +\infty$ when $\nu_1 \neq \nu_2$, the function u_λ is well defined with values in $[0, +\infty]$, and it vanishes if and only if $\nu_1 = \nu_2$. The value $+\infty$ is sometimes achieved. In fact, while the denominator is always strictly positive, if λ is normalized we have $\langle \lambda, \nu \rangle = +\infty$ if and only if $\lambda = \nu$ and $\alpha(\lambda) = +\infty$. In particular, u_λ takes only finite values if $\alpha(\lambda) < +\infty$, while it always takes finite values on $(\mathcal{V}_X \setminus \{\lambda\})^2$.

Notice that if ν_1 and ν_2 tend to the same valuation ν in the strong topology, then $\frac{\langle \lambda, \nu_1 \rangle \cdot \langle \lambda, \nu_2 \rangle}{\langle \nu_1, \nu_2 \rangle}$ tends to $\frac{\langle \lambda, \nu \rangle^2}{\alpha(\nu)}$. This value is finite as long as $\nu \neq \lambda$, and it is 0 if and only if $\alpha(\nu) = +\infty$. This always happens when ν is a curve semivaluation, and never happens for quasi-monomial valuations.

The natural extension of Theorem 1.3.2 to valuation spaces holds, and can be stated as follows.

Theorem 1.3.6. *Let X be a normal surface singularity. Then the following properties are equivalent:*

(a) For every semivaluation $\lambda \in \hat{\mathcal{V}}^*_X$, the function u_λ is an (extended) ultrametric distance on $\mathcal{V}_X$.

(b) There exists a semivaluation $\lambda \in \hat{\mathcal{V}}^*_X$, such that the function u_λ is an (extended) ultrametric distance on $\mathcal{V}_X$.

(c) The singularity X is arborescent.

1.3.4 Quadrangular inequalities on valuation spaces

Let (X, x_0) be a normal surface singularity. As we did before, for simplicity we fix the normalization given by the maximal ideal, and work with $\mathcal{V}_X$, but we could have worked with any other normalization, or directly with the set of normalizable semivaluations $\mathcal{V}^\bullet_X$.

In order to check the non-archimedean triangular inequality for u_λ, we are led to study inequalities of the form

$$u_\lambda(\nu_1, \nu_2) \leq u_\lambda(\nu_1, \nu_3),$$

which is equivalent to

$$\langle \lambda, \nu_2 \rangle \langle \nu_1, \nu_3 \rangle \leq \langle \lambda, \nu_3 \rangle \langle \nu_1, \nu_2 \rangle .$$

We are hence led to study when these quadrangular inequalities hold.

First, we recall that, given three normalized valuations $\mu, \nu_1, \nu_2 \in \mathcal{V}_X$, we say that μ separates the pair (ν_1, ν_2) if either $\mu \in \{\nu_1, \nu_2\}$, or if ν_1 and ν_2 belong to different connected components of $\mathcal{V}_X \setminus \{\mu\}$. In this situation, we have seen in Theorem 1.2.15 that

$$\langle \mu, \nu_1 \rangle \cdot \langle \mu, \nu_2 \rangle \leq \langle \mu, \mu \rangle \cdot \langle \nu_1, \nu_2 \rangle , \tag{1.5}$$

with equality if and only if μ separates (ν_1, ν_2) (for convenience, we switched the role of ν and μ valuations with respect to the presentation given in Theorem **??**).

The key to the proof of Theorem 1.3.6 is a generalization of Equation (1.5) to the case of four valuations $\nu_1, \dots, \nu_4$. This generalization is given by the following Proposition, where we also need an auxiliary fifth valuation μ.

Proposition 1.3.7. *Let X be a normal surface singularity, and $\nu_j \in \mathcal{V}_X$, for $j = 1, \dots, 4$, be four normalized semivaluations. Suppose that there exists $\mu \in \mathcal{V}_X$ that separates simultaneously the couple (ν_1, ν_2) and the couple (ν_3, ν_4). Then:*

$$\langle \nu_1, \nu_2 \rangle \cdot \langle \nu_3, \nu_4 \rangle \leq \langle \nu_1, \nu_3 \rangle \cdot \langle \nu_2, \nu_4 \rangle \,. \tag{1.10}$$

Moreover, the equality in Equation (1.10) holds if and only if μ also separates simultaneously the couple (ν_1, ν_3) and the couple (ν_2, ν_4).

Proof. $\boxed{\text{Case } \alpha(\mu) = +\infty.}$ In this case, μ is necessarily an end of $\mathcal{V}_X$, i.e., $\mathcal{V}_X \setminus \{\mu\}$ is connected. It follows that, up to permuting the roles of ν_1, ν_2 and of ν_3, ν_4, we have either $\nu_1 = \nu_3 = \mu$ or $\nu_1 = \nu_4 = \mu$.

In the first case, if either ν_2 or ν_4 coincides with μ, then both sides of Equation (1.10) are $+\infty$, and we have equality, in agreement with the statement. If both ν_2 and ν_4 differ from μ, the left hand side of Equation (1.10) is finite, while the right hand side is $+\infty$, again in agreement with the statement, since μ does not separate ν_2 and ν_4.

In the second case, the left and right hand sides of Equation (1.10) coincide, and in fact μ separates also the couple (ν_1, ν_3) and (ν_2, ν_4).

$\boxed{\text{Case } \alpha(\mu) < +\infty.}$ By Theorem 1.2.15, we have:

$$\langle \mu, \nu_1 \rangle \cdot \langle \mu, \nu_3 \rangle \leq \langle \mu, \mu \rangle \cdot \langle \nu_1, \nu_3 \rangle \,, \tag{1.11}$$
$$\langle \mu, \nu_2 \rangle \cdot \langle \mu, \nu_4 \rangle \leq \langle \mu, \mu \rangle \cdot \langle \nu_2, \nu_4 \rangle \,. \tag{1.12}$$

We want to prove the inequality:

$$\langle \nu_1, \nu_2 \rangle \cdot \langle \nu_3, \nu_4 \rangle \cdot \langle \mu, \mu \rangle \leq \langle \mu, \nu_2 \rangle \cdot \langle \mu, \nu_4 \rangle \cdot \langle \nu_1, \nu_3 \rangle \,, \tag{1.13}$$

which implies the statement Equation (1.10) by applying Equation (1.12). Now, again by Theorem 1.2.15, we have:

$$\langle \mu, \nu_1 \rangle \cdot \langle \mu, \nu_2 \rangle = \langle \mu, \mu \rangle \cdot \langle \nu_1, \nu_2 \rangle \,, \tag{1.14}$$
$$\langle \mu, \nu_3 \rangle \cdot \langle \mu, \nu_4 \rangle = \langle \mu, \mu \rangle \cdot \langle \nu_3, \nu_4 \rangle \,, \tag{1.15}$$

where the equalities are given by the fact that μ separates both couples (ν_1, ν_2) and (ν_3, ν_4). From these equalities, together with Equation (1.11), we deduce that:

$$\begin{aligned} \langle \nu_1, \nu_2 \rangle \cdot \langle \nu_3, \nu_4 \rangle \cdot \langle \mu, \mu \rangle^2 &= \langle \mu, \nu_1 \rangle \cdot \langle \mu, \nu_3 \rangle \cdot \langle \mu, \nu_2 \rangle \cdot \langle \mu, \nu_4 \rangle \\ &\leq \langle \mu, \mu \rangle \cdot \langle \nu_1, \nu_3 \rangle \cdot \langle \mu, \nu_2 \rangle \cdot \langle \mu, \nu_4 \rangle \,, \end{aligned}$$

which gives the desired inequality Equation (1.13).

Finally, by Theorem 1.2.15, the inequalities Equation (1.11) and Equation (1.12) are equalities if and only if μ separates both the couple (ν_1, ν_3) and the couple (ν_2, ν_4). This concludes the proof. □

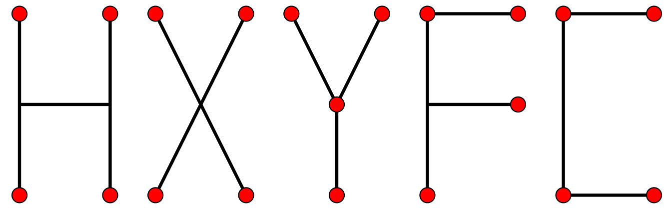

Figure 1.5: The 5 possible configurations of four valuations in their convex hull.

1.3.5 u_λ is an ultrametric for arborescent singularities

We proceed to the proof of Theorem 1.3.6. Clearly (a) $\Rightarrow$ (b). Here we prove the implication (b) $\Rightarrow$ (c), which generalizes the results of [GGP18] to the valuations setting.

Let $\lambda \in \mathcal{V}_X$ be any normalized semivaluation. Since by construction u_λ is symmetric and vanishes only on the diagonal, it is enough to show that the ultrametric triangular inequality holds.

Let $\nu_1, \nu_2, \nu_3 \in \mathcal{V}_X$, and assume that $c := \langle \lambda, \nu_1 \rangle \cdot \langle \lambda, \nu_2 \rangle \cdot \langle \lambda, \nu_3 \rangle \in [0, +\infty]$ is finite. This is guaranteed for example if the three semivaluations are taken in $\mathcal{V}_X \setminus \{\lambda\}$. Let us define I_1, I_2, I_3 by:

$$
\begin{aligned}
u_\lambda(\nu_1, \nu_2) &= \frac{\langle \lambda, \nu_1 \rangle \cdot \langle \lambda, \nu_2 \rangle}{\langle \nu_1, \nu_2 \rangle} = \frac{c}{\langle \nu_1, \nu_2 \rangle \cdot \langle \lambda, \nu_3 \rangle} =: \frac{c}{I_3}, \\
u_\lambda(\nu_1, \nu_3) &= \frac{\langle \lambda, \nu_1 \rangle \cdot \langle \lambda, \nu_3 \rangle}{\langle \nu_1, \nu_3 \rangle} = \frac{c}{\langle \nu_1, \nu_3 \rangle \cdot \langle \lambda, \nu_2 \rangle} =: \frac{c}{I_2}, \\
u_\lambda(\nu_2, \nu_3) &= \frac{\langle \lambda, \nu_2 \rangle \cdot \langle \lambda, \nu_3 \rangle}{\langle \nu_2, \nu_3 \rangle} = \frac{c}{\langle \nu_2, \nu_3 \rangle \cdot \langle \lambda, \nu_1 \rangle} =: \frac{c}{I_1}.
\end{aligned}
$$

We want to show that if X is arborescent, then among the quantities I_1, I_2, I_3, at least two coincide, and they are smaller or equal than the third one.

Since X is arborescent, we can consider the convex hull $S = \mathrm{Conv}(\nu_1, \nu_2, \nu_3, \lambda)$ of $\{\nu_1, \nu_2, \nu_3, \lambda\}$, defined by using the three structure of $\mathcal{V}_X$. It is easy to check that S has one of the shapes represented in Figure 1.5. We study case by case, according to the shape of S:

- **H-shaped.** Let μ be any point in the horizontal segment. It separates all couples, excepted at least one between ν_1, λ and ν_2, ν_3. By Proposition 1.3.7 we deduce that $I_3 = I_2 < I_1$.

- **X-shaped.** The branch point μ separates all couples, and $I_1 = I_2 = I_3$.

- **Y-shaped.** The branch point $\mu = \lambda$ separates all couples, and again $I_1 = I_2 = I_3$.

- **F-shaped.** Let μ be the branch point. It separates all couples, excepted ν_1, ν_2. We get $I_1 = I_2 < I_3$.

- C**-shaped.** Let μ be any point in the vertical segment. It separates all couples, excepted ν_1, ν_2 and ν_3, λ. We get $I_1 = I_2 < I_3$.

The case when some of the valuations $\nu_1, \nu_2, \nu_3, \lambda$ coincide is easier, and is left to the reader. We conclude that u_λ defines an (extended) ultrametric distance on $\mathcal{V}_X$.

1.3.6 Cycles prevent ultrametric inequalities

We conclude the proof of Theorem 1.3.6 by showing that (c) $\Rightarrow$ (a). We proceed by contradiction, and assume that X is not arborescent, i.e., the valuation space $\mathcal{V}_X$ has a loop S. Let $\lambda \in \mathcal{V}_X$ be given. We then chose $\nu_1 \in S$ to be a divisorial valuation so that λ belongs to the same connected component of $\mathcal{V}_X \setminus \{\nu_1\}$ as $S \setminus \{\nu_1\}$. Near ν_1, the loop S looks like a segment: we will pick ν_2 and ν_3 on S, sufficiently close to ν_1, and on opposite sites. As before, we set

$$I_1 = \langle \lambda, \nu_1 \rangle \cdot \langle \nu_2, \nu_3 \rangle\,, \qquad I_2 = \langle \lambda, \nu_2 \rangle \cdot \langle \nu_3, \nu_1 \rangle\,, \qquad I_3 = \langle \lambda, \nu_3 \rangle \cdot \langle \nu_1, \nu_2 \rangle\,.$$

In the limit situation where $\nu_1 = \nu_2 \neq \nu_3$, then μ separates simultaneously the pairs (λ, ν_1) and (ν_2, ν_3), as well as (λ, ν_2) and (ν_1, ν_3), but not (λ, ν_3); we deduce in this case that $I_1 = I_2 < I_3$. Similarly, when $\nu_1 = \nu_3 \neq \nu_2$, we get $I_1 = I_3 < I_2$.

We claim that, when $\nu_2 \neq \nu_1 \neq \nu_3$ are opportunely chosen, we get $I_1 < I_2$ and $I_1 < I_3$, against the assumption of u_λ being an ultrametric.

In order to set up the explicit computation, we can pick a good modification $\pi\colon X_\pi \to (X, x_0)$ so that $\nu_1 = \nu_E$ for some $E \in \Gamma_\pi^*$ an exceptional prime that intersects exactly two other exceptional primes F and G, at points p and q respectively. The valuations ν_2 and ν_3 are monomial valuations at p and q respectively, while the center of λ does not intersect E. Under these assumptions, all the intersections intervening in I_1, I_2, I_3 can be computed via the incarnations in the model X_π. We write $W = Z_\pi(\lambda)$, $Z_E = Z_\pi(\nu_E)$, $Z_F = Z_\pi(\nu_F)$, $Z_G = Z_\pi(\nu_G)$, and we have

$$Z_\pi(\nu_2) = (1-s)Z_E + sZ_F, \qquad Z_\pi(\nu_3) = (1-t)Z_E + tZ_G,$$

for $s, t \in [0,1]$. Notice that $\nu_2 = \nu_1$ exactly when $s = 0$, and similarly $\nu_3 = \nu_1$ when $t = 0$. A direct computation yields

$$\begin{aligned}
I_1 &= \langle W, Z_E \rangle \, \langle (1-s)Z_E + sZ_F, (1-t)Z_E + tZ_G \rangle =: a_1 st + b_1 s + c_1 t + d_1,\\
I_2 &= \langle W, (1-s)Z_E + sZ_F \rangle \, \langle Z_E, (1-t)Z_E + tZ_G \rangle =: a_2 st + b_2 s + c_2 t + d_2,\\
I_3 &= \langle W, (1-t)Z_E + tZ_G \rangle \, \langle Z_E, (1-s)Z_E + sZ_F \rangle =: a_3 st + b_3 s + c_3 t + d_3.
\end{aligned}$$

By analyzing the cases $s = 0$, $t > 0$ and $t = 0$, $s > 0$, we infer that $d_1 = d_2 = d_3 =: d$, $c_1 = c_2 < c_3$ and $b_1 = b_3 < b_2$. Assuming that $s > 0$ and $t > 0$, we have $I_1 < I_2$ if and only if $(a_2 - a_1)t + b_2 - b_1 > 0$, which happens for any $t \ll 1$, and similarly $I_1 < I_3$ if and only if $(a_3 - a_1)s + c_3 - c_1 > 0$, which happens for any $s \ll 1$.

1.3.7 Generalization on subset of branches and valuations

Instead of considering u_λ on pairs of valuations varying in the whole valuation space $\mathcal{V}_X$, we could restrict ourselves to a subset $\mathcal{F}$ of it, and see if u_λ defines a (possibly extended) distance there.

The construction of four valuations violating the ultrametric inequalities given above is quite general: the valuations λ and ν_1 are essentially arbitrary, and the valuations ν_2 and ν_3 need only be sufficiently close to ν_1 (and in a loop of $\mathcal{V}_X$). It turns out that, as long as we can avoid such a situation, we can show again that u_λ defines an ultrametric distance on $\mathcal{F}$.

Suppose first that $\lambda = \nu_L$ is a curve semi-valuation, and that $\mathcal{F}$ is a finite set of curve semi-valuations (containing ν_L). This allows to replace λ with L, $\mathcal{V}_X$ with $\mathcal{B}_X$, and $\mathcal{F}$ with a finite set F of branches. In this situation, it is possible to work directly on a model $\pi\colon X_\pi \to (X, x_0)$ which is a good resolution that is also an embedded resolution of the curves in F. The dual graph Γ_π can be turned into a tree, by contracting (non-trivial) any maximal cyclic subgraphs, called *brick*, to a point. The result of this procedure is called the *brick-vertex* tree associated to Γ_π. Then one can show that, if the convex hull of F in the brick-vertex tree has no brick vertices of valency ≥ 4, then u_L defines an ultrametric distance on F. This statement can be generalized to arbitrary subsets $\mathcal{F}$ of $\mathcal{V}_X$, by adapting the brick-vertex tree construction to arbitrary valuation spaces.

Notice that we do not get a complete characterization of the ultrametric character of u_λ on arbitrary subsets $\mathcal{F}$ of $\mathcal{V}_X$, since the answer would not depend only on the geometry of $\mathcal{V}_X$, but also on numerical properties (such as the self-intersections of exceptional primes in a good resolution of (X, x_0). For more details, we refer to [GGPR19].

Chapter 2

Valuative dynamics

Introduction

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https://bookstore.ams.org/.

Valuative spaces have found very fruitful applications in the study of algebraic and local dynamical systems. This line of research has been launched by Favre and Jonsson [FJ07], where they consider the global setting of polynomial endomorphisms $f\colon \mathbb{C}^2 \circlearrowleft$ of degree $d \geq 2$, and the local counterpart of superattracting germs $f\colon (\mathbb{C}^2, 0) \circlearrowleft$. By studying the action $f_\bullet$ induced by f on an opportune space of valuations $\mathcal{V}$ (valuations centered at the line at infinity in the global setting, and the one introduced in Chapter 1 in the local setting), and in particular the local attraction properties around some fixed points, called *eigenvaluations*, they are able to study normal forms for f up to birational conjugacy, and compute the dynamical degres, that are necessarily quadratic integers.

Since then, other functional spaces related to $\mathcal{V}_X$ have appeared. We mention the *Farey blow-up* of [HP08], which can be seen as a segment of monomial valuations in the valuation space $\mathcal{V}_X$ associated to $\mathbb{C}^2$; this space has been studied in order to compactify $\mathbb{C}^2$ and study the dynamics of Netwon maps in $\mathbb{C}^2$. We also recall the *Picard-Manin space*, introduced in [Man86], which can be seen as a cohomological space over the Zariski-Riemann space $\mathcal{ZR}(X, x_0)$, in the sense that it is obtained as a projective limit of the cohomologies of the modified spaces X_π. The action induced by birational transformations of $\mathbb{P}^2$ in this space is the fundamental tool for the study of dynamical and group-theoretic properties of the Cremoa group, initiated by [Can11]. This space, introduced from the point of view of b-divisors, is also the main tool in works such as [BFJ08a, FJ11], to study for example the existence of algebraically stable models for polynomial endomorphisms of $\mathbb{C}^2$ and the study of the sequence of degrees of the iterates of the endomorphism.

Our contribution to this subject is to adapt these result to the local setting in [GR14], and subsequently extending them to the singular setting in [GR21]. In particular, we are able to show the existence of algebraically stable models, and the existence of an integral linear recursion relation for the sequence of attraction rates, for all non-invertible germs $f\colon (X, x_0) \circlearrowleft$, with the notable exception of finite germs on cusp singularities, where counterexamples to these statements where known since [Fav10].

As for the global case, the techniques rely on proving that the action $f_\bullet$ on $\mathcal{V}_X$ enjoys some global attraction properties towards eigenvaluations. In order to achieve this goal, we need to proceed differently from the global setting: instead of looking at the action on b-divisors, we use the angular distance ρ_X to control the expanding behavior of $f_\bullet$. In particular, we show that $f_\bullet$ is non-expanding with respect to ρ_X.

This property is obtained as a consequence of the positivity properties of the intersection of valuations, the interpretation of valuations as b-divisors, and the study of pullbacks and pushforwards of b-divisors under the action of not necessarily finite germs, thus extending the finite case dealt by [Fav10].

In Section 2.1, we review some basics properties of the action $f_\bullet\colon \mathcal{V}_X \to \mathcal{V}_Y$ induced by a germ $f\colon (X, x_0) \to (Y, y_0)$, and study the action induced at the level of b-divisors. In Section 2.2, we use the results of the previous section to characterize the non-expanding and weak contraction behavior of $f_\bullet$, and apply this property to selfmaps $f\colon (X, x_0)$, in order to deduce the global attraction properties of $f_\bullet$ towards its eigenvaluations. Finally, in Section 2.3, we turn our attention to the applications to the construction of algebraically stable models and the study of the sequence of attraction rates.

2.1 Valuative analysis of maps between surface singularities

In this section, we consider a holomorphic germ $f\colon (X, x_0) \to (Y, y_0)$ between normal surface singularities. With this we mean a holomorphic map defined on a neighbohood of x_0, and such that $f(x_0) = y_0$. We will also assume f to be dominant, i.e., the image of f is not contained in any curve at y_0. We will introduce the action induced by f on valuation spaces, and discuss how this map interacts with b-divisors, skewness, log-discrepancy, angular distance.

2.1.1 Action induced on valuation spaces

Let $(R_X, \mathfrak{m}_X)$ and $(R_Y, \mathfrak{m}_Y)$ denote the completed local rings $\hat{\mathcal{O}}_{X,x_0}$ and $\hat{\mathcal{O}}_{Y,y_0}$ respectively, and let $f^*\colon R_Y \to R_X$ be the induced local ring homomorphism. In the smooth case $(X, x_0) = (Y, y_0) = (\mathbb{C}^2, 0)$, the complete local rings are $\hat{\mathcal{O}} = \mathbb{C}[\![x, y]\!]$, the map f is described by two (convergent) formal power series, and $f^*\phi = \phi \circ f$.

By duality, we get a map $f_*\colon \hat{\mathcal{V}}_X \to \hat{\mathcal{V}}_Y$ acting on centered valuations, defined by $f_*\nu := \nu \circ f^*$. This map f_* is continuous with respect to the weak topologies on $\hat{\mathcal{V}}_X$ and $\hat{\mathcal{V}}_Y$ and commutes with scalar multiplication of semivaluations.

If $\nu \in \hat{\mathcal{V}}_X^*$ is a finite semivaluation then $f_*\nu$ will again be finite, except in exactly one situation: if ν is a curve semivaluation associated to a curve germ $(C, x_0) \subset (X, x_0)$ which is contracted by f, i.e., $f(C) = y_0$, then $f_*\nu$ will not be finite.

We denote by $\mathcal{C}_f^b$ the (necessarily finite) set of branches C contracted to y_0 by f, and by $\mathcal{C}_f$ the corresponding set of contracted curve semi-valuations ν_C. The case when $\mathcal{C}_f^b = \emptyset$, or equivalently if $\mathcal{C}_f = \emptyset$, is rather special: in this case f is called *finite*.

From the discussion above, it follows that f_* induces a (continuous) map $f_\bullet \colon \mathcal{V}_X^\bullet \setminus \mathcal{C}_f \to \mathcal{V}_Y^\bullet$, which in fact extends by continuity to a map $f_\bullet \colon \mathcal{V}_X^\bullet \to \mathcal{V}_Y^\bullet$. We denote again by $f_\bullet$ the induced map $f_\bullet \colon \mathcal{V}_X^{\mathfrak{a}_X} \to \mathcal{V}_Y^{\mathfrak{a}_Y}$ on valuations normalized by some ideals $\mathfrak{a}_X$ and $\mathfrak{a}_Y$. In this case, we will write $f_*\nu = c_{\mathfrak{a}_X}^{\mathfrak{a}_Y}(f,\nu) f_\bullet(\nu)$, where

$$c_{\mathfrak{a}_X}^{\mathfrak{a}_Y}(f,\nu) := \frac{f_*\nu(\mathfrak{a}_Y)}{\nu(\mathfrak{a}_X)} \tag{2.1}$$

is called the *attraction rate* of f along ν. Notice that if $f^*\mathfrak{a}_Y \subseteq \mathfrak{a}_X$, then clearly $c_{\mathfrak{a}_X}^{\mathfrak{a}_Y}(f,\nu) \geq 1$ for all $\nu \in \mathcal{V}_X$. This happens for example when $\mathfrak{a}_X = \mathfrak{m}_X$ is the maximal ideal.

Unless otherwise specified, any singularity (X, x_0) is implicitly endowed with an ideal $\mathfrak{a}_X$ used for the normalization of valuations; the ideals will be omitted in the notations. For example, we will write $f_\bullet \colon \mathcal{V}_X \to \mathcal{V}_Y$ and $c(f,\nu)$ instead of $f_\bullet \colon \mathcal{V}_X^{\mathfrak{a}_X} \to \mathcal{V}_Y^{\mathfrak{a}_Y}$ and $c_{\mathfrak{a}_X}^{\mathfrak{a}_Y}(f,\nu)$.

In the next proposition, we collect some easy facts about the map $f_\bullet$ and the attraction rates $c(f,\nu)$. Proofs can be found in [FJ07, Fav10, GR21]

Proposition 2.1.1. *Let $f \colon (X,x_0) \to (Y,y_0)$ be any dominant holomorphic germ between normal surface singularities.*

(a) If $\nu \in \hat{\mathcal{V}}_X^* \setminus \mathcal{C}_f$ is not a contracted curve semivaluation, then $f_\bullet\nu$ is of the same type (divisorial, irrational, infinitely singular, curve) as ν. If ν is a contracted curve semivaluation, then $f_\bullet\nu$ is divisorial.

(b) Every $\nu \in \mathcal{V}_Y$ has at most $\operatorname{topdeg}(f)$ *preimages under $f_\bullet$, where*

$$\operatorname{topdeg}(f) := [\operatorname{Frac}(R_X) : f^* \operatorname{Frac}(R_Y)]$$

is the degree of the field extension $\operatorname{Frac}(R_X)/f^*\operatorname{Frac}(R_Y)$*, called the* topological degree *of f.*

(c) If f is finite, then $f_\bullet$ is surjective.

(d) Attraction rates are multiplicative, in the sense that if $g \colon (Y,y_0) \to (Z,z_0)$ is another dominant holomorphic germ between normal surface singularities, then

$$c(g \circ f, \nu) = c(g, f_\bullet\nu) \cdot c(f,\nu).$$

These properties are enough to show the special role played by the angular distance.

Proposition 2.1.2. *Let $f \colon (X,x_0) \to (Y,y_0)$ be any dominant holomorphic germ between normal surface singularities. For any $\nu_1, \nu_2 \in \mathcal{V}_X$ of finite skewness one has $\rho_Y(f_\bullet\nu_1, f_\bullet\nu_2) \leq \rho_X(\nu_1,\nu_2)$.*

Proof. Recalling Equation (1.8) and Proposition 1.2.18, we have

$$\rho_Y(f_\bullet\nu_1, f_\bullet\nu_2) = \log\left(\sup_{\mathfrak{a}} \frac{f_\bullet\nu_1(\mathfrak{a})}{f_\bullet\nu_2(\mathfrak{a})} \times \sup_{\mathfrak{a}} \frac{f_\bullet\nu_2(\mathfrak{a})}{f_\bullet\nu_1(\mathfrak{a})}\right) = \log\left(\sup_{\mathfrak{a}} \frac{\nu_1(f^*\mathfrak{a})}{\nu_2(f^*\mathfrak{a})} \times \sup_{\mathfrak{a}} \frac{\nu_2(f^*\mathfrak{a})}{\nu_1(f^*\mathfrak{a})}\right),$$

the suprema taken over all $\mathfrak{m}_Y$-primary ideals $\mathfrak{a} \subset R_Y$. The ideals $f^*\mathfrak{a}$ will not be $\mathfrak{m}_X$-primary if f is not finite, but in any case for large enough $n \in \mathbb{N}$ the ideals $\mathfrak{a}_n := f^*\mathfrak{a} + \mathfrak{m}_X^n$ are $\mathfrak{m}_X$-primary and satisfy $\nu_i(\mathfrak{a}_n) = \nu_i(\mathfrak{a})$ for $i = 1, 2$, and thus we can conclude that

$$\rho_Y(f_\bullet \nu_1, f_\bullet \nu_2) \leq \log \left(\sup_{\mathfrak{b}} \frac{\nu_1(\mathfrak{b})}{\nu_2(\mathfrak{b})} \times \sup_{\mathfrak{b}} \frac{\nu_2(\mathfrak{b})}{\nu_1(\mathfrak{b})} \right) = \rho_X(\nu_1, \nu_2),$$

the suprema taken over all $\mathfrak{m}_X$-primary ideals $\mathfrak{b} \subset R_X$. This completes the proof. □

For our goals, we will need to study in more details the contraction properties of $f_\bullet$ with respect to angular distances. This will require a careful study of the action induced by f at the level of b-divisors, that will be achieved in the next sections. For now, we end this section by recalling a criterion that allows to recover all the indeterminacy points the lift $\tilde{f}$ of a dominant germ $f\colon (X, x_0) \to (Y, y_0)$ with respect to given modifications $\pi\colon X_\pi \to (X, x_0)$ and $\varpi\colon Y_\varpi \to (Y, y_0)$.

The lift $\tilde{f} = \varpi^{-1} \circ f \circ \pi\colon X_\pi \dashrightarrow Y_\varpi$ is always meromorphic, being a composition of meromorphic maps, but it might not be regular, i.e., have indeterminacy points, whose set we denote by $\mathrm{Ind}\left(\tilde{f}\right)$. If $p \in \pi^{-1}(x_0)$ is a closed point, we denote by $U_\pi(p) \subset \mathcal{V}_X$ the subset of semivaluations ν whose center in X_π is p, i.e., $U_\pi(p) = \mathrm{cen}_\pi^{-1}(p)$. This is a weak open set for all closed points p. Similarly, if $q \in \varpi^{-1}(y_0)$ is a closed point, we set $U_\varpi(q) = \mathrm{cen}_\varpi^{-1}(q)$. Certainly if $\tilde{f}$ is holomorphic at a closed point $p \in \pi^{-1}(x_0)$, and if $q = \tilde{f}(p)$, then $f_\bullet(U_\pi(p)) \subseteq U_\varpi(q)$; this is simply a matter of unraveling definitions. As a consequence of the valuative criterion of properness (we use here the normality of the spaces X_π and Y_ϖ), the converse is also true (see [FJ07, Prop. 3.2] for a proof).

Proposition 2.1.3. *The lift $\tilde{f}$ is holomorphic at a closed point $p \in \pi^{-1}(x_0)$ and has $\tilde{f}(p) = q$ if and only if $f_\bullet(U_\pi(p)) \subseteq U_\varpi(q)$.*

2.1.2 Action on divisorial and curve valuations

From Proposition 2.1.1.(a) we deduce in particular that divisorial valuations are sent to divisorial valuations, while curve semi-valuations are sent to curve semi-vauations, unless they are contracted by f, in which case they are sent to divisorial valuations.

This subsection is devoted to giving some geometrical insights to this action, as well as to the attraction rates. For convenience of language, we introduce the following concept.

Definition 2.1.4. Let $f\colon (X, x_0) \to (Y, y_0)$ be a dominant germ, and let $V \subset \mathcal{V}_X$ be a finite set of divisorial or curve (semi-)valuations. We say that two good resolutions $\pi\colon X_\pi \to (X, x_0)$ and $\varpi\colon Y_\varpi \to (Y, y_0)$ give a *resolution of f with respect to V* if the following conditions are satisfied :

- For any divisorial valuation $\nu_E \in V$, we have that ν_E is realized by π, and $f_\bullet \nu_E$ is realized by ϖ.
- For any curve valuation $\nu_C \in V$ non contracted by f, we have that π is an embedded resolution of $(C, x_0) \subset (X, x_0)$ and ϖ is an embedded resolution of $(f(C), y_0) \subset (Y, y_0)$.

- For any contracted curve valuation $\nu_C \in V$, we have that π is an embedded resolution of $(C, x_0) \subset (X, x_0)$ and $f_\bullet \nu_C$ is realized by ϖ.
- The lift $\tilde{f} = \varpi^{-1} \circ f \circ \pi \colon X_\pi \to Y_\varpi$ is regular.

We summarize the data described above with the triplet $(\pi, \varpi, \widetilde{f} = \varpi^{-1} \circ f \circ \pi)$.

Such good resolutions always exist, and can be taken dominating any two given modifications. In fact, since V is finite, we can easily find good resolutions π and ϖ satisfying the first three conditions. By taking an opportune good resolution π' dominating π, we can assure that the lift $\tilde{f}$ is also regular.

We now come back to the geometric interpretation of the action of $f_\bullet$ on divisorial and curve valuations. Let $\pi \colon X_\pi \to (X, x_0)$ and $\varpi \colon Y_\varpi \to (Y, y_0)$ be good resolutions of f with respect to $\{\nu\}$, where ν is either divisorial or a curve valuation.

Denote by D the prime associated to ν (i.e., the exceptional prime E if $\nu = \nu_E$ is divisorial, or the strict transform C_π of C if $\nu = \nu_C$ is a curve semi-valuation), and set $D' = \widetilde{f}(D)$.

We attach to this situation two numbers:

- $k_D = k_{D \to D'} = k_{D \xrightarrow{f} D'} \geq 1$ the coefficient of D in the divisor $\tilde{f}^* D'$; in terms of valuations, k_D can be computed also as the ramification index $e(\mathrm{div}_D / f_* \mathrm{div}_D)$, i.e., the index of the value group of $f_* \mathrm{div}_D$ within the value group of $\mathrm{div}_{D'}$.
- If D is compact (i.e., ν is divisorial), we denote by $e_D = e_{D \to D'} = e_{D \xrightarrow{f} D'}$ the topological degree of the map $\widetilde{f}|_D \colon D \to D'$. If D is not compact, it only defines a germ at x_0, and e_D denotes the local degree of $\widetilde{f}|_D$, i.e., the integer e such that in suitable local coordinates at x_0 the map $\widetilde{f}|_D$ acts as $z \mapsto z^e$.

Notice that $k_{D \to D'}$ and $e_{D \to D'}$ do not depend on the good resolutions π and ϖ chosen, but only on the associated divisorial or curve semivaluations.

By unravelling the definitions of div_E and int_C, we get the following interpretation of the action of f_*.

Proposition 2.1.5 ([FJ07, Prop. 2.5-2.6])**.** *Let $f \colon (X, x_0) \to (Y, y_0)$ be a dominant germ between normal surface singularities. Let $\nu \in \mathcal{V}_X$ be either a divisorial or a curve semi-valuation, and let $(\pi, \varpi, \widetilde{f})$ be a resolution of f with respect to $\{\nu\}$.*

- *If $\nu = \mathrm{div}_E$ is divisorial, set $E' = \widetilde{f}(E)$; then*
$$f_* \mathrm{div}_E = k_{E \to E'} \mathrm{div}_{E'}.$$
- *If $\nu = \mathrm{int}_C$ is a curve semi-valuation not contracted by f, set $C' = f(C)$; then*
$$f_* \mathrm{int}_C = e_{C \to C'} \mathrm{int}_{C'}.$$

Remark 2.1.6. This geometrical interpretation of the image of divisorial valuations allows to describe tangent maps on valuation spaces. In fact, in the situation of Proposition 2.1.5, we may define the map $\mathrm{d}f_\bullet \colon \mathcal{T}_{\nu_E}\mathcal{V}_X \to \mathcal{T}_{\nu_{E'}}\mathcal{V}_Y$ by sending the tangent vector $\overrightarrow{v_p} \in \mathcal{T}_{\nu_E}\mathcal{V}_X$ corresponding to a point $p \in E$, to the tangent vector $\mathrm{d}f_\bullet \overrightarrow{v_p} \in \mathcal{T}_{\nu_{E'}}\mathcal{V}_Y$ corresponding to the point $q = \widetilde{f}|_E(p)$. Notice that if π and ϖ give a resolution of f with respect to $\{\nu_E\}$, then by Proposition 2.1.3 we have $f_\bullet(U_\pi(p)) \subseteq U_{\pi'}(q)$. This corresponds to the description of tangent maps on real trees described in [FJ07].

Remark 2.1.7. By plugging in the normalizations on $\mathcal{V}_X$ and $\mathcal{V}_Y$ associated to the ideals $\mathfrak{a}_X$ and $\mathfrak{a}_Y$ respectively, we get the following formulas for attraction rates at a divisorial valuation ν_E and a non-contracted curve semivaluation ν_C in $\mathcal{V}_X^{\mathfrak{a}_X}$:

$$c(f,\nu_E) = \frac{b_{E'}}{b_E} k_E, \qquad c(f,\nu_C) = \frac{m(C')}{m(C)} e_C,$$

where $b_E = b_E^{\mathfrak{a}_X} = \mathrm{div}_E(\mathfrak{a})$ and $m(C) = m^{\mathfrak{a}_X}(C) = \mathrm{int}_C(\mathfrak{a}_X)$ are, respectively, the generic multiplicity and multiplicity of E and C with respect to $\mathfrak{a}_X$; analogously for $b_{E'}$ and $m(C')$ with respect to $\mathfrak{a}_Y$.

If $\nu = \mathrm{int}_C$ is a curve semivaluation contracted by f, set G to be the exceptional prime containing $\widetilde{f}(C)$. Recall that a curve semivaluation ν_C induces a Krull valuation of rank 2. Its image by the map f_* induced on the Riemann-Zariski spaces of X and Y respectively is also a Krull valuation of rank 2, corresponding to the exceptional curve G as a germ at the point $q := \widetilde{f}(p)$, where $(\pi, \varpi, \widetilde{f})$ is a resolution of f with respect to $\{\nu_C\}$, and $p = C_\pi \cap \pi^{-1}(x_0)$.

The action $f_\bullet$ at the level of valuation spaces forgets the information given by q, and sends ν_C to $f_\bullet \nu_C = \nu_G$.

In this situation, computing $c(f,\nu)$ does not give any geometrical information on the action of $\widetilde{f}$ near C_π, since $c(f,\nu_C) = \infty$. What we can do instead, is to measure the speed at which $c(f,\nu) \to +\infty$ when $\nu \to \nu_C$.

Definition 2.1.8. Let $f\colon (X,x_0) \to (Y,y_0)$ be a dominant germ, and $\nu_C \in \mathcal{V}_X$ be a normalized curve semivaluation. We set

$$c_\alpha(f,\nu_C) = \lim_{\nu \to \nu_C} \frac{c(f,\nu)}{\alpha(\nu)}, \qquad c_A(f,\nu_C) = \lim_{\nu \to \nu_C} \frac{c(f,\nu)}{A(\nu)}.$$

Recall that for curve semivaluations, $\alpha(\nu_C) = A(\nu_C) = +\infty$. So $c_\alpha(f,\nu_C) = c_A(f,\nu_C) = 0$ unless ν_C is a contracted curve valuation. In the latter case, $c(f,\nu_C)$ is affine with respect to both the α and A parameterizations. Then the derivative of $c(f,\nu)$ is just the coefficient of the linear part.

By working on adapted coordinates on a resolution $(\pi, \varpi, \widetilde{f})$ of f with respect to $\{\nu_C\}$, one can compute explicitly these limits.

Proposition 2.1.9 ([GR21, Proposition 4.10])**.** *Let $f\colon (X,x_0) \to (Y,y_0)$ be a non-finite germ between normal surface singularities. Let $\nu_C \in \mathcal{C}_f$ be a contracted curve valuation, and set $\nu_G = f_\bullet \nu_C \in \mathcal{V}_Y$. Then*

$$c_\alpha(f,\nu_C) = m(C) b_G k_{C\to G}, \qquad c_A(f,\nu_C) = b_G k_{C \to G}.$$

2.1.3 Action on b-divisors

Let $f\colon (X,x_0) \to (Y,y_0)$ be a dominant germ between two surface singularities. This datum induces pushforward and pullback actions on the set of b-divisors (associated to valuations): let us describe these actions and their properties, starting with the pushforward.

Let us first fix $\pi\colon X_\pi \to (X,x_0)$ and $\varpi\colon Y_\varpi \to (Y,y_0)$ modifications of (X,x_0) and (Y,y_0) respectively. The germ f induces a pushforward operator $f_*\colon \mathcal{E}(\pi) \to \mathcal{E}(\varpi)$, defined as $f_*D = \widetilde{f}_*\eta^*D$, where $\pi'\colon X_{\pi'} \to (X,x_0)$ is any modification dominating π so that the lift $\widetilde{f}\colon X_{\pi'} \to Y_\varpi$ of f is regular, and $\eta = \eta_\pi^{\pi'}$.

If we start with a b-divisor $D = (D_\pi)_\pi \in b\text{-}\mathcal{E}(X)$, we can define its pushforward $f_*D = (D'_\varpi)_\varpi \in b\text{-}\mathcal{E}(Y)$, by taking, for any model $\varpi\colon Y_\varpi \to (Y,y_0)$, a model $\pi\colon X_\pi \to (X,x_0)$ so that the lift $\widetilde{f}\colon X_\pi \to Y_\varpi$ is regular. We can then define $D'_\varpi := \widetilde{f}_*D_\pi$. One can check that D'_ϖ does not depend on the choice of π, and the collection $(D'_\varpi)_\varpi$ satisfies the compatibility conditions that make it a b-divisor.

The case of pullbacks is trickier. First, assume that f is finite, so that we have a pullback operator $f^*\colon \mathcal{E}(\varpi) \to \mathcal{E}(\pi)$, defined as $f^*D' = \eta_*\widetilde{f}^*D$. Let now $D' = (D'_\varpi)_\varpi \in b\text{-}\mathcal{E}(Y)$ be a b-divisor. Assume it is Cartier, determined by the model ϖ_0. Then we pick $\pi\colon X_\pi \to (X,x_0)$ so that the lift $\widetilde{f}\colon X_\pi \to Y_\varpi$ is regular, and set f^*D' to be the Cartier b-divisor determined by $\widetilde{f}^*D'_{\varpi_0}$. In order to extend this definition to Weil b-divisor, one needs to work with modifications π for which the lift $\widetilde{f}$ defined above is not necessarily regular, but it does not contract any curve. In this case, f^* extends to a continuous pullback operator $f^*\colon b\text{-}\mathcal{E}(Y) \to b\text{-}\mathcal{E}(X)$; we refer to [BFJ08a] for further details.

If we restrict our attention to b-divisors associated to valuations, when $f\colon (X,x_0) \to (Y,y_0)$, for any normalized valuations $\nu \in \mathcal{V}_X$ and $\nu' \in \mathcal{V}_Y$, one gets:

$$f_*Z(\nu) = c(f,\nu)Z(f_\bullet\nu), \qquad f^*Z(\nu') = \sum_{f_\bullet\nu=\nu'} \frac{m(f,\nu)}{c(f,\nu)} Z(\nu), \tag{2.2}$$

where $m(f,\nu)$ are positive integers and their sum give the topological degree of f (see e.g. [Fav10, Lemma 1.10]). When $\nu = \nu_E$ is a divisorial valuation, these numbers are explicitly given by

$$m(f,\nu_E) = k_E e_E.$$

When the map f is not finite, the pullback operator f^* is not anymore defined on exceptional divisor, since the pullback of any exceptional divisor will have in general some non-trivial coefficient associated to the contracted curves.

One way to deal with this is to add contracted curves in the spaces of divisors we consider, and set

$$\mathcal{D}(\pi)_\mathbb{R} = \mathcal{E}(\pi)_\mathbb{R} \oplus \bigoplus_{C\in\mathcal{B}_X} \mathbb{R}C_\pi,$$

where C varies among the branches at (X,x_0), and C_π is the strict transform of C in X_π (one can define $\mathbb{Q}$-divisors and $\mathbb{Z}$-divisors analogously). Notice that the coefficients of a divisor are all vanishing, but for finitely many of them. When a map f is fixed, we can simply take C varying on the set $\mathcal{C}_f^b$ of branches contracted by f.

One can use these spaces as basis for the construction of Weil and Cartier b-divisors, whose spaces will be denoted by $b\text{-}\mathcal{D}(X)$ and $c\text{-}\mathcal{D}(X)$ respectively. We have a natural projection $\mathrm{pr}_{\mathcal{E}}\colon b\text{-}\mathcal{D}(X) \to b\text{-}\mathcal{E}(X)$, which forgets the non-exceptional part of the b-divisor, by sending all coefficients associated to branches to zero.

Notice that, within this setting, to a branch $C \in \mathcal{B}_X$, we can associate the Cartier b-divisor $\check{C} := (-\pi^{-1}(C))_\pi$, where $\pi^{-1}(C)$ is the total transform of C. In this case, $\mathrm{pr}_{\mathcal{E}}(\check{C}) = Z(\mathrm{int}_C)$, which suggests the notation $\check{C} =: \hat{Z}(\mathrm{int}_C)$. We extend the latter notation to multiples of int_C, and in particular to $\hat{Z}(\nu_C) = \frac{\hat{Z}(\mathrm{int}_C)}{m(C)}$.

In this setting, we can extend Equation (2.2) to the non-finite case.

Theorem 2.1.10 ([GR21, Theorem 4.18]). *Let $f\colon (X, x_0) \to (Y, y_0)$ be a dominant germ, and $\nu \in \mathcal{V}_X$ and $\nu' \in \mathcal{V}_Y$ be normalized valuations. Then we have*

$$f_* Z(\nu) = c(f,\nu) Z(f_\bullet \nu) - \sum_{\nu_C \in \mathcal{C}_f} \langle \nu_C, \nu \rangle \, c_\alpha(f, \nu_C) Z(f_\bullet \nu_C), \tag{2.3}$$

$$f^* Z(\nu') = \sum_{f_\bullet \nu = \nu'} \frac{m(f,\nu)}{c(f,\nu)} Z(\nu) + \sum_{\nu_C \in \mathcal{C}_f} \langle f_\bullet \nu_C, \nu' \rangle \, c_\alpha(f, \nu_C) \hat{Z}(\nu_C), \tag{2.4}$$

where $m(f,\nu) \in \mathbb{N}^*$ and their sum is bounded by the topological degree of f.

These formulas are obtaiend by first working out the expressions of $f_* Z(\mathrm{div}_E)$ and $f^* Z(\mathrm{div}_{E'})$, with the new contributions given by the non-exceptional b-divisors $\hat{Z}(\mathrm{int}_C)$ associated to contracted curve valuations in the latter case, and to the b-divisor associated to their divisorial image in the former case. These are obtained by working on resolutions of f with respect to the valuations involved in the formulas, and by using the duality provided by the intersection form. One can then find the general formulas by normalization and by continuity arguments.

2.1.4 Contraction properties with respect to angular distances

We can finally apply pushforward and pullback formulas, together with the positivity properties of Theorem 1.2.15, to give a characterization of pair of valuations whose angular distance is preserved by $f_\bullet$.

Theorem 2.1.11. *Let $f\colon (X, x_0) \to (Y, y_0)$ be a dominant non-invertible germ between two normal surface singularities. For any $\nu \neq \mu \in \mathcal{V}_X$ of finite skewness, we have $\rho_Y(f_\bullet \nu, f_\bullet \mu) \leq \rho_X(\nu, \mu)$. The equality holds if and only if f is finite, and moreover*

- *ν disconnects μ and any preimage of $f_\bullet \nu$, and symmetrically*
- *μ disconnects ν and any preimage of $f_\bullet \mu$.*

Proof. The proof of the 1-Lipschitzianity of $f_\bullet$ boils down to showing $\beta(f_\bullet \nu | f_\bullet \mu) \leq \frac{c(f,\mu)}{c(f,\nu)} \beta(\nu|\mu)$ for any μ, ν of finite skewness, or equivalently to proving

$$c(f,\nu) \langle f_\bullet \nu, f_\bullet \nu \rangle \langle \nu, \mu \rangle \leq c(f,\mu) \langle f_\bullet \nu, f_\bullet \mu \rangle \langle \nu, \nu \rangle . \tag{2.5}$$

By Equation (2.3), we get

$$c(f,\nu)Z(f_\bullet\nu) = f_*Z(\nu) + \sum_{\nu_C \in \mathcal{C}_f} \langle \nu_C, \nu \rangle \, c_\alpha(f,\nu_C) Z(f_\bullet \nu_C),$$
$$c(f,\mu)Z(f_\bullet\mu) = f_*Z(\mu) + \sum_{\nu_C \in \mathcal{C}_f} \langle \nu_C, \mu \rangle \, c_\alpha(f,\nu_C) Z(f_\bullet \nu_C).$$

Notice that the coefficients in front of $Z(f_\bullet \nu_C)$ in the expressions above are positive. By plugging these expressions in Equation (2.5), we get to the equivalent system of inequalities:

$$\big(- Z(f_\bullet\nu) \cdot f_*Z(\nu)\big) \langle \mu, \nu \rangle \leq \big(- Z(f_\bullet\nu) \cdot f_*Z(\mu)\big) \langle \nu, \nu \rangle \,, \tag{2.6}$$
$$\langle \nu_C, \nu \rangle \langle \mu, \nu \rangle \leq \langle \nu_C, \mu \rangle \langle \nu, \nu \rangle \,, \tag{2.7}$$

where the equality holds in Equation (2.5) if and only if the equality holds in all such inequalities. Recall that by Equation (2.4), we have

$$f^* Z(f_\bullet \nu) = \sum_{i=1}^{r} a_i Z(\nu_i) + \sum_{\nu_C \in \mathcal{C}_f} a_C Z(f_\bullet \nu_C)$$

for suitable $a_i, a_C > 0$, where $\{\nu_1, \ldots, \nu_r\}$ are the preimages of $f_\bullet\nu$. By using the projection formula and plugging in this pullback formula, Equation (2.6) is equivalent to the system of inequalities

$$\langle \nu_i, \nu \rangle \langle \mu, \nu \rangle \leq \langle \nu_i, \mu \rangle \langle \nu, \nu \rangle \,, \tag{2.8}$$
$$\langle f_\bullet\nu_C, \nu \rangle \langle \mu, \nu \rangle \leq \langle f_\bullet\nu_C, \mu \rangle \langle \nu, \nu \rangle \,. \tag{2.9}$$

All these inequalities (2.7), (2.8) and (2.9) hold by Theorem 1.2.15, and we deduce by symmetry that $\rho(f_\bullet\nu, f_\bullet\mu) \leq \rho(\nu,\mu)$.

Moreover, if f is finite, the set $\mathcal{C}_f$ is empty, and the inequalities (2.7) and (2.9) do not occur. Again by Theorem 1.2.15 we deduce the characterization of equality in the statement.

Assume now that f is not finite. Again by Theorem 1.2.15, the equality in Equation (2.7) holds if and only if ν disconnects ν_C and μ. We apply the same argument with the role of ν and μ interchanged, and for the angular distance to be preserved, we need that ν disconnects ν_C and μ, and μ disconnects ν_C and ν. It is easy to check that both situations cannot happen at the same time when $\nu \neq \mu$, and the angular distance strictly decreases in this case. □

2.1.5 Jacobian formula

We present here another fundamental tool to control the action of $f_\bullet$, not by angular distance, but by log-discrepancy.

Proposition 2.1.12 (The Jacobian Formula). *Let (X, x_0) and (Y, y_0) be two normal surface singularities, and let $f \colon (X, x_0) \to (Y, y_0)$ be a dominant holomorphic map. Then for any $\nu \in \hat{\mathcal{V}}^*_X$, we have*

$$A_Y(f_*\nu) = A_X(\nu) + \nu(R_f), \tag{2.10}$$

where R_f is the Weil divisor on X defined by

$$R_f = \mathrm{Div}(f^*\omega) - f^* \mathrm{Div}(\omega), \tag{2.11}$$

with ω any 2-form on (Y, y_0) (in the sense of Kähler differntials).

A proof of this result can be found in [GR21, Section 4.5], see also [Fav10, Proposition 1.9] for the finite case.

The divisor R_f is often referred to as the *Jacobian divisor* (and Equation (2.10) as the *Jacobian formula*), due to the fact that when (X, x_0) and (Y, y_0) are regular, or when f is finite, the divisor R_f is none other than the divisor defined by the Jacobian determinant of f (an easy consequence of the pullback formula of 2-forms). In particular, in these cases R_f is effective, and $\nu(R_f) \geq 0$ for any $\nu \in \hat{\mathcal{V}}^*_X$. By Equation (2.11), we can also interpret R_f as the canonical divisor relative to the map f. Notice that, in presence of contracted curves, R_f need not be effective. As an example, let (X, x_0) be any non log canonical singularity. Let $\pi\colon X_\pi \to (X, x_0)$ be any good desingularization of (X, x_0), and let E be an exceptional prime for π so that $A(E) < 0$. By continuity of log discrepancy on dual graphs, for any $p \in E$ there exists a valuation $\nu \in \mathcal{V}_{X_\pi}$ centered at p such that $A(\pi_*\nu) < 0$. The Jacobian formula Equation (2.10) applied to $\pi\colon (X_\pi, p) \to (X, x_0)$ at ν gives

$$\nu(R_\pi) = A(\pi_*\nu) - A(\nu) < 0,$$

where the last inequality follows from $A(\nu) > 0$ since X_{π_0} is smooth. In particular, R_π is not effective in this case.

Remark 2.1.13. An alternative proof of Proposition 2.1.12 with respect to the ones given in op.cit. can be obtained starting from the definition of log-discrepancy $A_Y(f_\star\nu) = K_Y^{\log} \cdot Z(f_*\nu)$, applying Equation (2.3), and the projection formula. The computation boils down to computing the pullback $f^*K_Y^{\log}$ as $K_X^{\log}$ plus a b-divisor of the form $\sum_C a_C \check{C}$, where C varies among a finite set of branches at (X, x_0), and the coefficients a_C are not necessarily positive.

Notice that the Jacobian formula uses in an essential way the fact that we are working in characteristic zero. We can consider normalized valuations, and in this case we get

$$c(f, \nu) A_Y(f_\bullet \nu) = A_X(\nu) + \nu(R_f). \tag{2.12}$$

where $c(f, \nu)$ is the asymptotic rate computed with respect to the given normalizations in $\mathcal{V}_X$ and $\mathcal{V}_Y$.

2.1.6 Critical skeleton

In the previous sections we have seen several functionals $\hat{e}\colon \hat{\mathcal{V}}^*_X \to \mathbb{R} \cup \{+\infty\}$ on the set of finite semivaluation $\hat{\mathcal{V}}^*_X$, or their restriction e on normalized semivaluations $\mathcal{V}_X$: the evaluation $\nu \mapsto \nu(\phi)$ on an element $\phi \in R_X$, or $\nu \mapsto \nu(\mathfrak{a})$ on an ideal $\mathfrak{a}$; the asymptotic rate $\nu \mapsto c(f, \nu)$; the evaluation on the Jacobian divisor $\nu \mapsto \nu(R_f)$; the log-discrepancy

$\nu \mapsto A_X(\nu)$. These functionals are all 1-homogeneous, meaning that $\hat{e}(\lambda\nu) = \lambda\hat{e}(\nu)$ for any $\lambda > 0$ and $\nu \in \hat{\mathcal{V}}_X^*$, and in fact all of the form $\hat{e} = \hat{e}_Z \colon \nu \mapsto Z(\nu) \mapsto Z$ for a suitable (possibly non-exceptional) b-divisor $Z \in b\text{-}\mathcal{D}(X)$.

Under suitable conditions on Z, the functional e_Z is locally constant on $\mathcal{V}_X \setminus \mathcal{S}(Z)$, where $\mathcal{S}(Z)$ is a finite subgraph $\mathcal{V}_X$, which depends on Z. By definition, $\mathcal{S}(Z)$ will always contain the skeleton $\mathcal{S}_X$ of (X, x_0). If $V \subset \mathcal{V}_X$ is a finite set of semivaluations, we will denote by

$$\mathcal{S}_X(V) := \mathcal{S}_X \cup \bigcup_{\nu \in V} [r_X\nu, \nu]$$

the *skeleton generated by* V. Associated to thee skeleta, there is a natural retraction map $r_X(V) \colon \mathcal{V}_X \to \mathcal{S}_X(V)$.

In order to define the analogous property for non-normalized valuations, we need to introduce a few notations. Let $\pi \colon X_\pi \to (X, x_0)$ be a good resolution of (X, x_0).

We extend the maps $\mathrm{ev}_\pi \colon \mathcal{V}_X \to \Gamma_\pi$ and $\mathrm{emb}_\pi \colon \Gamma_\pi \to \mathcal{V}_X$ introduced in Section 1.1.9 to maps $\widehat{\mathrm{ev}}_\pi \colon \hat{\mathcal{V}}_X^* \to \widehat{\Gamma}_\pi$ and $\widehat{\mathrm{emb}}_\pi \colon \widehat{\Gamma}_\pi \to \hat{\mathcal{V}}_X^*$, where $\widehat{\Gamma}_\pi$ is the cone over Γ_π in $\hat{\mathcal{V}}_X^*$. The evaluation map is simply given by $\widehat{\mathrm{ev}}_\pi(\nu) = Z_\pi(\nu)$. The embedding map sends $\check{E}$ to div_E for any $E \in \Gamma_\pi^*$, and $r\check{E} + s\check{F}$ to the monomial valuation $\nu_{r,s}$ at the point $E \cap F$, for any $r, s > 0$. Similarly, we set $\hat{r}_\pi : \widehat{\mathrm{emb}}_\pi \circ \widehat{\mathrm{ev}}_\pi$, which gives a retraction of $\hat{\mathcal{V}}_X^*$ onto cone over $\mathcal{S}_\pi$, that we denote by $\widehat{\mathcal{S}}_\pi$.

These notions extends naturally to retractions $\hat{r}_X(V) \colon \hat{\mathcal{V}}_X^* \to \widehat{\mathcal{S}}_X(V)$ to the cone over the skeleton $\mathcal{S}_X(V)$ generated by a finite set V of semi-valuations.

Then we say that a 1-homogeneous functional $\mathrm{ev} \colon \hat{\mathcal{V}}_X^* \to \mathbb{R} \cup \{+\infty\}$ is *determined on* $\widehat{\mathcal{S}}_X(V)$ if we have that $\hat{e}(\nu) = \hat{e}(\hat{r}_X(V)(\nu))$ for any $\nu \in \hat{\mathcal{V}}_X^*$.

Finally we have all we need to describe the properties of $\hat{e}_Z$ and e_Z. We start with $Z = Z(\nu)$, with $\nu \in \hat{\mathcal{V}}_X^*$. With abuse of notation, we denote by ν also its representative in any normalized space $\mathcal{V}_X$. As a direct consequence of Proposition 1.2.10, the functional $\hat{e}_{Z(\nu)}$ is determined on $\widehat{\mathcal{S}}_X(\nu)$.

Let now $\mathfrak{a}$ be an ideal, that we consider at first $\mathfrak{m}_X$-primary. We denote by $\mathcal{R}(\mathfrak{a})$ set of *Rees valuations* associated to $\mathfrak{a}$, i.e., the finite set of divisorial valuations associated to the exceptional primes appearing in the normalized blow-up of $\mathfrak{a}$. Then $\hat{e}_{Z(\mathfrak{a})}$ is determined on $\widehat{\mathcal{S}}_X(\mathcal{R}(\mathfrak{a}))$.

The case of $\mathfrak{a}$ not $\mathfrak{m}_X$-primary, is similar: in this case we need to add to $\mathcal{R}(\mathfrak{a})$ all curve semi-valuations associated to irreducible curves in the support of $\mathfrak{a}$. This argument applies also to the functional $\nu \mapsto \nu(R_f)$, which is determined on $\widehat{\mathcal{S}}_X(V)$ with V the set of curve semi-valuations associated to curves lying in the support of R_f.

Since normalization by an ideal $\mathfrak{a}$ corresponds to dividing a valuation ν by the value $\nu(\mathfrak{a}) = Z(\nu) \cdot Z(\mathfrak{a})$, from a functional $\hat{e} \colon \hat{\mathcal{V}}_X^* \to \mathbb{R} \cup \{+\infty\}$ determined on a cone $\widehat{\mathcal{S}} = \widehat{\mathcal{S}}_X(V)$, one can deduce that the normalization e on $\mathcal{V}_X^{\mathfrak{a}}$ is locally constant outside the subgraph $\mathcal{S}_X(V \cup \mathcal{R}(\mathfrak{a}))$.

We apply this remark to the evaluation of the attraction rate $c(f, \nu)$ of f along ν. By interpreting the definition Equation (2.1) in terms of intersection of b-divisors, we get

$$c_{\mathfrak{a}_X}^{\mathfrak{a}_Y}(f, \nu) = \frac{Z(\nu) \cdot \mathrm{pr}_{\mathcal{E}}\, f^* Z(\mathfrak{a}_Y)}{Z(\nu) \cdot Z(\mathfrak{a}_X)}, \tag{2.13}$$

where $\mathrm{pr}_{\mathcal{E}} \colon b\text{-}\mathcal{D}(X) \to b\text{-}\mathcal{E}(X)$ is the natural projection to the exceptional part of a b-divisor. We deduce that $c_{\mathfrak{a}_X}^{\mathfrak{a}_Y}(f,\cdot)$ is locally constant outside $\mathcal{S}_X(V)$ with $V = \mathcal{R}(\mathfrak{a}_X) \cup f_\bullet^{-1}\mathcal{R}(\mathfrak{a}_Y)$.

When normalizations are clear from context, we denote the latter skeleton as $\mathcal{S}_f$, and refer to it as the *critical skeleton*. We refer to [GR21, Section 4.6] for further details.

Remark 2.1.14. The log-discrepancy case is rather special, since it is associated to the b-divisor $K_X^{\log}$, that is not Cartier (not even as a non-exceptional b-divisor). One can nevertheless exploit its increasing properties, and determine the log-discrepancy functional with respect to its value on the skeleton $\mathcal{S}_X$ of X, which we recall being the skeleton associated to a minimal resolution $\pi_0 \colon X_{\pi_0} \to (X, x_0)$. This is done, roughly speaking, via the Jacobian formula Equation (2.10) applied to π_0.

2.2 Dynamical systems and global basins of attraction

We now focus on the case of (dominant) self-maps $f : (X, x_0) \circlearrowleft$, and the action $f_\bullet$ induced on $\mathcal{V}_X$ (considered to be normalized with respect to some ideal $\mathfrak{a}_X$).

2.2.1 Global attractors for $f_\bullet$

We give here complete description of the action $f_\bullet$ induced by a non-invertible dominant germ $f \colon (X, x_0) \circlearrowleft$ on the normalized valuation space $\mathcal{V}_X$ associated to a surface singularity (X, x_0).

To state the main result, it is useful to recall two properties that a local dynamical system can have:

- f is *finite* if f contracts no curves, or equivalently if f_* has no contracted curve valuations.
- f is *superattracting* if there exists $n \gg 0$ so that $(f^n)^*\mathfrak{m}_X \subseteq \mathfrak{m}_X^2$, or equivalently so that $c(f^n, \cdot)$ is bounded below by some constant $C > 1$ on $\mathcal{V}_X$. If f is non-invertible and not superattracting, it is called *semi-superattracting*.

Finally, we recall that a set $S \subseteq \mathcal{V}_X$ is *totally invariant* for $f_\bullet$ if $f_\bullet^{-1}(S) = S$. This is stronger than usual notion of invariance $f_\bullet(S) \subseteq S$. If $f_\bullet$ is surjective and S is totally invariant, then $f_\bullet(S) = S$.

Theorem 2.2.1. *For the dynamical system $f_\bullet \colon \mathcal{V}_X \circlearrowleft$, exactly one of the following statements holds.*

- *The map f is not finite. In this case, there is a unique semivaluation $\nu_\star \in \mathcal{V}_X$ for which $f_\bullet^n \nu \to \nu_\star$ weakly as $n \to \infty$ for all $\nu \in \mathcal{V}_X$ of finite skewness. If $\nu_\star$ is of finite skewness, then in fact $f_\bullet^n \nu \to \nu_\star$ in the strong topology for each $\nu \in \mathcal{V}_X$ of finite skewness.*

- *The map f is finite and $R_f \neq 0$. In this case, (X, x_0) is a quotient singularity (or regular). Moreover, either we have a unique fixed point attracting all finite skewness valuations (an in the non-finite case), or there exists a subset $I \subset \mathcal{V}_X$, homeomorphic to a closed interval, which is totally invariant, and $\nu \in \mathcal{V}_X$ of finite skewness, $f_\bullet^{2n}\nu \to r_I\nu$ weakly as $n \to \infty$, where r_I is the natural retraction to I. If this $\nu_\star$ itself has finite skewness, then in fact $f_\bullet^{2n}\nu \to \nu_\star$ in the strong topology.*

- *The map f is finite and $R_f = 0$. In this case, f is superattracting, and (X, x_0) is log-canonical but not log-terminal. The set $S := A^{-1}(0) \subset \mathcal{V}_X$ is totally invariant for $f_\bullet$, and $f_\bullet|_S$ is an isometry for the angular distance ρ. The set S is either a point (for simple elliptic singularities or their quotients), a segment (quotient-cusp singularities) or a circle (cusp singularities). Denote by $r_S\colon \mathcal{V}_X \to S$ the natural retraction. Then for every $\nu \in \mathcal{V}_X$ of finite skewness we have $\rho_X(f_\bullet^n\nu, f_\bullet^n r_S\nu) \to 0$ as $n \to \infty$.*

The proof of this theorem relies on the contraction properties of $f_\bullet$ described by Theorem 2.1.11. In the non-finite case, the map $f_\bullet$ is a weak contraction: distances strictly decrease. While this is not enough to get a fixed point theorem (being $\mathcal{V}_X$ not compact for the strong topology), we can still get a fixed point attracting all quasi-monomial valuations, by working on skeleta associated to good resolutions. In the finite case, we rely on Wahl's classification of surface singularities admitting a finite endomorphism, and argument case by case.

2.2.2 Non-finite maps

We have seen that when a selfmap $f\colon (X, x_0) \circlearrowleft$ is not finite, then $f_\bullet$ acts on $\mathcal{V}_X$ as a weak contraction with respect to the angular distance ρ_X, i.e., $\rho_X(f_\bullet\nu, f_\bullet\mu) < \rho_X(\nu, \mu)$, for any ν, μ so that $0 < \rho_X(\nu, \mu) < +\infty$. Our goal is to show that this weak contraction property is enough to get a unique fixed point attracting all valuations with finite skewness.

In general we do not have much control over curve semi-valuations, and we might have other fixed points with infinite skewness, which have more of a repelling behavior with respect to nearby valuations. As an easy example, one can think of the map $f(x, y) = (x^2, xy)$ in $(\mathbb{C}^2, 0)$, for which $f_\bullet$ fixes the divisorial valuation $\nu_\star = \mathrm{ord}_0$. This valuation attracts all valuations of finite skewness, but curve valuations associated to lines $y = \theta x$ are all fixed by $f_\bullet$ (see also [Gig14] for a more involved example).

In order to distinguish "good" fixed points from "bad" ones (where good means that they enjoy suitable attraction properties), we call the former *eigenvaluations*.

Formally, one can define the set of eigenvaluations of f as the closure of the set of fixed points $\nu_\star$ for which there exists a weakly open set U containing $\nu_\star$ at the boundary, and for which any $\nu \in U$ of finite skewness has its orbit weakly converging to $\nu_\star$. In fact, any fixed point of finite skewness is automatically an eigenvaluation: and the local attraction property stated above is needed when $\nu_\star$ is an isolated eigenvaluation of infinite skewness, in which case one can take the open U to be a neighborhood of $\nu_\star$. In this case, we say that $\nu_\star$ is a *weakly attracting end.*

We now give a sketch of the proof of Theorem 2.2.1 in the non-finite case.

Construction of $\nu_\star$. In order to construct an eigenvaluation $\nu_\star$ for a non-finite selfmap $f\colon (X,x_0) \circlearrowleft$, we consider the maps $F_\pi := r_\pi \circ f_\bullet \colon \mathcal{S}_\pi \circlearrowleft$ induced on the skeleton $\mathcal{S}_\pi$ associated to π.

As a consequence of Theorem 2.1.11, F_π is also a weak contraction. Being $\mathcal{S}_\pi$ compact, we deduce the existence and uniqueness of a fixed point $\nu_\pi \in \mathcal{S}_\pi$.

One can then show that if π' is another good resolution that dominates π, then $r_\pi \nu_{\pi'} = \nu_\pi$. The main property used here is the graph structure of $\mathcal{V}_X$, and in particular the fact that $r_\pi^{-1}(\nu_\pi)$ is a union of connected components of $\mathcal{V}_X \setminus \{\nu_\pi\}$.

Remark 2.2.2. This argument is quite different from the construction of fixed points one can find in [FJ07]: in op.cit., the authors take advantage of the tree structure of $\mathcal{V}$ in the case of (X,x_0) regular in order to get a fixed point. Here $\mathcal{V}_X$ can have a very complicated topology, and in fact fixed points might not exist when f is finite (in this case we have necessarily a cusp singularity (X,x_0), explicit examples are given in the next chapter).

Basin of attraction.

Denote by $\mathcal{V}_X^\alpha := \{\nu \in \mathcal{V}_X \mid \alpha_X(\nu) < +\infty\}$ the set of normalized valuations with finite skewness. Inside of it, we consider the basin of attraction of $\nu_\star$, defined by

$$B^\alpha_{\nu_\star} = \{\nu \in \mathcal{V}_X^\alpha \mid f_\bullet^n \nu \text{ converges to } \nu_\star\}.$$

The convergence is considered to be strong when $\alpha(\nu_\star) < +\infty$, and weak otherwise, and the arguments we use slightly differ depending on the finiteness of $\alpha(\nu_\star)$.

As a consequence of the non-expansion of $f_\bullet$ with respect to ρ_X, we deduce that $B^\alpha_{\nu_\star}$ is closed in the strong topology. Theorem 2.2.1 is obtained by showing that $B^\alpha_{\nu_\star}$ is also open. This boils down to showing that $B^\alpha_{\nu_\star}$ contains an open neighborhood of $\nu_\star$.

Case $\alpha(\nu_\star) = +\infty$ We show that there exists a weakly open neighborhood U of $\nu_\star$ of valuation whose orbit converges to $\nu_\star$. This is fairly easy: since $\nu_\star$ has infinite skewness, it does not belong to the critical skeleton $\mathcal{S}_f$ of f (where we fix on $\mathcal{V}_X$ the normalization given by the maximal ideal). Then we pick as U the connected component of $\mathcal{V}_X \setminus \mathcal{S}_f$ containing $\nu_\star$, and use the fact that $f_\bullet$ is increasing to show that U is $f_\bullet$-invariant, and the orbits there converge to $\nu_\star$ (in the weak topology).

Case $\alpha(\nu_\star) < +\infty$ In this case, we show that f is necessarily superattracting. In fact, if f is not finite and $c(f,\nu_\star) = 1$ (condition equivalent to being semi-superattracting), by applying Equation (2.3) and Equation (2.4) to $Z(\nu_\star)$, and arrive to an equation of the form

$$\alpha(\nu_\star) = m(f,\nu_\star)\alpha(\nu_\star) + k$$

where $k > 0$ is a positive constant, consequence of the non-finiteness of f. We deduce that $\alpha(\nu_\star) = +\infty$ as desired.

Hence, we may assume that $c(f,\nu_\star) = c > 1$. In this case, we can find an open neighborhood U of $\nu_\star$ in $\mathcal{V}_X$, such that all valuations of finite skewness have orbits strongly converging to $\nu_\star$. The analysis here is divided case by case, according to the type of the eigenvaluation, the case of a divisorial eigenvaluation being the hardest.

Let us fix the normalization on $\mathcal{V}_X$ induced by the maximal ideal. As above, a central role for the convergence estimates is played by the critical skeleton, or more precisely by the union $\mathcal{S}$ of the critical skeleton $\mathcal{S}_f$ and the skeleton associated to the Jacobian divisor R_f.

In fact, $\mathcal{S}_f$ is compact, and the angular distance to the eigenvaluation strictly decreases under the action of $f_\bullet$ here. Outside of it, both $\nu \mapsto c(f,\nu)$ and $\nu \mapsto \nu(R_f)$ are locally constant, and the Jacobian formula Equation (2.12) gives the contraction property

$$A(f_\bullet\nu) - A(\nu_\star) = \frac{A(\nu) - A(\nu_\star)}{c} < A(\nu) - A(\nu_\star).$$

These are the main ingredients to build the (strongly open) neighborhood U. For example, when $\nu_\star = \nu_{E_\star}$ is divisorial, one can take U of the form

$$U = \{\nu_\star\} \cup \bigcup_{p \in E_\star} U_p(\varepsilon_p),$$

where $U_p(\varepsilon_p)$ denotes the strongly open set of valuations ν tangent to the direction $\overrightarrow{v_p} \in \mathcal{T}_{\nu_\star}\mathcal{V}_X$ associated to $p \in E$ and at angular distance $\rho_X(\nu_\star, \nu) < \varepsilon_p$, and $\inf_{p \in E_\star} \varepsilon_p > 0$. The map $p \mapsto \varepsilon_p$ is determined by recursion as follows. Firstly, we set small enough values for ε_p when $\overrightarrow{v_p}$ varies inside the finite set $S = \mathcal{T}_{\nu_\star}\mathcal{S}$ of tangent directions tangent to $\mathcal{S}$; here we mainly use the contraction properties of the angular distance. Then, we extend ε to the orbit of S via the tangent map $\mathrm{d}f_\bullet$; here we use the contraction properties with respect to both angular distance and log-discrepancy. Finally, we set $\varepsilon_p = +\infty$ for the other tangent directions.

2.2.3 Finite maps

Let now $f\colon (X,x_0) \circlearrowleft$ be a non-invertible finite germ. Firstly, we show that f is necessarily superattracting. This is again a consequence of the push

The dynamical study of $f_\bullet$ in the finite case strongly relies on Wahl's classification of surface singularities admitting non-invertible finite germs.

Theorem 2.2.3 ([Wah90, Fav10]). *Let $f\colon (X,x_0) \circlearrowleft$ be a non-invertible finite germ.*

Then (X,x_0) is necessarily log-canonical. More precisely, we are in exactly one of the following situations:

- *$R_f > 0$, and (X,x_0) is log-terminal;*
- *$R_f = 0$, and (X,x_0) is log-canonical but not log-terminal.*

We report here the proof when f is superattracting, i.e., $c(f^n,\nu) \geq 2$ for $n \gg 1$.

Proof when f is superattracting. Consider the action $f_\bullet\colon \mathcal{V}_X \circlearrowleft$ induced by f on the valuation space. By the Jacobian formula Equation (2.12), we have

$$c(f,\nu)A_X(f_\bullet\nu) = A_X(\nu) + \nu(R_f).$$

Recall that $c(f^n, \nu) \geq 2$ for $n \gg 0$ Recall that, since f is finite, we have that $f_\bullet$ is surjective, and R_f is effective.

Let $\nu_0 \in \mathcal{V}_X$ be minimizing A_X. Let ν_{-1} be any valuation in $f_\bullet^{-1}(\nu_0)$. In this case, we have

$$c(f, \nu_{-1}) A_X(\nu_0) = A_X(\nu_{-1}) + \nu_{-1}(R_f).$$

Suppose first that $R_f > 0$, and by contradiction that $A_X(\nu_0) \leq 0$. Then $A_X(\nu_0) \leq c(f, \nu_{-1}) A_X(\nu_0) < A_X(\nu_{-1})$, in contradiction with the minimality of A_x at ν_0.

Suppose now that $R_f = 0$ and $A_X(\nu_0) \neq 0$. By iterating this procedure, we obtain a family $(\nu_n)_{n \in \mathbb{Z}}$ such that $f_\bullet(\nu_n) = \nu_{n+1}$ for all $n \in \mathbb{Z}$, and

$$A_X(\nu_n) = \begin{cases} c(f^{-n}, \nu_n) A_X(\nu_0) & \text{if } n \leq 0, \\ c(f^n, \nu_n)^{-1} A_X(\nu_0) & \text{if } n \geq 0. \end{cases}$$

We pick $n \gg 0$ so that cf^n, and we get a contradiction with the minimality of $A_X(\nu_0)$ by considering either ν_n or ν_{-n}, depending on the sign of $A_X(\nu_0)$. ☐

The general case is obtained by showing that, when $R_f = 0$, then f is necessarily superattracting (see [Fav10, Theorem B] or [GR21, Theorem 6.3]).

Thanks to Theorem 2.2.3, we can reduce our study to log-canonical singularities, which are relatively mild, and completely classified in dimension 2.

Log-terminal singularities In particular, when $R_f > 0$, then (X, x_0) is log-terminal, i.e., a quotient singularity. The valuation space $\mathcal{V}_X$ has trivial topology (it is a tree), and all results valid on $\mathcal{V}_{\mathbb{C}^2}$ extends effortlessly to $\mathcal{V}_X$. Even simpler, one can take advantage of the geometric description $(X, x_0) \cong (\mathbb{C}^2, 0) \big/ G$, with G a finite group acting freely on $\mathbb{C}^2 \setminus \{0\}$. In this setting, any finite germ $f \colon (X, x_0) \circlearrowleft$ lifts to a G-invariant finite germ $g \colon (\mathbb{C}^2, 0) \circlearrowleft$, and one can apply the results known in the regular case to g.

Regardless of the approach followed, the main tools used here are:

- a procedure to guarantee of existence of fixed points for $f_\bullet$, that takes advantage of the tree structure of $\mathcal{V}_X$ (see [FJ07, Theorem 4.5]);
- a control of the monotonicity of $f_\bullet$, encoded by the critical tree $\mathcal{S}_f$ (possibly enlarged by $\mathcal{S}_{R_f}$),
- the geometric description of $f_\bullet$ and its action via pushforward and pullback of b-divisors, via Equation (2.2). This allows to bound the geometry of $\mathrm{Fix}(f_\bullet)$, and show that it is a segment. See [GR14, Section 4] for further details.

Log-canonical not log-terminal singularities

When $R_f = 0$, then (X, x_0) is log-canonical but not log-terminal, i.e., a simple-elliptic singularity, a cusp singularity, or a finite quotient of these. One can lift finite germs to finite quotients as for the case of quotient singularities, so we can reduce to simple-elliptic and cusp singularities.

Directly from the Jacobian formula, one deduces that the set $S = \{\nu \in \mathcal{V}_X \mid A_X(\nu) = 0\}$ of valuations that realize the log-canonical threshold must be totally invariant, and in fact, $f_\bullet$ acts on it as an isometry.

In the simple elliptic case, S consists of a unique valuation ν_E, where E is the elliptic curve appearing in the minimal resolution (and in any other resolution in fact). We deduce that ν_E is an eigenvaluation for f, and conclude this study as in the non-finite case.

In the cusp case, S is topologically a circle. In this case, $f_\bullet$ acts as an isometry with respect to the angular distance ρ_X. Up to replacing f by its second iterate, we may suppose that $f_\bullet$ is a rotation on S (identified with the circle $\mathbb{S}^1$). The global contraction properties towards S can be obtained by working with the distance to S induced by log-discrepancy. The arithmetic properties of f depend strongly on the rotation number of $f_\bullet|_S$, as we will present in the next sections.

2.3 Algebraically stable models and attraction rates

2.3.1 Algebraic stability

As we have seen, in order to study the local dynamics induced by a selfmap $f\colon (X, x_0) \circlearrowleft$ (or the global dynamics of a rational map on some variety X), we are led to consider bimeromorphic models X_π obtained from X by blow-ups, and the induced dynamics $f_\pi\colon X_\pi \circlearrowleft$. In general f_π has an indeterminacy set $\mathrm{Ind}\,(f_\pi)$. Algebraic stability is a condition that, roughly speaking, controls the degeneracy of the dynamics of f_π in relation with its indeterminacy set.

Several algebraic stability notions are present in literature, and the lexic is not necessarily uniform. Hence we clarify here the concept we work with.

Definition 2.3.1. Let $f\colon (X, x_0) \circlearrowleft$ be a dominant non-invertible germ. We say that a model $\pi\colon X_\pi \to (X, x_0)$ is *geometrically stable* for f if the lift $f_\pi\colon X_\pi \dashrightarrow X_\pi$ has the following property. For every exceptional prime E of X_π, $f_\pi^n(E)$ is an indeterminacy point of f_π for at most finitely many n.

Notice that geometric stability does allow for $f_\bullet^n(E)$ to be a curve, that might or might not contain indeterminacy points of $f_\bullet$. But if E is contracted to a point p_n by some iterate f_π^n, then geometric stability requires for the following iterates of p_n to remain points (at least for $n \gg 0$.

The dynamics of indeterminacy points is related to the behavior of the pullback of exceptional divisors, as resumed by the following statement.

Lemma 2.3.2. *Let $f\colon (X, x_0) \to (Y, y_0)$ and $g\colon (Y, y_0) \to (Z, z_0)$ be two dominant germs between normal surface singularities. Let $\pi\colon X_\pi \to (X, x_0)$, $\varpi\colon Y_\varpi \to (Y, y_0)$ and $\eta\colon Z_\eta \to (Z, z_0)$ be good resolutions. Set $\widetilde{f} = \varpi^{-1} \circ f \circ \pi\colon X_\pi \dashrightarrow Y_\varpi$ and $\widetilde{g} = \eta^{-1} \circ g \circ \varpi\colon Y_\varpi \dashrightarrow Z_\eta$. If for any exceptional prime $E \in \Gamma_\pi^*$ we have that $\widetilde{f}(E)$ is not an indeterminacy point of $\widetilde{g}$, then $(g \circ f)^* = f^* \circ g^*$.*

As a consequence, we obtain the following interpretation (although, the condition obtained is stronger in general) of geometric stability in terms of pullbacks of exceptional divisors.

Proposition 2.3.3. *Let $f\colon (X,x_0) \circlearrowleft$ be a dominant non-invertible germ. Let $\pi\colon X_\pi \to (X,x_0)$ be a proper bimeromorphic map, and denote by $f_\pi\colon X_\pi \dashrightarrow X_\pi$ the lift of f to X_π. If π is geometrically stable for f, then there exists $N \in \mathbb{N}$ so that for any $n \geq N$, we have*

$$(f_\pi^n)^* = (f_\pi^N)^*(f_\pi^*)^{n-N} \qquad \textit{and} \qquad \mathrm{pr}_{\mathcal{E},\pi} \circ (f_\pi^n)^* = \mathrm{pr}_{\mathcal{E},\pi} \circ (f_\pi^N)^*(\mathrm{pr}_{\mathcal{E},\pi} \circ f_\pi^*)^{n-N}, \quad (2.14)$$

where, for any b-divisor $Z \in b\text{-}\mathcal{D}(X)$, $\mathrm{pr}_{\mathcal{E},\pi}(Z)$ denotes the incarnation in π of $\mathrm{pr}_{\mathcal{E}}(Z)$.

Notice that if f is finite, then the two expressions in Equation (2.14) coincide, and this condition is often called *algebraic stability*.

Remark 2.3.4. To be more precise, algebraic stability is the above condition when we can take $N = 0$, while the condition we used is sometimes called *eventually algebraically stable*. With this in mind, our definition of geometric stability should be called *eventual geometric stability*, while the geometric stability would correspond to having that $f_\pi^n(E)$ is never an indeterminacy point of f_π.

2.3.2 Existence of geometrically stable models

The main result of this section is a theorem of existence of geometrically (and hence algebraically) stable models, dominating any given model, for any dominant germ $f\colon (X,x_0) \circlearrowleft$ on a normal surface singularity (X,x_0), with the notable exception of finite germs on cusp singularities (that we treat in Section 3.2).

Theorem 2.3.5. *Let (X,x_0) be a normal surface singularity, and let $f\colon (X,x_0) \circlearrowleft$ be a dominant non-invertible selfmap. Assume that f is not a finite germ at a cusp singularity inducing an irrational rotation on the essential skeleton. Then for any modification $\pi\colon X_\pi \to (X,x_0)$, one can find another modification $\pi'\colon X_{\pi'} \to (X,x_0)$ dominating π for which f is geometrically stable. In general $X_{\pi'}$ may have cyclic quotient singularities. It can be taken smooth up to replacing f by an iterate.*

Notice that the singular surfaces $X_{\pi'}$ will only have (cyclic) quotient singularities. Such surfaces can be also described with the formalism of *orbifolds* (also called V-manifolds). Hence Theorem 2.3.5 could be also restated as a result on the existence of geometrically stable orbifolds.

Sketch of proof of Theorem 2.3.5. Our goal is to show the existence of geometrically stable models π' dominating a given model π, at least for an iterate f^n of f (in order to avoid technicalities). The analysis is split according to the dynamics of $f_\bullet$, as described by Theorem 2.2.1:

(a) $f_\bullet$ admits a unique eigenvaluation $\nu_\star$, which is not quasimonomial; every valuation $\nu \in \mathcal{V}_X^\alpha$ weakly converges to $\nu_\star$.

(b) $f_\bullet$ admits a unique eigenvaluation $\nu_\star$, which is irrational; every valuation $\nu \in \mathcal{V}_X^\alpha$ strongly converges to $\nu_\star$.

(c) There exists $S \subseteq \mathcal{V}_X$ which is either a divisorial point, a segment with divisorial or curve endpoints, or a circle, so that $f_\bullet^k|_S = \mathrm{id}_S$, and every valuation $\nu \in \mathcal{V}_X^\alpha$ converges strongly to S.

Cases (a,b) These are the easy cases. Being $\nu_\star$ non-divisorial, the center $p_\star = p_\star(\pi')$ of $\nu_\star$ in any model π' is a closed point, which may be assumed to be a free point when $\nu_\star$ is not quasimonomial, and a satellite poit when ν is an irrational valuation. The infinitely near point $p_\star \in X_{\pi'}$ identifies a weakly open subset $U := U_{\pi'}(p_\star)$ of $\mathcal{V}_X$ containing $\nu_\star$.

In case (a), $\nu_\star$ is a weakly attracting end: up to replacing π' with a higher model, we my suppose that U does not intersect the critical skeleton $\mathcal{S}_f$. We deduce that $f_\bullet$ is increasing on U and that $f(U) \subseteq U$.

In case (b), we use instead the weak contraction properties of $f_\bullet$ with respect to the angular distance $\nu_\star$, and deduce again that $f_\bullet(U) \subseteq U$.

By Proposition 2.1.3, we deduce that $p_\star$ is a regular fixed point for $f_{\pi'}$. Moreover, by Theorem 2.2.1, all divisorial valuations converge to $\nu_\star$ by the action of $f_\bullet$. In particular, for any exceptional prime $E \in \Gamma_{\pi'}^*$, $f_\bullet^n(\nu_E) \in U$ for all $n \gg 0$, which translates to $f_{\pi'}^n(E) = p_\star$ for any such n. We have showed that π' is geometrically stable.

Case (c) Up to replacing f by an iterate, we may assume that the set S contains a divisorial eigenvaluation $\nu_\star = \nu_{E_\star}$, which is fixed by $f_\bullet$.

Let us assume for simplicity that $\nu_\star$ is the unique eigenvaluation (the other cases are dealt similarly, but they are technical more involved).

Let $\pi' \colon X_{\pi'} \to (X, x_0)$ be a model dominating π and so that $E_\star \in \Gamma_{\pi'}^*$. The fact that $f_\bullet \nu_\star = \nu_\star$ tells us that $E_\star$ is left invariant by $f_{\pi'}$: we set $h = f_{\pi'}|_{E_\star} : E_\star \circlearrowleft$.

Since $\nu_\star$ attracts all divisorial valuations, we have that for any exceptional prime $E \in \Gamma_{\pi'}^*$, for $n \gg 0$ we have that $f_{\pi'}^n(E_\star)$ is either a closed point $p_n \in E_\star$, or the prime $E_\star$ itself. While the latter case does not conflict with geometric stability, the previous does whenever p_n belongs to $\mathrm{Ind}\,(f_{\pi'})$ for infinitely may n.

Since $\mathrm{Ind}\,(f_{\pi'})$ is a finite set, the obstructions to the geometric stability of $f_{\pi'}$ are the indeterminacy points in $E_\star$ that are periodic for h.

Let $p_0 \in \mathrm{Ind}\,(f_{\pi'}) \cap E_\star$ be such an indeterminacy point, with orbit $p_0, p_1, \ldots, p_{n-1}, p_n = p_0$, where n is the period of p_0 (we will use cyclic notations from now on).

The idea is to blow-up along this orbit in order to eliminate all indeterminacy points. More precisely, for each j, we blow-up p_j, obtaining a divisor $E_{j,1}$, then the intersection $p_{j,1}$ between $E_{j,1}$ and the strict transform of $E_\star$, obtaining $E_{j,2}$. We proceed recursively, performing m_j blow-ups, with the values $m_j \in \mathbb{N}$ to be determined.

The main difficulty comes from the fact that, by blowing up along $p_j, p_{j,1}$ etcetera, we could create new indeterminacies at p_{j-1}: we need a global control of indeterminacies along the orbit.

Recall that each point $p_j \in E_\star$ is related to a tangent vector $\vec{v_j}$ at $\nu_{E_\star}$. The divisorial valuations $\nu_{E_{j,m}}$ belong to the tangent direction $\vec{v_j}$, and get closer and closer to ν_j (with respect to the strong topology) when $m \to +\infty$.

In view of Proposition 2.1.3, we can control the creation of new indeterminacies thanks to the contraction properties of $f_\bullet$ with respect to the angular distance ρ_X along these tangent directions.

We refer directly to [GR21, Section 7.1] for further details. □

With the next example, we display how the construction of a geometrically stable model works in the case of a unique divisorial eigenvaluation.

Example 2.3.6. Consider the map $f\colon (\mathbb{C}^2, 0) \circlearrowleft$ given by $f(x,y) = (y + x^3, x^2 y)$. We consider the space $\mathcal{V} = \mathcal{V}^{\mathfrak{m}}$ of valuations normalized by the maximal ideal $\mathfrak{m}$. In this case, $c(f,\nu)$ is locally constant outside the finite tree with ends the curve semivaluations ν_x, ν_y and ν_{y+x^3} (in fact, the critical skeleton $\mathcal{S}_f$ is strictly contained in it).

We study the dynamics of $f_\bullet$ in this tree, and get the following picture.

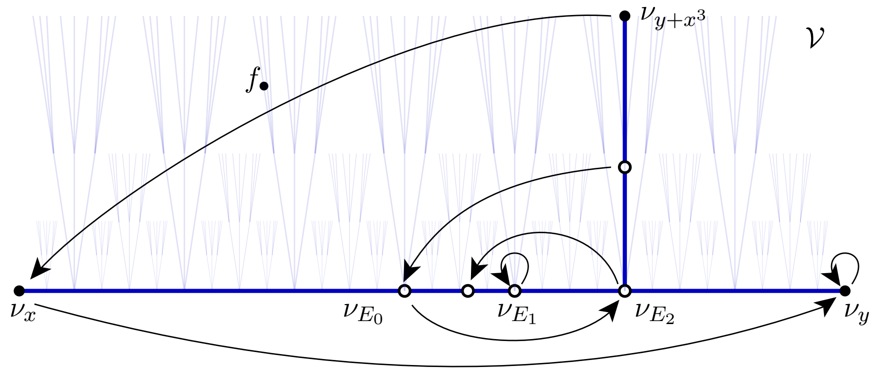


Figure 2.1: Action of $f_\bullet$ on $\mathcal{V}$.

Here, the exceptional primes E_0, E_1, E_2 are obtained one after another by blowing up the origin 0, then the intersection p_0 between E_0 and the strict transform of $\{y = 0\}$, and finally the intersection p_1 between E_1 and the strict transfor of $\{y = 0\}$ in this new model. We denote by $\pi_j\colon X_{\pi_j} \to (\mathbb{C}^2, 0)$ the models where E_j appears first, for $j = 0, 1, 2$.

We notice that ν_{E_1} is fixed by $f_\bullet$, hence it is a divisorial eigenvaluation for f. We can also show that the eigenvaluation is unique, and that the action of $f_\bullet$ is weakly contracting for the angular distance ρ.

In the model π_0, f_{π_0} is not geometrically stable, since E_0 is sent to p_0, which is an indeterminacy point for f_{π_0}.

The model π_1 is where $E_1 = E_\star$ is realized. The action $h = f_{\pi_1}|_{E_1}$ is an involution in this case, and E_0 is sent to p_1 by f_{π_1}. We deduce that f_{π_1} is not geometrically stable either.

Following the algorithm depicted above, we need to blow-up along p_1 and $h(p_1)$. In this case, it suffices to just blow-up p_1 once: while other indeterminacy points appear in E_2, E_1 is now free of indeterminacy points, and f_{π_2} is geometrically stable.

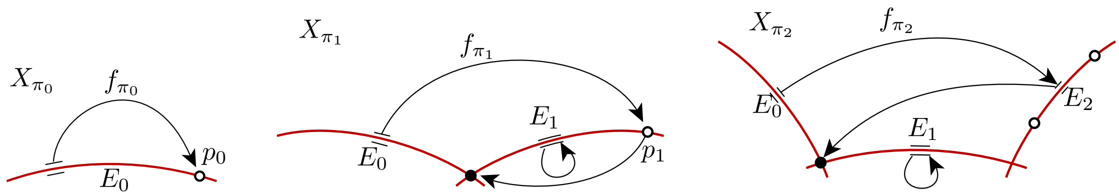


Figure 2.2: Search for a geometrically stable model.

2.3.3 The sequence of attraction rates

Let $\nu \in \mathcal{V}_X$ be a valuation. We consider the sequence $(c_n := c(f^n, \nu))_n$ of attraction rates of the iterates of f, along ν. The *Poincaré series* associated to this sequence is $\gamma(f, \nu)(t) = \sum_{n=1}^{+\infty} c_n t^n$. The asymptotic behavior of the sequence $(c_n)_n$ is encoded by an important birational invariant, called the *first dynamical degree* $c_\infty(f)$.

Proposition-Definition 2.3.7 ([GR21, Proposition 8.1]). *Let $f\colon (X, x_0) \circlearrowleft$ be a dominant germ on a normal surface singularity. Let $\nu \in \mathcal{V}_X^\alpha$ be any normalized valuation of finite skewness. Then $c_\infty(f, \nu) = \lim_{n\to\infty} \sqrt[n]{c(f^n, \nu)}$ exists, belongs to $[1, +\infty)$, and it does not depend on ν. This common limit $c_\infty(f)$ is called the* (first) dynamical degree *of f.*

The proof is based on a comparison between the sequences associated to two valuations ν and μ via the relative Izumi constants $\beta(\nu|\mu)$ and $\beta(\mu|\nu)$, plus the existence of a ν for which $(c(f^n, \nu))$. When f admits an eigenvaluation $\nu_\star$ of finite skewness, we simply take $\nu = \nu_\star$. When f admits a locally attracting end $\nu_\star$, we use the fact that $c(f, \nu)$ is locally constant at $\nu_\star$.

When f does not admit eigenvaluations, we necessarily are in the situation of a finite map on a cusp singularity. The skeleton $\mathcal{S}_X$ is a circle, and we may assume that the action of $f_\bullet$ on $\mathcal{S}_X$ is an irrational rotation. In this case, the convergence of $\sqrt[n]{c(f^n, \nu_\star)}$ with $\nu_\star \in \mathcal{S}_X$ is a consequence of the ergodic theorem.

The dynamical degree describes the speed of convergence of the orbits of X towards x_0. It is a fundamental invariant of (bimeromorphic) conjugacy for global dynamics as well as in our setting, see e.g. [DF01].

We state here the main theorem about the sequence of attraction rates and the related objects, that we obtain in [GR14] in the smooth case, and in [GR21] in the singular case.

Theorem 2.3.8. *Let (X, x_0) be a normal surface singularity, and let $f\colon (X, x_0) \circlearrowleft$ be a dominant non-invertible germ. Assume we are not in the case of a finite germ f at a cusp singularity (X, x_0) inducing an irrational rotation. Then for any quasimonomial valuation $\nu \in \mathcal{V}_X$, the sequence of attraction rates $c_n := c(f^n, \nu)$ eventually satisfies an integral linear recursion relation. More precisely, there exist integers a, b, N, and m with $N, m \geq 1$ such that $c_{n+2m} = a c_{n+m} + b c_n$ for all $n \geq N$. In particular, $\gamma(f, \nu)(t)$ is rational, and $c_\infty(f)$ is a quadratic integer.*

In the local smooth setting (as well as in the global setting for polynomial endomorphisms of $\mathbb{C}^2$), $c_\infty(f)$ was known to be a quadratic integer since [FJ07]. In [FJ11], the authors prove that the sequence of degrees satisfies a recursion relation in case of polynomial endomorphisms of $\mathbb{C}^2$. Our theorem can be seen as a local counterpart of Favre-Jonsson's result, and an extension to the singular setting. The rationality of $\gamma_\nu(t)$ was also known in the smooth case, under an additional transversality condition, by [CAR11].

Sketch of proof of Theorem 2.3.8. Recall that by Equation (2.13) we have

$$c(f,\nu) = -Z(\nu)\cdot \mathrm{pr}_{\mathcal{E}}\, f^* Z(\mathfrak{m}_X).$$

By using the fact that cf,ν is locally constant outside the critical skeleton $\mathcal{S}_f$, that $f_\bullet$ is continuous and that divisorial valuations are dense in $\mathcal{V}_X$, we may assume that $\nu = \nu_E$ is divisorial. Let $\pi\colon X_\pi \to (X, x_0)$ be a good resolution high enough to have E as an exceptional prime. By Theorem 2.3.5, up to taking a higher model, we may assume that π is geometrically stable for f. Denote by $f_\pi = \pi^{-1}\circ f \circ \pi\colon X_\pi \dashrightarrow X_\pi$ the lift of f to X_π.

By Proposition 2.3.3, there exists $N\in\mathbb{N}$ so that for any $n \geq N$, we have

$$\mathrm{pr}_{\mathcal{E},\pi}\circ (f_\pi^n)^* = \mathrm{pr}_{\mathcal{E},\pi}\circ (f_\pi^N)^* (\mathrm{pr}_{\mathcal{E},\pi}\circ f_\pi^*)^{n-N},$$

where $\mathrm{pr}_{\mathcal{E},\pi}$ associates to $Z\in b\text{-}\mathcal{D}(X)$ the exceptional part of its incarnation in $\mathcal{E}(\pi)$. Notice that $\mathrm{pr}_{\mathcal{E},\pi}\circ f_\pi^*\colon \mathcal{E}(\pi)_\mathbb{R}\to\mathcal{E}(\pi)_\mathbb{R}$ is a $\mathbb{Z}$-linear map on the finite-dimensional $\mathbb{R}$-vector space $\mathcal{E}(\pi)_\mathbb{R}$. As a consequence of Cayley-Hamilton's theorem, we find the desired linear recursion (of order bounded by the dimension of $\mathcal{E}(\pi)_\mathbb{R}$.

From this result we directly derive the rationality of $\gamma_\nu(t)$, and the fact that $c_\infty(f)$ is an algebraic integer.

In order to get a recursion relation of order 2 for the sequence of attraction rates for f (or one of its iterates), which would imply that $c_\infty(f)$ is a quadratic integer, we need a finer study of the dynamics of $f_\bullet$ on the valuative space.

The study depends on the type of eigenvaluation $\nu_\star$ that attracts the valuation ν.

- If $\nu_\star$ is a locally attracting end, then $c(f,\nu)$ is locally constant at $\nu_\star$, and we get $c_{n+1} = c_\infty c_n$ for all $n \gg 0$.
- If $\nu_\star$ is irrational, one can find a model $\pi\colon X_\pi\to(X,x_0)$ such that the center of $\nu_\star$ in X_π is at the intersection of two exceptional primes, written in local coordinates as $\{x=0\}$ and $y=0$, and f_π is monomial in such coordinates, i.e., of the form $f_\pi(x,y) = (x^a y^b u, x^c y^d v)$ for some matrix $A = \begin{pmatrix} a & b\\ c & d\end{pmatrix}$ with integer entries, and u, v unities. In this case, the recursion relation is of order 2, with the coefficients given by the trace and determinant of A (up to sign).
- If $\nu_\star$ is divisorial, the situation is more complicated. When $n\gg 0$, we have that $f_\bullet^n$ defines a tangent vector $\vec{v_n}$ at $\nu_\star$. The behavior of the sequence $c(f^n,\nu)$ depends on the behavior of the sequence $\vec{v_n}$, and in particular on how many times $\vec{v_n}$ is tangent to the critical skeleton $\mathcal{S}_X$.

$\square$

Chapter 3

Dynamical properties of surface singularities

Introduction

[FR14] C.Favre and M.Ruggiero: "Normal surface singularities admitting contracting automorphisms". Annales Mathématiques de la faculté des sciences de Toulouse, Volume 23, no. 4, pp. 797–828, 2014. http://afst.cedram.org/.

[GR21] W.Gignac and M.Ruggiero: "Local dynamics of non-invertible maps near normal surface singularities". Memoirs of the AMS 272, no. 1337, xi+100 pages, 2021. https://bookstore.ams.org/....

[FFR20] L.Fantini, C.Favre and M.Ruggiero: "Links of sandwiched surface singularities and self-similarity". Manuscripta Mathematica, Volume 162, Issue 1-2, pp. 23–65, 2020. https://link.springer.com/.

This chapter is dedicated to exploring the interactions between the dynamical properties of selfmaps acting on a normal surface singularity, and the geometry of the singularity that supports such dynamical systems.

As we already did in the previous chapter, we restrict our study to dominant self-maps, in order to keep the dynamical phenomena 2-dimensional in nature. It is fairly easy to construct selfmaps on any normal surface singularity (X, x_0). On the one hand, we can always consider a resolution $\pi\colon X_\pi \to (X, x_0)$. On the other hand, we can consider an embedding $\varphi\colon (X, x_0) \hookrightarrow (\mathbb{C}^d, 0)$, followed by a generic projection $\ell\colon (\mathbb{C}^d, 0) \to (\mathbb{C}^2, 0)$ to a plane. Finally, we pick any (dominant) morphism $\sigma\colon (\mathbb{C}^2, 0) \to (X_\pi, p)$, where $p \in \pi^{-1}(x_0)$. The composition $f = \pi \circ \sigma \circ \ell \circ \varphi : (X, x_0) \circlearrowleft$ gives a dominant selfmap. Maps built with this procedure are far from showing all possible dynamical behavior of selfmaps: they are never finite, and they often have high topological degree. For example, to have topological degree 1, both σ and the generic projection ℓ must have topological degree 1, and this implies in particular that (X, x_0) is in fact regular.

Even if we restrict the study to automorphisms, a theorem of Müller [Mül87, p.230–231] states that any singularity (X, x_0) carries countably many analytic vector fields tangent to (X, x_0) and linearly independent. By taking the flow of such vector fields, we deduce that the automorphism group $\mathrm{Aut}(X, x_0)$ is always "infinite dimensional".

But when we impose some special dynamical properties of the selfmap $f : (X, x_0) \circlearrowleft$, it turns out that the singularity (X, x_0) has to be rather special. In particular, we present here three results:

Theorem. *A normal surface singularity (X, x_0) admits:*

- *a non-trivial finite selfmap if and only if it is log-canonical.*
- *a contracting automorphism if and only if it is quasi-homogeneous.*
- *a non-trivial Kato datum if and only if it is sandwiched.*

In this theorem, "non-trivial" means that the selfmap f is not an automorphism.

The case of finite selfmaps is due to Wahl [Wah90], and we already presented a more precise statement in Theorem 2.2.3, which is due to Favre [Fav10].

We recall that 2-dimensional log-canonical singularities are finite quotients of either $(\mathbb{C}^2, 0)$, a cusp singularity (the exceptional divisor of the minimal resolution is a cycle of rational curves), or a simple elliptic singularity (the exceptional divisor of the minimal resolution is a smooth elliptic curve). The study portrayed in Chapter 2 gives a complete picture of the local dynamics of finite selfmaps (existence of geometrically stable models, eventual linear recurrence of the sequence of attraction rates, quadratic integrality of the first dynamical degree), as long as they admit at least an eigenvaluation. The situation that remains to deal with is the one of finite germs f on a cusp singularity (X, x_0), and for which $f_\bullet$ acts on the skeleton $\mathcal{S}_X$ as an irrational rotation.

In the first section, we recall the arithmetic and toric costructions of cusp sigularities; following [Fav10], we use this interpretation to construct a special family of finite germs on cusps singularities, that we refer as *toric*. We then show that for any finite germ f, there exists a toric finite germ $\widetilde{f}$ so that the action of $f_\bullet$ and $\widetilde{f}_\bullet$ coicide on $\mathcal{S}_X$. Thanks to this interpretation, we are able to prove that, in this case, while the first dynamical degree remains a quadratic integer (and in fact, the square root of a positive integer), the sequence of attraction rates satisfies no eventual integral recursion relations. We deduce also the non-existence of geometric stable models for such maps, that can be built on any cusp singularities, extending the results in [Fav10]. We conclude this section with a discussion about normal forms of finite germs on cusp singularities.

The second result is contained in [FR14], and extends previous results of Orlik and Vagreich [OW71], and of Camacho, Movasati, Scardua [CMS09]. In this case, we are also able to describe explicitly the normal forms of contracting automorphisms. The techniques here are different in nature, and rely on the study of the local dynamics induced by f on suitable good resolutions of the singularity (X, x_0). The first step consists in showing that the dual graph of such a resolution is necessarily star-shaped, i.e., there is at most a branching point, corresponding to an exceptional prime $E_\star$. When (X, x_0) is not a cyclic quotient sigularity, we can construct an invariant foliation $\mathcal{F}$ transverse to $E_\star$, by collectiong all stable manifolds at $E_\star$, and extending to a neighborhood of the exceptional divisor by following the dynamics. The existence of this foliation provides the rigidity needed to show that (X, x_0) is, up to finite quotient, a *cone* singularity, i.e., obtained from the contraction of the zero section of a negative line bundle (which is related to the normal bundle of $E_\star$).

The third result is contained in [FFR20]. A Kato datum (π, σ) consists of a modification $\pi\colon X' \to (X, x_0)$, and a local isomorphism $\sigma\colon (X, x_0) \to (X', x_1)$, where $x_1 = \sigma(x_0) \in \pi^{-1}(x_0)$. This datum induces a selfmap $f = \pi \circ \sigma$ of topological degree 1 (and in fact one can show that all topological degree 1 selfmaps on a normal surface singularity come from a Kato construction). The proof presented in [FFR20], that works over fields of any characteristic, relies on extending the plumbing techniques of [Spi90] to our setting, on analytic non-archimedean techniques on Berkovich spaces and non-archimedean links, and the combinatorics of self-similar dual graphs. Here we present an alternative proof working over $\mathbb{C}$, which is more geometrical in nature, and relies in part on the description of the valuative dynamics of f portrayed in Chapter 2.

These two results are related to the construction of non-kahler surfaces, obtained as compactifications of the space of orbits of a germ $f\colon (X, x_0) \circlearrowleft$ of topological degree 1. In the case of a contracting automorphism $f\colon (X, x_0) \circlearrowleft$, one can consider the orbit space $S = S(f)$ obtained by quotienting $X \setminus \{x_0\}$ by the action of f. Notice that $S(f)$ is a compact complex surface, which is a Hopf surface when (X, x_0) is non-singular. Similarly, a Kato datum (π, σ) allows to construct a compact complex surface $S = S(\pi, \sigma)$, called a Kato surface.

When working with a germ f of topological degree 1 (i.e., either contracting automorphisms or contracting Kato germs), the link L (and in fact, the shell) of the singularity (X, x_0) naturally embeds in the surface S, without disconnecting it. Since L is the boundary of a Stein manifold, it is a strongly pseudoconvex hypersurface, and S has a global strongly pseudoconvex hypersurface (GSPH), in the terminology of Kato [Kat79]. In op.cit., Kato classifies singularities admitting a GSPH (using strongly Kodaira's classification of compact complex surfaces).

In our works, we are able to recover Kato's result directly from the geometry of quasi-homogeneous and log-canonical singularities.

3.1 Links, shells and compactifications of orbit spaces

We start by recalling a few generalities on links (and shells) of singularities, in relation to the construction of suitable non-kahler compact surfaces, obtained as the compactification of the orbit spaces of dynamical systems $f\colon (X, x_0) \circlearrowleft$ of topological degree 1.

3.1.1 Link, shell of a singularity, and CR-structures.

Let (X, x_0) be a normal surface singularity. Fix an embedding $\varphi : (X, x_0) \hookrightarrow (\mathbb{C}^d, 0)$ of this singularity to a complex Euclidean space.

For any $\varepsilon > 0$ small enough, the sphere $\mathbb{S}_\varepsilon^{2d-1} = \{z \in \mathbb{C}^d \ : \ \|z\| = \varepsilon\}$ centered at 0 of radius ε intersects $\varphi(X, x_0)$ transversely.

In particular, the set

$$L_\varphi^\varepsilon(X, x_0) := \varphi^{-1}\big(\varphi(X, x_0) \cap \mathbb{S}_\varepsilon^{2d-1}\big)$$

is a smooth real 3-dimensional surface inside X, called the *link* of the singularity (X, x_0).

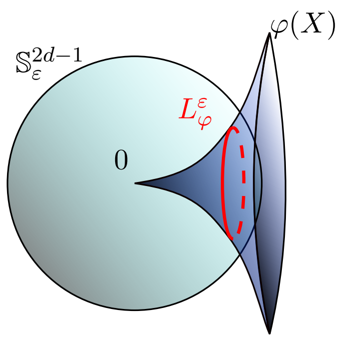


Figure 3.1: Link of a singularity

The $\mathcal{C}^\infty$ structure of the link $L_\varphi^\varepsilon(X, x_0)$ does not depend on $\varepsilon \ll 1$, nor on the choice of the embedding φ. Its $\mathcal{C}^\infty$-diffeomorphism class is simply denoted by $L(X, x_0)$.

By the celebrated Conical Structure Theorem, the link of the singularity completely determines the topological class of a singularity: (X, x_0) is homeomorphic to the (real) cone over its link. The topological class of (X, x_0) is, in turn, completely determined by the dual graph Γ_π of any resolution $\pi\colon X_\pi \to (X, x_0)$, decorated with genus and self-intersection of any exceptional prime.

In what follows, we also denote by

$$X_\varphi^\varepsilon := \{x \in X \mid \|\varphi(x)\| < \varepsilon\}$$

the open neighborhood of $x_0 \in X$ delimitated by the link $L_\varphi^\varepsilon(X, x_0)$, and analogously $\overline{X_\varphi^\varepsilon}$ for its closure in X.

Finally, we consider the germ (X, L) of the analytic space X around its link $L = L_\varphi^\varepsilon(X, x_0)$; we refer to $S_\varphi^\varepsilon(X) = (X, L)$ as the *shell* of the singularity (X, x_0) (with respect to the embedding φ and the radius ε). We can always assume that a shell $S_\varphi^\varepsilon(X)$ is represented by $(X_\varphi^{\varepsilon^-,\varepsilon^+}, L_\varphi^\varepsilon)$, where $0 < \varepsilon^- < \varepsilon < \varepsilon^+ \ll 1$ and $X_\varphi^{\varepsilon^-,\varepsilon^+} = X_\varphi^{\varepsilon^+} \setminus \overline{X_\varphi^{\varepsilon^-}}$.

Remark 3.1.1. By construction, X_φ^ε is a Stein manifold, and its boundary L_φ^ε is strongly pseudoconvex: the map $\log\|\varphi\|$ is a strongly psh exhaustion.

Since the link L_φ^ε is a real hypersurface in its shell, it inherits a natural CR-structure. Since embeddable strongly pseudoconvex CR manifolds admit a unique Stein filling (see [HL75]), one can recover X_φ^ε from the link L_φ^ε (considered as a CR-3-fold), and hence the analytic type of (X, x_0).

However, the CR-structure of the link strongly depends on the choices of the embedding φ, and of the radius ε, see for example [Sch86].

3.1.2 Kato data

Definition 3.1.2. Let (X, x_0) be a normal surface singularity. A *Kato datum* over (X, x_0) is a pair (π, σ), where $\pi \colon X' \to (X, x_0)$ is a modification (with X' normal), and $\sigma \colon (X, x_0) \to X'$ is a local isomorphism such that $x_1 := \sigma(x_0) \in \pi^{-1}(x_0)$.
The Kato datum is said to be *trivial* if π is trivial, i.e., an isomorphism.
The selfmap $f := \pi \circ \sigma : (X, x_0) \circlearrowleft$ is called a *Kato germ*.

A Kato datum (π, σ) is called *contracting* if there exists an open neighborhood of x_0 in X so that $\sigma(U)$ is relatively compact in $\pi^{-1}(U)$, or equivalently if $f(U)$ is relatively compact in U, where $f = \pi \circ \sigma$ is the Kato germ associated to the Kato datum.

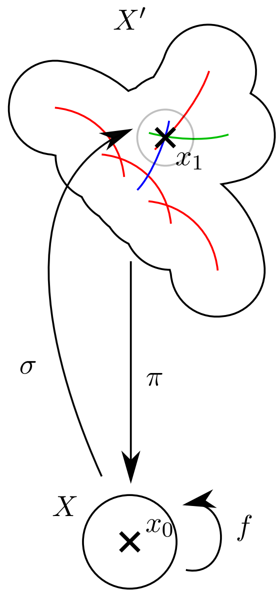


Figure 3.2: Contracting Kato datum.

In the smooth setting $(X, x_0) = (\mathbb{C}^2, 0)$, (contracting) Kato data have been introduced by Masahide Kato, in order to construct interesting compact complex surfaces, now called *Kato surfaces*. One can compute some invariants of Kato surfaces: they have negative Kodaira dimension, their first Betti number is $b_1(S) = 1$, while the second Betti number $b_2(S)$ equals the number of exceptional primes in π, and is hence strictly positive. By picking the Kato datum to be minimal (in a certain sense), we end up with a minimal surface S that has still strictly positive second Betti number. Such surfaces are said to be in *Kodaira's class VII_0*.

The constuction of Kato surfaces goes as follows. We take two open neighborhoods $U^+ \supseteq U^- \ni x_0$ in X, so that $\sigma(U^+)$ is relatively compact in $\pi^{-1}(U^-)$. We then consider the complex surface

$$S = \pi^{-1}(U^+) \setminus \sigma(\overline{U^-}) \Big/ \sigma \circ \pi \colon \pi^{-1}(U^+ \setminus \overline{U^-}) \to \sigma(U^+ \setminus \overline{U^-}). \tag{3.1}$$

It turns out that the class of isomorphism of S does not depend on the choice of the neighborhoods U^+ and U^-, but only on the Kato datum (π, σ) itself. In particular, we may (and will) assume that $U^\pm = X_\varphi^{\varepsilon^\pm}$ for some embedding $\varphi \colon (X, x_0) \to (\mathbb{C}^d, 0)$, and some $0 < \varepsilon^- < \varepsilon^+$.

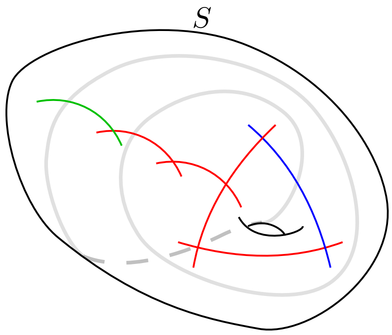


Figure 3.3: Kato surface associated to a (contracting) Kato datum.

Remark 3.1.3. In the following, we will perform several gluing procedures in the fashion of Equation (3.1). To simplify notations, we will not describe explicitly the choice of the embedding φ, nor of the radii $0 < \varepsilon^- < \varepsilon^+$, and simply write

$$S = \pi^{-1}(X) \setminus \sigma(X) \Big/ \sigma \circ \pi \colon \pi^{-1}(\partial X) \to \sigma(\partial X),$$

instead of

$$S = \pi^{-1}(X_\varphi^{\varepsilon^+}) \setminus \sigma(\overline{X_\varphi^{\varepsilon^-}}) \Big/ \sigma \circ \pi \colon \pi^{-1}(X_\varphi^{\varepsilon^-, \varepsilon^+}) \to \sigma(X_\varphi^{\varepsilon^-, \varepsilon^+}).$$

Notice that there is a natural embedding $i : S_\varphi^\varepsilon(X) \hookrightarrow S$ from the shell of the singularity (X, x_0) to the corresponding Kato surface S, since the latter contains a copy of $X_\varphi^{\varepsilon^-, \varepsilon^+}$ inside of it. Moreover, the $S \setminus i(L_\varphi^\varepsilon)$ is connected: we say that S admits a *global strongly pseudoconvex shell*.

Remark 3.1.4. With respect to Kato's classification [Kat79], this situation corresponds to case β. In this case, Kato shows that $S(f)$ contains a global spherical shell, which in this case tells us that any Kato surface $S = S(\pi, \sigma)$ associated to a Kato datum (π, σ) on a normal surface singularity (X, x_0) is isomorphic to a Kato surface $\hat{S} = S(\hat{\pi}, \hat{\sigma})$ associated to a Kato datum $(\hat{\pi}, \hat{\sigma})$ on $(\mathbb{C}^2, 0)$.

In the following sections, we will see how to construct explicitly such a Kato datum (see Remark 3.4.7).

3.2 Finite endomorphisms and log-canonical singularities

3.2.1 Toric construction of cusp singularities

We describe here the arithmetic/toric construction of cusps singularities, referring to [Oda88, Section 4.1] for further details.

We recall that cusp singularities are defined as the one whose minimal resolution has an exceptional divisor which form a cycle of rational curves (with the extreme case of a single nodal rational curve). We write these rational curves in a cyclic order, $E_0, \ldots, E_{r-1}$, for some $r \geq 1$ (if $r = 1$, we have only a singular rational curves that selfintersects transversely), and denote by $k_j := -E_j \cdot E_j$ be the opposite of the self-intersections of these primes, which are integers ≥ 2 and not all 2 (since the intersection form must be negative definite, and we have no -1-curves by minimality). Being cusp singularities taut by Laufer's classification [Lau73], the r-uple $k := (k_0, \ldots, k_{r-1})$ identifies uniquely the isomorphism class of the cusps (X, x_0). Notice that we have still some freedom on the r-uple k: we can shift the cyclic order, or flip it.

The toric construction of (X, x_0) starts by building a regular toric surface (not of finite type) $X(\Sigma)$ associated to an infinite fan Σ inside $N_{\mathbb{R}} \cong \mathbb{R}^2$, whose support fills an open strongly convex cone $\mathcal{C}^+$ with irrational slopes. The surface $X(\Sigma)$ contains an infinite chain of rational curves, with opposite self-intersections that form a r-periodic sequence $k_\bullet := (k_n)_{n \in \mathbb{Z}}$ which extends our initial r-uple $k = (k_0, \ldots, k_{r-1})$.

The explicit construction of the one-parameter lattice N and of the fan Σ are related to arithmetic properties of a quadratic algebraic number ω obtained from k via Hirzebruch-Jung continued fractions. In particular, $N = N_\omega$ can be seen as an additive subgroup of $\Bbbk = \mathbb{Q}(\omega)$.

The multiplication by a suitable element $\varepsilon \in \Bbbk$ leaving $N_\omega \cap \mathcal{C}^+$ invariant induce automorphisms g_ε acting transitively on $X(\Sigma)$. One can pick ε so that the quotient $X(\Sigma) \big/ \langle g_\varepsilon \rangle$ is a regular surface having a cycle of rational curves with opposite self-intersections $(k_0, \ldots, k_{r-1})$. This is the minimal resolution of the cusp singularity we were looking for.

3.2.2 Arithmetic construction of cusp singularities

Let $r \in \mathbb{N}^*$ and $k_0, \ldots, k_{r-1}$ be a finite sequence of integer numbers ≥ 2, not all equal to 2. Set $k_{nr+j} := k_j$ for all $n \in \mathbb{Z}$, so that $k_\bullet := (k_j)_{j \in \mathbb{Z}}$ defines a periodic sequence of integers ≥ 2 (not all equal to 2), that we may assume of exact period r. We denote by $\omega = [k_0, k_1, \ldots] = \left[\,\overline{k_0, \ldots, k_{r-1}}\,\right]$ the Hirzebruch-Jung continued fraction

$$\omega = k_0 - \cfrac{1}{k_1 - \cfrac{1}{k_2 - \cfrac{1}{\ddots}}}. \tag{3.2}$$

Then ω is a quadratic irrational number, $\omega = a + b\sqrt{d}$, where $a, b \in \mathbb{Q}_{>0}$ are positive rational numbers and $d \in \mathbb{N}^*$ is a positive and square-free integer. Moreover $\omega > 1 > \omega' > 0$, where ω' is the conjugate of ω in $\mathbb{Q}(\sqrt{d}) =: \Bbbk$. Set now

- $N = N_\omega := \mathbb{Z} \oplus \omega\mathbb{Z}$ the lattice of rank 2 in $\Bbbk$ generated by 1 and ω;
- $\mathfrak{o}$ is the group of units in the ring of integers of $\Bbbk$; $\mathfrak{o}^> = \mathfrak{o} \cap \mathbb{R}_{>0}$ is the subgroup of *positive* units of $\Bbbk$, and $\mathfrak{o}^+$ is the subgroup of *totally positive* units of $\Bbbk$ (i.e., $u \in \mathfrak{o}^>$ so that $u' > 0$);
- $\mathfrak{o}^>_\omega$ and $\mathfrak{o}^+_\omega$ are the groups of positive and totally positive units u satisfying $uN_\omega = N_\omega$.

By Dirichlet's unit theorem, $\mathfrak{o}^>$ and $\mathfrak{o}^+$ are infinite cyclic groups, and $\mathfrak{o}^+$ has index either 1 or 2 in $\mathfrak{o}^>$. We denote by $\varepsilon^>$ and ε^+ the generators of $\mathfrak{o}^>$ and $\mathfrak{o}^+$ that are > 1. The same property holds for unities preserving the lattice N_ω. Let $\varepsilon^>_\omega$ and ε^+_ω be the generators of $\mathfrak{o}^>_\omega$ and $\mathfrak{o}^+_\omega$ that are > 1.

As usual in toric geometry, we set $N_\mathbb{R} := N_\omega \otimes_\mathbb{Z} \mathbb{R} \cong \mathbb{R}^2$. The $\mathbb{Z}$-linear map $N_\omega \to \mathbb{R}^2$ given by $\xi \mapsto (\xi, \xi')$ extends to a linear isomorphism $\Phi = (\phi_1, \phi_2)\colon N_\mathbb{R} \to \mathbb{R}^2$, represented by the matrix

$$\begin{pmatrix} 1 & \omega \\ 1 & \omega' \end{pmatrix}.$$

Denote by $\mathcal{E}^+ := \mathbb{R}^2_{>0}$ the first quadrant in $\mathbb{R}^2$, and by

$$\mathcal{C}^+ := \Phi^{-1}(\mathcal{E}^+) = \{x + y\omega \mid x + \omega y > 0, x + \omega' y > 0\}$$

the corresponding open cone in $N_\mathbb{R}$. Notice that $\mathcal{C}^+$ is a strictly convex cone with irrational slopes (the edges of the cone are generated by $(-\omega', 1)$ and $(\omega, -1)$).

Take Δ to be the convex hull of $N_\omega \cap \mathcal{C}^+$, and set $S = \partial\Delta \cap N_\omega$.

Explicitly, set $e_0 = (1, 0)$, $e_1 = (0, 1)$, and recursively e_n for all $n \in \mathbb{Z}$ by imposing $e_{n-1} + e_{n+1} = k_n e_n$. Then $S = \{e_n \mid n \in \mathbb{Z}\}$.

The fan Σ generated by S, i.e., whose only 0-face is the origin, the 1-faces are $\mathbb{R}_{\geq 0} e_n$ and the 2-faces are $\mathbb{R}_{\geq 0} e_n + \mathbb{R}_{\geq 0} e_{n+1}$ for all $n \in \mathbb{Z}$, is a regular fan of the open cone $\mathcal{C}^+_0 := \mathcal{C}^+ \cup \{0\}$.

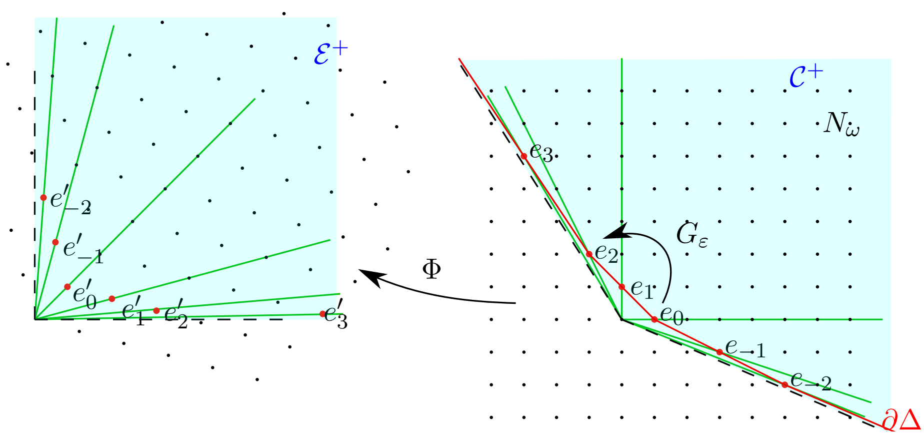


Figure 3.5: Arithmetic/toric construction for $\omega = \left[\,\overline{3,2}\,\right] = \frac{3+\sqrt{3}}{2}$, with $G_\varepsilon = \begin{pmatrix} -1 & -3 \\ 2 & 5 \end{pmatrix}$.

d	$\varepsilon^>$	$\mathfrak{o}^>:\mathfrak{o}^+$	ε^+	k	ω	$\mathfrak{o}^>:\mathfrak{o}^>_\omega$	$\varepsilon^>_\omega$	$\mathfrak{o}^>_\omega:\mathfrak{o}^+_\omega$	ε^+_ω
2	$1+\sqrt{2}$	2	$3+2\sqrt{2}$						
				$[\overline{4,2}]$	$2+\sqrt{2}$	1	$\omega-1$	2	$2\omega-1$
				$[\overline{2,4}]$	$1+\frac{1}{2}\sqrt{2}$	1	$2\omega-1$	2	$4\omega-1$
3	$2+\sqrt{3}$	1							
				$[\overline{4}]$	$2+\sqrt{3}$	1	ω	1	
				$[\overline{3,2}]$	$\frac{3}{2}+\frac{1}{2}\sqrt{3}$	1	$2\omega-1$	1	
5	$\frac{1}{2}+\frac{1}{2}\sqrt{5}$	2	$\frac{3}{2}+\frac{1}{2}\sqrt{5}$						
				$[\overline{3}]$	$\frac{3}{2}+\frac{1}{2}\sqrt{5}$	1	$\omega-1$	2	ω
				$[\overline{5,4}]$	$\frac{5}{2}+\sqrt{5}$	6	$4\omega-1$	1	
6	$5+2\sqrt{6}$	1							
				$[\overline{4,3}]$	$2+\frac{2}{3}\sqrt{6}$	1	$3\omega-1$	1	
				$[\overline{3,3,2}]$	$\frac{8}{5}+\frac{2}{5}\sqrt{6}$	1	$5\omega-3$	1	
7	$8+3\sqrt{7}$	1							
10	$3+\sqrt{10}$	2	$19+6\sqrt{10}$						
11	$10+3\sqrt{11}$	1							
13	$\frac{3}{2}+\frac{1}{2}\sqrt{13}$	2	$\frac{11}{2}+\frac{3}{2}\sqrt{13}$						
14	$15+4\sqrt{14}$	1							
15	$4+\sqrt{15}$	1							
17	$4+\sqrt{17}$	2	$33+8\sqrt{17}$						
19	$170+39\sqrt{19}$	1							
21	$\frac{5}{2}+\frac{1}{2}\sqrt{21}$	1							
				$[\overline{5}]$	$\frac{5}{2}+\frac{1}{2}\sqrt{21}$	1	ω	1	
				$[\overline{3,2,2}]$	$\frac{3}{2}+\frac{1}{6}\sqrt{21}$	1	$3\omega-2$	1	

Figure 3.4: This table gives the list of generators of $\mathfrak{o}^>$ and $\mathfrak{o}^+$ for $K=\mathbb{Q}(\sqrt{d})$ for $d \leq 21$. We also give explicit computations of the generators of $\mathfrak{o}^>_\omega$ and $\mathfrak{o}^+_\omega$ for some ω coming from the construction above.

The toric surface $X(\Sigma)$ associated to Σ is regular, and it contains an infinite chain of rational curves E_n corresponding to the points e_n, and with opposite self-intersection k_n, since by construction $e_{n-1} + e_{n+1} = k_n e_n$.

For any element $\varepsilon \in \mathfrak{o}_\omega^+$, consider the map $G_\varepsilon \colon \mathcal{C}_0^+ \circlearrowleft$ induced by the multiplication by ε in $\Bbbk$. Since $\varepsilon N_\omega = N_\omega$, the map G_ε leaves N_ω invariant, and induces an automorphism $g_\varepsilon \colon X(\Sigma) \circlearrowleft$ (the inverse given by $g_{\varepsilon^{-1}} = g_{\varepsilon'}$), which (as far as $\varepsilon \neq 1$), acts freely and properly discontinuously. Hence the quotient $X_\varepsilon^+ = X(\Sigma)/\langle g_\varepsilon \rangle$ is a complex surface, which has a cycle of rational curves, with opposite self-intersections given by s copies of $(k_0, \dots, k_{r-1})$ if $\varepsilon = (\varepsilon_\omega^+)^s$ (see [Oda88, Proposition 4.1]). We may contract this cycle in X_ε, obtaining a cusp singularity (X, x_0).

Remark 3.2.1. Notice that $\Phi_\omega \colon \mathbb{Q}^2 \to \mathbb{R}^2$ is $\mathbb{Q}$-linear, and it can be extended by continuity to a $\mathbb{R}$-linear map $\Phi_\omega \colon \mathbb{R}^2 \to \mathbb{R}^2$. Analogously, consider the quadratic form $Q(\alpha) = \alpha\alpha'$ on K. The map $Q_\omega = L_\omega \circ Q \colon \mathbb{Q}^2 \to \mathbb{R}$ can be extended to a continuous map $Q_\omega \colon \mathbb{R}^2 \to \mathbb{R}$. Notice that $Q_\omega(x, y) > 0$ as far as $\Phi_\omega(x, y) \in \mathcal{C}$.

If we denote by $\mathcal{S}_X$ the skeleton of (X, x_0), and by $\hat{\mathcal{S}}_X^*$ the cylinder $\mathcal{S}_X \times \mathbb{R}_+^*$, then there is a natural isomorphism $\tau \colon \hat{\mathcal{S}}_X \to \mathcal{C}/\langle g_\varepsilon \rangle =: \mathcal{C}_\varepsilon$. Denote by $[\cdot]$ the natural projection $\mathcal{C} \to \mathcal{C}_\varepsilon$. Then τ has the following properties :

- For any exceptional prime E_n, $n = 0, \dots, rs - 1$, $\tau(\mathrm{ord}_E) = [\Phi_\omega(e_n)]$.
- For any valuation $\nu \in \hat{\mathcal{S}}_X^*$ and any $\lambda > 0$, we have $\tau(\lambda\nu) = \lambda\tau(\nu)$.
- For any intersection point p_n of two exceptional primes E_n and E_{n+1}, the monomial valuation $\nu_{r,s}$ at p_n satisfying $Z_{\pi_0}(\nu_{r,s}) = r\check{E}_n + s\check{E}_{n+1}$ is sent by τ to $[\Phi_\omega(re_n + se_{n+1})]$.

Notice that two valuations $\nu, \mu \in \hat{\mathcal{S}}_X^*$ are proportional if and only if $\tau(\nu)$ and $\tau(\mu)$ belong to the same ray of $\mathcal{C}_0$. A normalization of valuations in $\hat{\mathcal{S}}_X^*$ corresponds to taking a non-vanishing section of the set of rays of $\mathcal{C}_0$. We can take for example the section given by $Q \equiv 1$. Notice that since $Q(\varepsilon) = 1$, this section is g_ε-invariant, and it induces a section of the rays in $\mathcal{C}_\varepsilon$.

3.2.3 Finite endomorphisms on cusps

We now describe an arithmetic construction which produces finite endomorphisms of cusps (X, x_0) constructed as above (see [Fav10, §2.5]).

We denote by $\mathfrak{c}_\omega^+$ the set of (totally) positive elements $\alpha \in \Bbbk$ so that $\alpha N_\omega \subseteq N_\omega$. For any $\alpha \in \mathfrak{c}_\omega^+$, the map $g_\alpha \colon \mathcal{C}_0 \circlearrowleft$ induces a morphism $G_\alpha \colon X_\infty \circlearrowleft$. This map will have in general indeterminacy points, and it is not an automorphism as far as α is not a unit. Since g_α commutes with g_ε, G_α induces a map on the quotient $X_\varepsilon = X_\infty/\langle G_\varepsilon \rangle$. By contracting the cycle of rational curves, we obtain a finite endomorphism $f_\alpha \colon (X, x_0) \circlearrowleft$.

Remark 3.2.2. The fact that $\alpha N_\omega \subset N_\omega$ guarantees that f_α can be expressed as a formal (rational) map (with respect to suitable coordinates), while the fact that the cone $\mathcal{C}$ is invariant assures that f_α defines a holomorphic map at x_0. Notice also that α needs to be an integer in $\Bbbk$. In fact, since α leaves N_ω invariant, we have in particular $\alpha^n \in N_\omega$ for all n. If α is not an integer, then $Q(\alpha^n)$ would have unbounded denominators, in contradiction with belonging to N_ω. Finally, we notice that f_α is finite, of topological degree $Q(\alpha) = \alpha\alpha' \in \mathbb{N}^*$.

The map $\gamma(t) = (\varepsilon^{2t}, 1) \cdot \mathbb{R}_{\geq 0}$, where $t \in \mathbb{R}$, gives a parameterization of the rays in $\mathcal{C}_0$ (see [Fav10, p. 413]). The action of g_ε on the rays of $\mathcal{C}_0$ corresponds to the translation by 1 on $\mathbb{R}$. By Remark 3.2.1 we deduce that the rays in $\mathcal{C}_0$, quotiented by the action of g_ε, are in 1-to-1 correspondence with the skeleton $\mathcal{S}_X$. In fact, the normalization of valuations, which corresponds to taking a section of $\hat{\mathcal{S}}^*_X$, here correspond to taking a non-vanishing section of the rays of $\mathcal{C}_0$ which is g_ε-invariant. We deduce that γ induces a parameterization $\gamma_X \colon \mathbb{R}/\mathbb{Z} \to \mathcal{S}_X$.

The action of g_α corresponds, with respect to this parameterization, to a translation by the value

$$\beta = \frac{\log\alpha - \log\alpha'}{2\log\varepsilon}. \tag{3.3}$$

By generalizing the examples found in [Fav10, §2.5], one can show the following.

Proposition 3.2.3 ([GR21, Proposition 6.6])**.** *Any cusp singularity (X, x_0) admits a finite germ $f \colon (X, x_0) \circlearrowleft$ whose action on $\mathcal{S}_X$ is conjugated to an irrational rotation.*

Proof. We may assume that (X, x_0) is constructed as in §3.2.2, and we look for f of the form f_α for some totally positive element $\alpha \in N_\omega \cap \varepsilon^{-1} N_\omega$.

To prove the statement, it suffice to find α so that the number β defined by Equation (3.3) belongs to $\mathbb{R} \setminus \mathbb{Q}$.

Notice that $\beta \in \mathbb{Q}$ if and only if some power of α/α' is an integer multiple of a power of ε^2. If we write $\varepsilon = a + b\sqrt{d}$, then one can show that $\alpha = p + b\sqrt{d}$, with $p - a \in \mathbb{N}^*$, preserves N_ω and is totally positive, hence it induces a finite germ f_α acting on (X, x_0). Moreover, by a direct computation, one can show that α/α' and its powers are not integral in $\mathbb{Q}(\sqrt{d})$ whenever $\left|p^2 - db^2\right| > 2bp$, which happens for $p \gg a$. For such values, we deduce that β is irrational, and we are done. $\square$

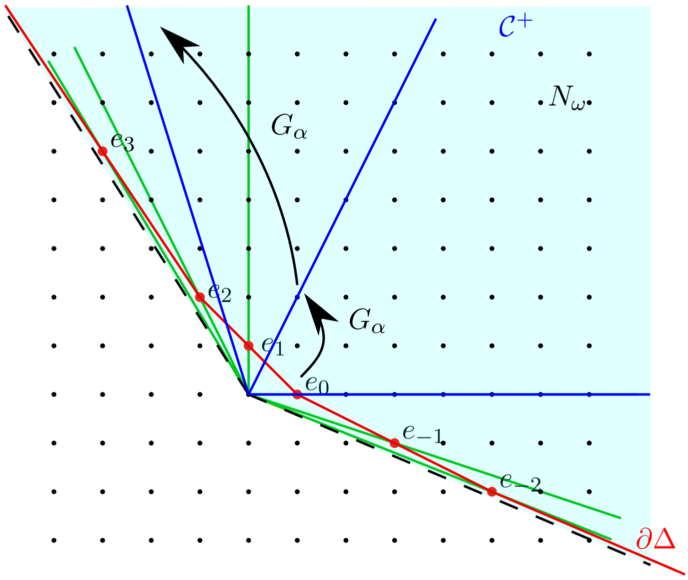


Figure 3.6: Action of the multiplication by $\alpha = \varepsilon + 2 = 2\omega + 1 = 4 + \sqrt{3}$, represented by $G_\alpha = \begin{pmatrix} 1 & -3 \\ 2 & 7 \end{pmatrix}$. The induced finite germ f_α has topological degree $13 = \det G_\alpha$, while $\beta = \frac{\log\left(\frac{19+8\sqrt{3}}{13}\right)}{2\log(2+\sqrt{3})} \in \mathbb{R} \setminus \mathbb{Q}$.

As a consequence of the existence of maps acting as irrational rotations on $\mathcal{S}_X$, we deduce that the angular distance on $\mathcal{S}_X \cong \mathbb{R}/\mathbb{Z}$ induces on it a Haar measure. Concretely, we get that for all $t, u \in \mathbb{R}/\mathbb{Z}$,

$$\rho(\gamma_X(t), \gamma_X(u)) = K d_{\mathbb{Z}}(t, u)$$

for some constant $K > 0$, where $d_{\mathbb{Z}}$ denotes the distance induced by the euclidean distance on $\mathbb{R}/\mathbb{Z}$.

This allows to show that the arithmetic construction of finite germs $f_\alpha \colon (X, x_0) \circlearrowleft$ given above is quite general, in the sense of the following statement.

Proposition 3.2.4. *Let $f \colon (X, x_0) \circlearrowleft$ be a finite germ on a cusp singularity (X, x_0) constructed as in Section 3.2.2. Suppose that $f_\bullet$ preserves the orientation of the circle $\mathcal{S}_X$ (it may be always assumed by replacing f by its second iterate). Then there exists $\alpha \in N_\omega$ so that the action of f_* on $\hat{\mathcal{S}}_X^*$ corresponds to the action of g_α on $\mathcal{C}$.*

Proof. The action of $f_\bullet$ corresponds, through the parameterization given by γ_X, to the translation by some $\beta \in \mathbb{R}$. Let $\alpha \in \mathbb{R}$ be so that $\varepsilon^{2\beta} = \alpha^2/Q(\alpha)$. Notice that this defines α up to positive multiplicative constants.

By local monomialization, we know that f_* acts on $\mathcal{C}^+$ as a piecewise linear map with integer coefficients. This implies that, up to rescaling by a suitable multiplicative constant,

f_* acts as the linear map g_α. Since f is holomorphic at x_0, we must have $g_\alpha N_\omega \subseteq N_\omega$, and $g_\alpha \mathcal{C}^+ \subset \mathcal{C}^+$, i.e., α is totally positive. $\square$

3.2.4 Sequence of attraction rates and non-existence of algebraically stable models

This section is devoted to study the sequence of attraction rates for finite germs on cusp singularities. The only situation that is not covered by Theorem 2.3.8 is when $f_\bullet$ acts on the circle $S = \{\nu \in \mathcal{V}_X \mid A(\nu) = 0\}$ as an irrational rotation. We shall show that in this case, no linear recursion relations are satisfied.

Theorem 3.2.5. *Let $f\colon (X, x_0) \circlearrowleft$ be a superattracting germ on a cusp which induces an irrational rotation on $\mathcal{S}_X$. Let $\nu \in \mathcal{V}_X^\alpha$ be any valuation of finite skewness. Then the sequence $c(f^n, \nu)$ does not satisfy any linear recursion relation. In particular, for any good resolution $\pi\colon X_\pi \to (X, x_0)$, the lift f_π of f at X_π is not algebraically stable.*

Sketch of proof. Reduction to $\nu \in \mathcal{S}_X$ First of all, we reduce ourself to the case of $\nu \in \mathcal{S}_X$: the attraction rate functional $c(f, \cdot)$ is locally constant outside the critical skeleton $\mathcal{S}_f$, which contains the skeleton $\mathcal{S}_X$. Being $\mathcal{S}_f$ a finite graph, $\mathcal{S}_f \setminus \mathcal{S}_X$ is contained in finitely many tangent directions. Being the action of $f_\bullet$ on $\mathcal{S}_X$ an irrational rotation, $f_\bullet^n(\nu)$ avoids these directions for $n \gg 0$, and in this case the the values of $c(f, \cdot)$ at $f_\bullet^n \nu$ concides with the one of its projection to $\mathcal{S}_X$.

Main argument In the next step, we adapt a strategy due to [HP07] to our setting.

The valuation ν corresponds to a point $[p] \in \mathcal{C}_\varepsilon$. Take any representative $p \in \mathcal{C}$. The sequence $c_n = c(f^n, \nu)$ can be computed as

$$c_n = (f_*^n \nu)(\mathfrak{m}) = \mathrm{ev}_{\mathfrak{m}}(g_\alpha^n p),$$

where $g_\alpha\colon \mathcal{C} \circlearrowleft$ is the multiplication by α, induced by the map $f_\alpha\colon (X, x_0) \circlearrowleft$ given by Proposition 3.2.4. The evaluation $\mathrm{ev}_{\mathfrak{m}}$ of the maximal ideal on $\mathcal{C}$ is piecewise linear: denote by L any of these linear maps, and define $d_n = L(g_\alpha^n p)$. Note that d_n satisfies a linear recursion relation. Suppose that c_n also eventually satisfies a recursion relation. Then so would the sequence $c_n - d_n$. By Skolem-Mahler-Lech's theorem, we conclude that $\{n : c_n = d_n\}$ consists of a finite union of arithmetic progressions. Since f gives an irrational rotation, the only way this can happen is if $\mathrm{ev}_{\mathfrak{m}} \equiv L$, which implies $c(f, \cdot)$ is constant on the cycle $\mathcal{S}_X$.

$c(f, \cdot)$ is not constant on $\mathcal{S}_X$. Suppose by contradiction that $c(f, \cdot) \equiv C$ is constant on $\mathcal{S}_X$. Let $\pi\colon X_\pi \to (X, x_0)$ be a log-resolution of $\mathfrak{m}_X$. We can write

$$Z(\mathfrak{m}_X) = \sum_{E \in \Gamma_\pi^*} \lambda_E Z(\nu_E),$$

for some $\lambda_E \geq 0$. Denote by $\Gamma_\pi^*(\mathcal{S}_X)$ the exceptional primes $E \in \Gamma_\pi^*$ for which $\nu_E \in \mathcal{S}_X$. Since the divisorial valuation of the exceptional primes in a minimal resolution of (X, x_0) lie in $\mathcal{S}_X$, there exists $F \in \Gamma_\pi^*(\mathcal{S}_X)$ such that $\lambda_F > 0$.

Recall that $f_\bullet$ acts as an isometry on $\mathcal{S}_X$, and has infinite order: up to replacing f with an iterate, we may assume that the center of $f_\bullet^{-1}(\nu_F)$ in X_π is a point $p = D_1 \cap D_2$ at the intersection of two exceptional primes $D_1, D_2 \in \Gamma_\pi^*(\mathcal{S}_X)$.

Let now μ be a monomial valuation at p, of weights $s_1, s_2 > 0$ satisfying $s_1 b_{D_1} + s_2 b_{D_2} = 1$. In this case, we obtain the contradiction

$$C = c(f,\mu) = \langle \mu, f^*\mathfrak{m}_X\rangle > s_1 b_{D_1} \langle \nu_{D_1}, f^*\mathfrak{m}_X\rangle + s_2 b_{D_2} \langle \nu_{D_2}, f^*\mathfrak{m}_X\rangle = s_1 b_{D_1} C + s_2 b_{D_2} C = C,$$

where the inequality is strict because $\lambda_E > 0$, and $f_\bullet^{-1}(\nu_E)$ and μ have the same center in X_π. □

Remark 3.2.6. Even though $c(f^n,\nu)$ does not satisfy any linear recursion relation, it turns out that the first dynamical degree $c_\infty(f)$ is a quadratic integer. In fact, if f acts as f_α on $\mathcal{S}_X$, then $c_\infty(f)$ is the spectral radius of the matrix $A := G_\alpha$ representing the action of multiplication by α with respect to the basis $(1,\omega)$. This can be seen by noticing that any valuation $\nu \in \mathcal{S}_X$ correspond to the orbit by the multiplication by G_ε of a vector $v \in \mathcal{C}^+$. Normalization corresponds to multiplication by a scalar, which gives another element in $[v] := \mathbb{R}_{>0} v$. The value $c(f^n,\nu)$ is then computed as

$$c(f^n,\nu) = \frac{\|A^n v\|}{m[A^n v]} \cdot \frac{m[v]}{\|v\|},$$

where $m[v]$ corresponds to the normalization factor: it is the value of $\|v\|$ such that the corresponding valuation ν satisfies $\nu(\mathfrak{m}_X) = 1$. By continuity and compactness of $\mathcal{S}_X$, the functional $[v]$ is bounded away from 0 and ∞, and we deduce that $\sqrt{c(f^n,\nu)}$ converges to the spectral radius of A (regardless if the action on $\mathcal{S}_X$ is an irrational rotation or not).

3.2.5 Dual cusps and Inoue-Hirzebruch surfaces

Proposition 3.2.4 states that for any finite germ f on a cusp singularity for which the action on the circle $\mathcal{S}_X$ is a rotation, then this action coincides with the one of a finite germ f_α constructed arithmetically as above. Of course, in general f and f_α do not coincide. In fact, any singularity (X,x_0) admits lots of non-commuting automorphisms, coming from the time-1 flow of vector fields tangent to the singularity (see [Mül87]). We may compose f_α by any such automorphism of (X,x_0), obtaining another germ f. Notice that this operation does not change the action induced on $\mathcal{S}_X$. In particular there exists infinitely many different finite germs whose action on $\mathcal{S}_X$ coincide. Nevertheless, we may wonder if all such germs are analytically conjugated one to another.

Question 3.2.7. *Is any finite germ $f\colon (X,x_0) \to (X,x_0)$ on a cusp singularity (X,x_0), whose action preserves the orientation of the circle $\mathcal{S}_X$, analytically conjugated to a finite germ of the form f_α, for some $\alpha \in N_\omega$?*

In this section we present a strategy to attack this question, and some of the difficulties that arise.

Cusp singularities come in dual pairs: given (X, x_0) a cusp singularity, we can construct, from the combinatorial data of the opposite self-intersections $(k_0, \ldots, k_{r-1})$, another cusp singularity $(\check{X}, \check{x}_0)$, as follows.

Suppose that (X, x_0) is given by its toric construction, associated to the quadratic irrational number ω, and the totally positive unit ε.

Instead of considering the totally positive cone $\mathcal{C}^+ = \Phi^{-1}(\mathcal{E}^+)$, we consider its supplementary cone

$$\mathcal{C}^- := \Phi^{-1}(\mathcal{E}^-) = \{x + y\omega \mid x + \omega y > 0, x + \omega' y < 0\},$$

where $\mathcal{E}^- = \mathbb{R}_{>0} \times \mathbb{R}_{<0}$. The action induced by the multiplication by the totally positive unit ε leaves $\mathcal{C}^-$ invariant, since $(\varepsilon\xi)' = \varepsilon'\xi' < 0$ for any ξ so that $\xi' < 0$. As above, we can build a fan $\check{\Sigma}$, and a toric surface $X(\check{\Sigma})$, with a free and properly discontinuous action $\check{g}_\varepsilon$ induced by the multiplication by ε. The quotient $X_\varepsilon^- = X(\check{\Sigma})/\langle \check{g}_\varepsilon \rangle$ is a complex surface having a cycle of rational curves, with opposite self-intersections $(\check{k}_0, \ldots, \check{k}_{s\check{r}-1})$, and we get the dual cusp $(\check{X}, \check{x}_0)$.

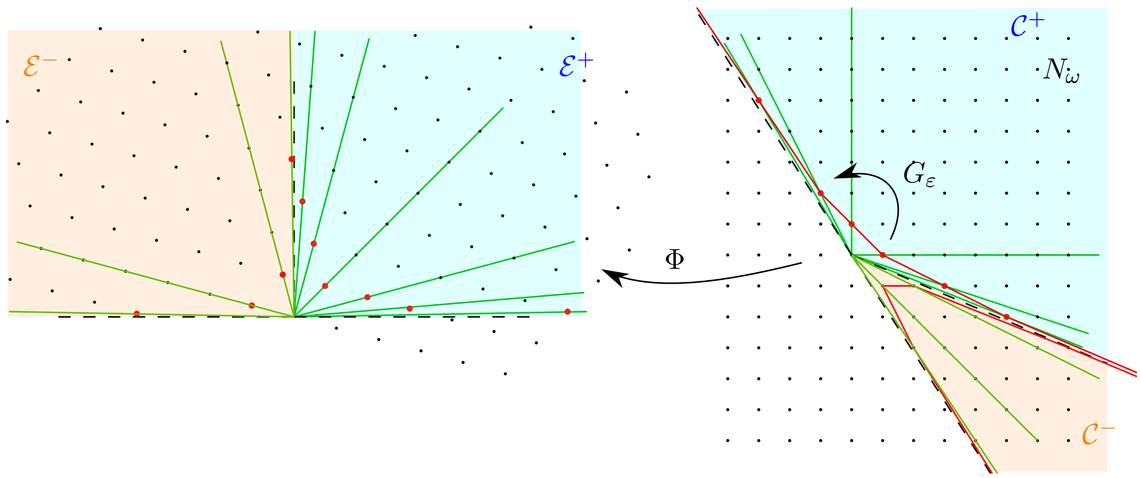


Figure 3.7: Dual cusp for $k = (3, 2)$, $\check{k} = (4)$.

Remark 3.2.8. One can compute combinatorially the self-intersections $(\check{k}_0, \ldots, \check{k}_{s\check{r}-1})$ starting from $(k_0, \ldots, k_{r-1})$ as follows.

Up to relabelling the indices (in cyclic order), we may assume that $k_0 \geq 3$. Then we can regroup the vector $(k_0, \ldots, k_{r-1})$ into a sequence of the form

$$3 + n_1, \overbrace{2, \ldots, 2}^{m_1}, 3 + n_2, \overbrace{2, \ldots, 2}^{m_2}, \cdots, 3 + n_h, \overbrace{2, \ldots, 2}^{m_h},$$

where $n_j, m_j \geq 0$. Then the dual cusp is associated to the vector

$$\overbrace{2,\dots,2}^{n_1}, 3+m_1, \overbrace{2,\dots,2}^{n_2}, 3+m_2, \cdots, \overbrace{2,\dots,2}^{n_h}, 3+m_h.$$

For example, the dual of $k = (3, 2)$ is $\check{k} = (4)$, while the dual of $k = (5, 4, 2, 2)$ is $\check{k} = (2, 2, 3, 2, 5)$.

Pairs of dual cusps can be also obtained as the contraction of the two rational cycles in a special class of Kato surfaces, called Inoue-Hirzebruch surfaces (or hyperbolic Inoue surfaces). They are obtained from a *monomial* Kato germ $g\colon (\mathbb{C}^2, 0) \circlearrowleft$, given by $g(x, y) = (x^a y^b, x^c y^d)$, with $ad - bc = 1$.

A finite germ $f_\alpha\colon (X, x_0) \circlearrowleft$ defined as in Section 3.2.3 induces, always by multiplication by α on $\mathcal{C}^-$, a finite germ $\check{f}_\alpha$ on $(\check{X}, \check{x}_0)$, that we can imagine as the dual of f_α acting on (X, x_0). These germs can be also constructed from the Kato construction, by considering a monomial germ $h_\alpha\colon (\mathbb{C}^2, 0) \circlearrowleft$, commuting with g.

The focus of the problem can be then shifted to constructing a germ h, that coincides with h_α up to invertibles, and inducing the germ f_α on (X, x_0). If this is possible, we can conclude by studying the simultaneous monomialization of commuting germs.

3.3 Contracting automorphisms and quasi-homogeneous singularities

3.3.1 Singularities admitting a contracting automorphism

We focus here on contracting automorphisms on normal surface singularities.

Definition 3.3.1. An automorphism $f : (X, x_0) \circlearrowleft$ on a normal surface singularity is a germ admitting a representant X such that $f(X)$ is relatively compact in X, and $\bigcup_{n\in\mathbb{N}} f^n(X) = \{x_0\}$.

Our goal is to classify singularities admitting contracting automorphisms, and to find normal forms for such contracting automorphisms. We start with two classes of examples.

Example 3.3.2. Let $\lambda, \mu \in \mathbb{C}^*$ be such that $1 > |\lambda| \geq |\mu| > 0$. Consider the contracting automorphism $\widetilde{f}\colon (\mathbb{C}^2, 0) \circlearrowleft$ in its *Poincaré-Dulac normal form*, i.e., given by

$$\widetilde{f}(x, y) = (\lambda x, \mu y + \varepsilon x^u),$$

where $\varepsilon \in \mathbb{C}$, and $u \in \mathbb{N}^*$ satisfies $\varepsilon(\mu - \lambda^u) = 0$.

Let now G be the finite group generated by $g : (x, y) \mapsto (\zeta x, \zeta^q y)$, where ζ is a primitive p-th root of unity, and p, q are coprime. Assume that $\widetilde{f}$ leaves G invariant, i.e., $\widetilde{f} \circ g = g^k \circ \widetilde{f}$ for some $k \in \{1, \dots, p\}$. Then $\widetilde{f}$ passes to the quotient, and defines a contracting automorphism $f : (X, x_0) \circlearrowleft$ on the cyclic quotient singularity $(X, x_0) = (\mathbb{C}^2, 0)/G$.

By a direct computation, the G-invariance condition is satisfied if and only if either $\varepsilon = 0$ (linear case), or $\varepsilon \neq 0$ and $u \equiv q \pmod{p}$ (resonant case).

Example 3.3.3. Pick $w = (w_1, \dots, w_d) \in (\mathbb{N}^*)^d$ and suppose we are given a family of weighted homogeneous polynomials $P_1, \dots, P_k$ satisfying $P_i(t^{w_1}x_1, \dots, t^{w_n}x_n) = t^{d_i}P_i(x_1, \dots, x_n)$ for all $t \in \mathbb{C}^*$ and all $x = (x_1, \dots, x_d) \in \mathbb{C}^d$ and for some $d_i \in \mathbb{N}^*$.

Any normal surface singularity isomorphic to the common zero locus $X = \bigcap_{i=1}^k \{x \in \mathbb{C}^d \mid P_i(x) = 0\}$ at the origin $x_0 = 0$ is called *quasi-homogeneous* (of weighted homogeneous), with respect to the weight w.

The map $f_t(x) = \left(t^{w_1}x_1, \dots, t^{w_n}x_n\right)$ is then an automorphism of $(\mathbb{C}^d, 0)$ that is contracting as soon as $|t| < 1$, and leaves X invariant. Hence f_t induces a contracting automorphism on (X, x_0).

It turns out that these two classes of examples are essentially all the possible constructions of contracting automorphisms on normal surface singularities.

Theorem 3.3.4. *A normal surface singularity (X, x_0) admits a contracting automorphism if and only if it is quasi-homogeneous.*

Moreover, if $f : (X, x_0) \circlearrowleft$ is a contracting automorphism, then one of its iterates is conjugated to one of the maps described in Examples 3.3.2, 3.3.3.

This statement is a simplified version of [FR14, Theorem 7.5], where we explicit normal forms directly for the contracting automorphisms, and not up to iterates as we did here.

3.3.2 Description of the orbit spaces

Given a contracting automorphism $f : (X, x_0) \circlearrowleft$ on a normal surface singularity, we may consider its space of orbit $S(f)$, defined as

$$S(f) := X \setminus \{x_0\} \big/ \langle f \rangle.$$

Corollary 3.3.5. *Let $f : (X, x_0) \circlearrowleft$ be a contracting automorphism on a (quasi-homogeneous) surface singularity. Denote by $S(f)$ the complex surface obtained by quotienting $X \setminus \{x_0\}$ by the action of f.*

(a) The surface $S(f)$ is compact non-Kähler.

(b) Either $S(f)$ is isomorphic to a Hopf surface (in the case of Example 3.3.2), or it is the quotient of a principal elliptic fibration L over a smooth curve E, by the action of a finite group G acting freely on L and preserving the fibration.

(c) Depending on the Kodaira dimension $\kappa(S(f))$, we have the following situations:

- *If $\kappa(S(f)) = -\infty$, then $S(f)$ is a Hopf surface.*
- *If $\kappa(S(f)) = 0$, then $S(f)$ is a Kodaira surface.*
- *If $\kappa(S(f)) = 1$, then $S(f)$ is the quotient of an elliptic fibration over an hyperbolic curve.*

This result is a consequence of Theorem 3.3.4: up to finite quotients, we either have that (X, x_0) is a cyclic quotient singularity, and in this case we get Hopf surfaces, or (X, x_0) is a cone singularity over a smooth curve E, and f acts on the fibers of the cone as the multiplication by λ with $|\lambda| < 1$. In this case, we get the principal elliptic fibration over E. Moreover, the Kodaira dimension of $S(f)$ is exactly the Kodaira dimension of E. If E is a rational curve, we get again a Hopf surface; if E is an elliptic curve, we get a Kodaira surface; the third case corresponds exactly to E being a hyperbolic curve.

Remark 3.3.6. With respect to Kato's classification [Kat79], this situation corresponds to case α. In this case, Kato shows that $S(f)$ can be embeded into a higher-dimensional Hopf manifold, and concludes from there.

3.3.3 Nice resolutions and construction of a Green function

Let $f : (X, x_0) \circlearrowleft$ be a contracting automorphism of a normal surface singularity. As usual, we are led to consider good resolutions $\pi\colon X_\pi \to (X, x_0)$; the automorphism f lifts to a bimeromorphic map $f_\pi\colon X_\pi \circlearrowleft$, which may have indeterminacy points.

But among possible resolutions, we can find special ones that have the property that any automorphism of (X, x_0) lifts to an automorphism of X_π.

Definition 3.3.7. Let (X, x_0) be a normal surface singularity, and denote by $\mathrm{Aut}(X, x_0)$ its group of automorphisms. Let $\pi\colon X_\pi \to (X, x_0)$ be a modification. We say that $\mathrm{Aut}(X, x_0)$ *lifts regularly* to π (or to X_π) if the lift $f_\pi := \pi^{-1} \circ f \circ \pi$ is regular for any $f \in \mathrm{Aut}(X, x_0)$.

If moreover π is a log-resolution of the maximal ideal $\mathfrak{m}_X$, we say that π is a *nice resolution*.

Proposition 3.3.8. *Any normal surface singularity admits a nice resolution.*

Proof. Let (X, x_0) be a normal surface singularity, and denote by $\mathfrak{m}_X$ the associated maximal ideal. Since $\mathrm{Aut}(X, x_0)$ leaves $\mathfrak{m}_X$ invariant, it lifts regularly to the (normalized) blow-up $\pi_\mathfrak{m}\colon X_\mathfrak{m} \to (X, x_0)$ of $\mathfrak{m}_X$ (this is a consequence of the universal property of the blow-up of an ideal).

The space $X_\mathfrak{m}$ might be singular. In this case, its singularities are either fixed by the action of $\mathrm{Aut}(X, x_0)$, or two such singularities are isomorphic, the isomorphism given by the lift of an element of $\mathrm{Aut}(X, x_0)$. We deduce that the action $\mathrm{Aut}(X, x_0)$ lifts regularly to the model $X_{\pi'}$ obtained from $X_\mathfrak{m}$ by taking the minimal resolutions of the singularities of $X_\mathfrak{m}$.

We might still have that the exceptional divisor of π' is not SNC. But again such singularities are either fixed or isomorphic by the action of $\mathrm{Aut}(X, x_0)$, and we can get a good resolution $\pi\colon X_\pi \to (X, x_0)$ by blowing up (maybe more then once) those points where $(\pi')^{-1}(x_0)$ is not SNC. □

From now on, we fix a nice resolution $\eta\colon X_\eta \to (X, x_0)$. What we need to do next is to have a way to control the contracting behavior of automorphisms on the model X_ν. This is done by constructing a plurisubharmonic (psh) function $g\colon X_\eta \to [-\infty, +\infty)$, for which $\nu^{-1}(x_0) = g^{-1}(-\infty)$, which measures in some sense the distance from the exceptional divisor, and reduces uniformly under the action of the lift f_η of the automorphism we started with.

Proposition 3.3.9. *Let (X, x_0) be a normal surface singularity, and let $\eta\colon X_\eta \to (X, x_0)$ be a nice resolution. There exists a psh function $g\colon X_\eta \to [-\infty, +\infty)$ that is smooth on $X \setminus W$, and, for any contracting automorphism f satisfies the following conditions:*

- *g is bounded below on $X_\eta \setminus f_\eta^n(X_\eta)$ for all $n \geq 1$;*
- *for any constant $C > 0$, there exists an integer $n \geq 1$ such that $g \circ f_\eta \leq g - C$ on X_η.*

The proof of Proposition 3.3.9 is technical in nature, but it is rather easy to say what g is. We start from a set $\chi_1, \ldots, \chi_r$ of generators of $\mathfrak{m}_X$, and take

$$g_0(x) := \log\left(\sum_{i=1}^{r} |\chi_i(x)|^2\right).$$

The function g_0 is psh, an is smooth on $X \setminus \{x_0\}$. We pull it back to X_η, and take $g := g_0 \circ \eta$.

While it is a direct computation to show that g satisfies the first property of Proposition 3.3.9, it is more delicate to show the last property: we need to compare g with another psh function obtained by taking the upper envelope of psh functions on X_η that are more singular than g_0 along $\eta^{-1}(x_0)$, following [RS05] (see [FR14, Section 3.2] for further details).

3.3.4 The dual graph is star-shaped

Using the contracting property of the map f_η on X_η, as measured by the Green function g, we can finally show how the mere existence of a contracting automorphism strongly costricts the shape of the dual graph Γ_η.

This is done by studying the action of $F := f_\eta$ on the support W of the exceptional divisor $\eta^{-1}(x_0)$. First of all, up to replacing f by one of its iterates, we may assume that F fixes all exceptional primes of η. In this situation, $h := F|_E \colon E \circlearrowleft$ is an automorphism for any $E \in \Gamma_\eta^*$.

We recall here a few properties of automorphisms h of a Riemann surface E.

- If E is an hyperbolic curve, then the group of automorphisms of E has finite order.
- If E is an elliptic curve, then either $h \in \mathrm{Aut}(E)$ has finite order, or it is an infinite order translation (in particular, it has no fixed points in this case).
- The automorphisms of $\mathbb{P}^1$ are the Möbius transformations. A Möbius transformation $h \in \mathrm{Aut}(\mathbb{P}^1)$ always fixes at least one point, that we may assume to be ∞, so that h is written as $h(z) = az + b$ for some $a \in \mathbb{C}^*$ and $b \in \mathbb{Z}$. If $a = 1$ and $b \neq 0$, we have a translation (elliptic case). If $a \neq 1$, we have another fixed point, that we may assume to be 0, i.e., $b = 0$. In this case, either $|a| \neq 1$ (hyperbolic case), and we have exactly two fixed points, or $|a| = 1$: h is a rotation, and we have either exactly two fixed points, or h has finite order.

For convenience, we say that an automorphism $h \in \mathrm{Aut}(E)$ of a compact Riemann surface E is *hyperbolic* if E is the Riemann sphere and h is hyperbolic in the sense described above, i.e., it has one contracting and one repelling fixed points. Notice that if the automorphism h is not hyperbolic, then $|h'(p)| = 1$ for any fixed point p.

We also recall that a *chain* (resp., *cycle*) of rational curves on X is a finite collection $E_1, \ldots, E_n$ of smooth rational curves intersecting transversally whose dual graph is a segment (resp., a cycle).

Moreover, a graph is said to be *star-shaped* if it is a tree and admits at most one branched point.

Finally, a rational exceptional prime E is called *negative* if $E \cdot E \leq -2$.

We can finally state the main result of this subsection.

Theorem 3.3.10. *Let (X, x_0) be a normal surface singularity, $\eta \colon X_\eta \to (X, x_0)$ be a nice resolution, and $f : (X, x_0) \circlearrowleft$ be a contracting automorphism. Denote by $W = \eta^{-1}(x_0)$ the support of exceptional divisor, and by $F = f_\eta = \eta^{-1} \circ f \circ \eta$ the lift of f to X_η. Suppose that $F(E) = E$ for any exceptional prime $E \in \Gamma_\eta^*$. Then we are in one of the following situations.*

(a) W is a chain of negative rational curves. The action of $F|_E$ is hyperbolic for all irreducible components E of W, but at most one.

(b) There exists a component $E_\star$ with $E_\star \cdot E_\star \leq -1$ such that $F|_{E_\star}$ has finite order. In this case, the closure of $W \setminus E_\star$ is a disjoint union of a finitely many chains of negative rational curves, and $F|_E$ is hyperbolic for any component E of W different from $E_\star$.

Notice that cases (a) and (b) are not mutually exclusive: the case of the support of W being a chain of negative rational curves, with one component $E_\star$ such that $F|_{E_\star}$ has finite order, satisfies both (a) and (b).

A key element of the proof of Theorem 3.3.10 is the following lemma.

Lemma 3.3.11. *Pick any point $p \in W$ that is fixed by F.*

- *When p belongs to a unique exceptional component E, then $dF(p)$ admits an eigenvalue of modulus < 1 whose eigenvector is transverse to E.*
- *When p is the intersection point of two irreducible components E, E' of W, then we have*
$$\frac{\log \left|dF|_E(p)\right|}{b_E} + \frac{\log \left|dF|_{E'}(p)\right|}{b_{E'}} < 0, \tag{3.4}$$
where $b_E = \mathrm{div}_E(\mathfrak{m}_X)$, and similarly for $b_{E'}$.

Proof. We treat only the second case, the first being completely analogous and easier. We denote by g the Green function given by Proposition 3.3.9.

Pick local coordinates (z, w) at p such that $E = \{z = 0\}$ and $E' = \{w = 0\}$. Since η is a log-resolution of the maximal ideal, in these coordinates we have
$$\left|g(z, w) - b_E \log |z| - b_{E'} \log |w|\right| \leq C_1$$

for some $C_1 > 0$.

Set $\lambda = (dF|_{E'})(p)$ and $\mu = (dF|_E)(p)$. Then for any integer $n \geq 0$, one can write

$$F^n(z, w) = \big(\lambda^n z(1 + \delta_n), \mu^n w(1 + \varepsilon_n)\big)$$

with $\delta_n(0) = \varepsilon_n(0) = 0$. Pick $C > 2C_1$. By Proposition 3.3.9, for any $C > 0$ there exists $n \gg 0$ so that

$$g(F^n(z, w)) \leq g(z, w) - C.$$

It follows

$$b_E \log|\lambda^n z| + b_{E'} \log|\mu^n w| + o(1) - C_1 \leq b_E \log|z| + b_{E'} \log|w| + C_1 - C,$$

and hence

$$n\big(b_E \log|\lambda| + b_{E'} \log|\mu|\big) \leq 2C_1 - C + o(1).$$

Letting $(z, w) \to 0$, we get $b_E \log|\lambda| + b_{E'} \log|\mu| < 0$.

We conclude by dividing the last relation by $b_E b_{E'} > 0$. □

Proof of Theorem 3.3.10. Suppose that there exists $E_\star$ so that $h_\star := F|_{E_\star}$ is not hyperbolic. Let E be any exceptional component intersecting $E_\star$ at a point p, and set $h := F|_E$. By assumption, $|h'_\star(p)| = 1$, and by Lemma 3.3.11, we infer $\log|h'(p)| < 0$, i.e., h is hyperbolic, with an attracting fixed point in p. In particular, there is at most one other exceptional component E' in W intersecting E in the other fixed point $q \neq p$ of h (which is repelling). Since $|h'(q)| > 1$, again by Lemma 3.3.11 we infer that $F|_{E'}$ is hyperbolic. By applying recursively this argument, we show that Γ_η is star-shaped in this case.

Notice that we cannot form cycles in this situation, since we would need for $E_\star$ to belong to the cycle, and have an attracting fixed point for $h_\star$, against the hypothesis.

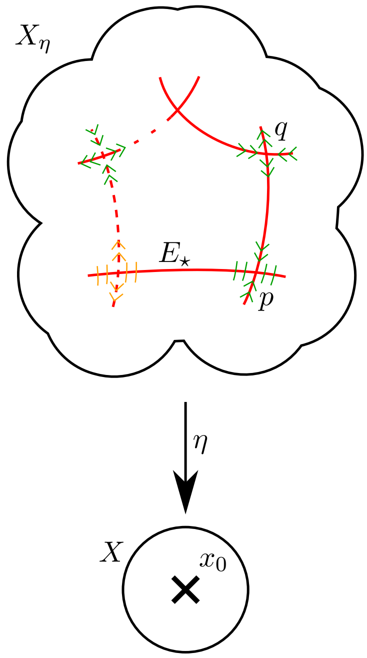


Figure 3.8: Dynamics of F on a nice resolution $\eta\colon X_\eta \to (X, x_0)$, when $F|_{E_\star} = \mathrm{id}_{E_\star}$. The dynamical situation on the dashed component described in orange contradicts Lemma 3.3.11, which prevents the existence of cycles.

Another situation to avoid is when $h_\star$ is not hyperbolic, and also not of finite order. In this case, we necessarily have that $W = E_\star$ is elliptic, and $h_\star$ is a translation. This situation can be ruled out by remarking that the action of $h_\star$ via pullback leaves the normal bundle $N_\star$ of $E_\star$ in X_η invariant. If $h_\star$ has infinite order, this implies that the degree of $N_\star$ is 0, a contradiction, since this degree must be negative by Grauert [Gra62].

We are hence in case (b) (apart from the part of the statement regarding the negativity of the rational curves).

Suppose now that $F|_E$ is hyperbolic on any exceptional component. Arguing as above, we can show that W is either a chain or a cycle of rational curves. The first situation is contained in case (a). The latter case can be ruled out by applying again Lemma 3.3.11, and using the fact that if h is an hyperbolic automorphism, and p, q are the two fixed points of h, then $h'(p)h'(q) = 1$, hence $\log|h'(p)| = -\log|h'(q)|$. We obtain a contradiction by plugging in this information into the sum of equations Equation (3.4) applied to the points p in each intersection of two adjacent exceptional components in the cycle.

To conclude, the part of the statement about the negativity of rational curves is obtained by noticing that, since the intersection form on W is negative definite, all self-intersections must be ≤ -1, and if E is a rational curve belonging to a chain with self-intersection -1, we can contract E to a point, shortening the chain by one. $\square$

3.3.5 Orbifolds and orbibundles

We have shown that the dual graph Γ_η of a nice resolution $\eta\colon X_\eta \to (X, x_0)$ of a singularity admitting a contracting automorphism is star-shaped. It is also known by the works of Orlik and Wagreich [OW71] that quasi-homogeneous singularities have this property. But star-shaped graphs are not taut: in other terms, having a star-shaped dual graph is not sufficient to conclude that the singularity is quasi-homogeneous.

In order to prove that we indeed have quasi-homogeneous singularities, we proceed as follows. Suppose first that we are in case (a) of Theorem 3.3.10. In this case, we can contract the chain of rational curves, and get a cyclic quotient singularity (also known as *Hirzebruch-Jung singularity*). Hence $(X, x_0) \cong {(\mathbb{C}^2,0)}\big/{G}$, where G is the finite group generated by $g : (x, y) \mapsto (\zeta x, \zeta^q y)$ for a primitive p-th root of unity, and p, q coprime. By studying G-invariant Poincaré-Dulac normal forms (up to iterate), we fall exactly in the situation of Example 3.3.2.

Suppose now that we are in case (b) of Theorem 3.3.10. Up to replacing f by an iterate if necessary, we may assume that $F|_{E_\star} = \mathrm{id}_{E_\star}$. Also in this case we contract all the chains of rational curves, obtaining a model $\widetilde{X}$ with a single exceptional component, that we denote by $\widetilde{E}_\star$, and finitely many Hirzebruch-Yung singularities $\widetilde{p}_1, \dots, \widetilde{p}_r$, one for each chain. The map F descends to the quotient to an automorphism $\widetilde{F}$ of $\widetilde{X}$.

At any singularity p_j, we can describe $(\widetilde{X}, \widetilde{p}_j) \cong {(\mathbb{C}^2,0)}\big/{G_j}$, and find coordinates at $(\mathbb{C}^2, 0)$ such that the map $\widetilde{F}$ lifts to a map of the form $(x, y) \mapsto (x, \lambda y)$, where $\{y = 0\}$ projects to $\widetilde{E}_\star$, and $|\lambda| < 1$.

Similarly, at any smooth point $\widetilde{p}$ of $\widetilde{E}_\star$, we can find local coordinates (x, y) so that $\widetilde{E}_\star = \{y = 0\}$ and $\widetilde{F}(x, y) = (x, \lambda y)$.

In any of these coordinates, the curves $x = \text{const}$ are in fact the stable manifolds at the corresponding fixed point $\widetilde{p} \in \widetilde{E}_\star$, $\widetilde{F}$ has two eigenvalues, one that equals 1, corresponding to the action $\widetilde{F}|_{\widetilde{E}_\star} = \mathrm{id}_{\widetilde{E}_\star}$, and another λ of modulus < 1 by Lemma 3.3.11, which is constant along $\widetilde{E}_\star$ by compactness.

We deduce that these coordinates patch together along $\widetilde{E}_\star$, and show that Y is in fact isomorphic to the neighborhood of the zero section of an orbibundle $\widetilde{L}$ over $\widetilde{E}_\star$.

We recall that an orbifold (of dimension 1 over $\mathbb{C}$) locally modeled by sets of the form ${(\mathbb{C},0)}\big/{z \mapsto \zeta z}$, where ζ is a m-th root of unity. Since this local model is topologically equivalent to $(\mathbb{C}, 0)$, this boils down to taking a Riemann surface E, and an effective divisor $\sum_i m_i p_i$, where $m_i \in \mathbb{N}^*$ and $p_i \in E$.

Not all orbifolds can be obtained as the quotient of a Riemann surface $\hat{E}$ by the global action of a group $\hat{G}$. When this happens, the orbifold is called *good*. In our setting, all orbifolds are good, but for weighted projective spaces, i.e., for orbifolds over $\mathbb{P}^1_{\mathbb{C}}$ with associated divisor $m_1 p_1 + m_2 p_2$.

An orbibundle over an orbifold E is locally modeled on $\mathbb{D}\times\mathbb{C}/(\zeta z, \zeta^q w)$, where ζ is a m-th root of unity, and m and q are coprime. Moreover, the natural projection on the first coordinate gives the identification of $\mathbb{D}/(\zeta z)$ with the orbifold chart on E.

When an orbibundle L is over a good orbifold E, it turns out that there exists a (genuine) compact Riemann surface E', a line bundle L' over E', and a finite group G' acting linearly on the fibers of L', such that L is isomorphic to the quotient of L' by G'. We refer to [FS92, RT11, Sat57, Sco83] for further details on this topic.

In our case, the orbibundle we obtained is over $\widetilde{E}_\star$. By assumption, $\widetilde{E}_\star$ is either of genus ≥ 1, or rational with at least three singularities (since if we have ≤ 2 singularities, the fall back to the case of W being a chain of rational curves, that we have already dealt with). Hence $\widetilde{E}_\star$ is a good orbifold, $\widetilde{L}$ is an orbibundle that can be described as the quotient by a group G' of a line bundle L' over $E'_\star \cong E_\star$. By replacing f by its $|G'|$-th iterate, we get the case of a contracting automorphism f on a cone singularity (X, x_0) (i.e., a singularity obtained by contracting the zero section of a negative-degree line bundle over a curve), which can be described within the framework of Example 3.3.3.

3.4 Kato germs and sandwiched singularities

3.4.1 Selfsimilarity for normal surface singularities

In [FFR20], we shift the focus from Kato surfaces to the Kato datum, and to the singularities they are defined on.

Definition 3.4.1. A normal surface singularity (X, x_0) is called *selfsimilar* if there exists a Kato datum (π, σ) on (X, x_0), with π that is not a local diffeomorphism.

In other terms, there exists a non-trivial modification $\pi\colon X' \to (X, x_0)$, and a point $x_1 \in \pi^{-1}(x_0)$, so that (X', x_1) is isomorphic to (X, x_0).

Our main result is the classification of selfsimilar normal surface singularities.

Theorem 3.4.2 ([FFR20, Theorem A])**.** *Let (X, x_0) be a normal surface singularity. Then (X, x_0) is selfsimilar if and only if it is sandwiched.*

Of course, this is useless if we don't recall what a sandwiched singularity is.

Definition 3.4.3. A normal surface singularity (X, x_0) is *sandwiched* if it birationally dominates $(\mathbb{C}^2, 0)$.

Remark 3.4.4. Since all surface singularities (X, x_0) admit a resolution $\mu\colon Z \to (X, x_0)$, being sandwiched corresponds to having the diagram:

$$\begin{tikzcd} Z \arrow[d, "\mu"] \\ (X, x_0) \arrow[d, "\eta"] \\ (\mathbb{C}^2, 0) \end{tikzcd},$$

where μ is a resolution as above, and η is a proper bimeromorphic map. This picture give to sandwiched singularities their name: they are bimeromorphically between two smooth models.

Sandwiched singularities have been introduced by Spivakovski [Spi90], where they play a crucial role in the proof of resolution of singularities of surfaces via Nash transformations. Several authors have further contributed to the study of sandwiched singularities, for example their deformation theory has been investigated by De Jong and Van Straten [dJvS98], while their Milnor fibers have been described by Némethi and Popescu-Pampu [NP10].

Remark 3.4.5. A priori, being selfsimilar is a condition on a singularity (X, x_0) that depends on its analytical class (we ask for (X, x_0) to have an analytically isomorphic copy of itself in a bimeromorphic model strictly dominating the identity). Since being a sandwiched singularity depends only on the (decorated) dual graph of a resolution (see [Spi90, Section II.4]), we deduce from Theorem 3.4.2 that being selfsimilar is, in fact, a topological condition.

Remark 3.4.6. In [FFR20] we prove the classification of selfsimilar singularities over any (algebraically closed) field $\Bbbk$. The full statement is more complicated and technical in nature, it involves the analytic structure of the valuation space $\mathcal{V}_X$ seen as a Berkovich space (and often referred to as the *non-archimedean link* of (X, x_0), see [Fan18]. In this setting, we also need to work with algebraic singularities, and we need to modify the definition of selfsimilarity by allowing Kato data (π, σ) to be formal, in the sense that σ defines a formal isomorphism between (X, x_0) and (X', x_1): the formal complation of the ring of regular functions of X at x_0 and X' at x_1 are isomorphic.

In this manuscript, we keep the presentation simpler by working on the field of complex numbers, which allows to avoid several technicalities, and give a more geometrical flavor to the proof of Theorem 3.4.2.

To prove Theorem 3.4.2, we proceed as follows. Firstly, we prove that sandwiched singularities are selfsimilar. This is done by considering a Kato datum (π, σ) on $(\mathbb{C}^2, 0)$, where $\pi = \eta \circ \mu$ is the birational map induced by the sandwiched structure on (X, x_0). From (π, σ) we construct a Kato datum $(\widetilde{\pi}, \widetilde{\sigma})$ on (X, x_0).

Secondly, we proceed to show the harder implication: selfsimilar singularities are necessarily sandwiched. In order to do that, we consider the minimal resolution $\mu \colon Z \to (X, x_0)$ of a normal surface singularity admitting a non-trivial Kato datum (π, σ). From this datum we build a tower of Kato data $(\pi^{(n)}, \sigma^{(n)})$ over (X, x_0) by introducing the composition of Kato data. We claim that for n big enough, the natural map $X^{(n)} := (\pi^{(n)})^{-1}(X) \to Z$ is regular at $x_n := \sigma^{(n)}(x_0)$. This statement is proved by applying the study of the dynamics induced on valuation spaces, and a combinatorial argument on suitable (decorated) dual graphs.

3.4.2 Sandwiched singularities are selfsimilar

Sandwiched singularities are all obtained as follows (see [Spi90, Section II.1]).

- First, we take a good resolution $\pi\colon Y' \to (Y, y_0)$, where $(Y, y_0) = (\mathbb{C}^2, 0)$ is a regular point. In other terms, π is a finite sequence of point blow-ups above $0 \in \mathbb{C}^2$.

- We select any exceptional divisor D, whose support $|D| \subseteq \pi^{-1}(y_0)$ is connected. By Grauert's criterion [Gra62], we can contract D, to obtain a bimeromorphic map $\mu\colon Y' \to X$, where X has a normal singularity at $x_0 := \mu(D)$ (notice that the normality condition ensures the uniqueness of μ up to isomorphisms).

- We contract the part of the exceptional divisor $\pi^{-1}(y_0)$ that we haven't contracted before, thus completing the diagram $\pi = \eta \circ \mu$ with $\eta\colon X \to (Y, y_0)$ the bimeromorphic map that gives to (X, x_0) the sandwiched structure.

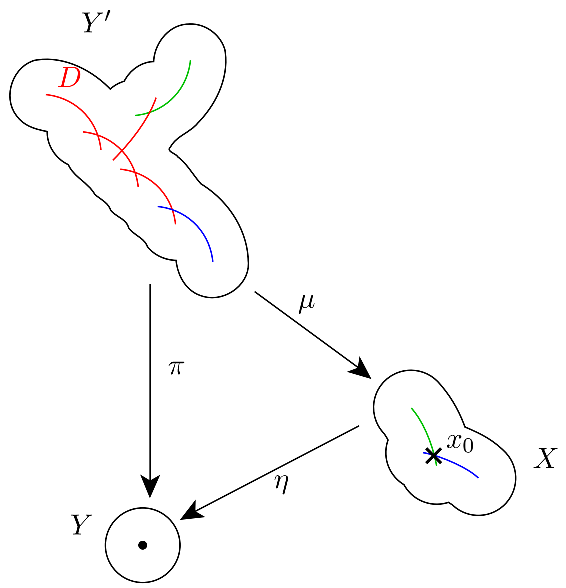


Figure 3.9: Construction of a sandwiched singularity

Let now $\sigma\colon (Y, y_0) \to (Y', y_1)$ be any local isomorphism, where $y_1 \in |D|$ is any point in the support of D. In particular, the pair (π, σ) is a Kato datum over (Y, y_0).

In this setting, we can lift the Kato datum (X, x_0) via the bimeromorphic map η, as follows. We will build a Kato datum over (X, x_0) by a fibered product process between the Kato datum on (Y, y_0) and the bimeromorphic map η.

- We construct the space X', that will be the total space of the Kato datum over (X, x_0), by fibered product with respect to π and η:
$$X' := Y' \setminus \sigma(Y) \sqcup X \Big/ \sigma \circ \eta \colon \partial X \to \sigma(\partial Y).$$
- We denote by $\widetilde{\sigma} \colon X \to X'$ the natural identification between X and its copy inside X'. Clearly $\widetilde{\sigma}$ is an isomorphism.
- We define the map $\widetilde{\pi} \colon X' \to X$ as:
$$\widetilde{\pi} \equiv \begin{cases} \mu & \text{su } Y' \setminus \sigma(Y), \\ \mu \circ \sigma \circ \eta & \text{su } \widetilde{\sigma}(X). \end{cases}$$

The map $\widetilde{\pi}$ is a non-trivial bimeromorphic map from X' to X, and the pair $(\widetilde{\pi}, \widetilde{\sigma})$ is a Kato datum over (X, x_0), as desired.

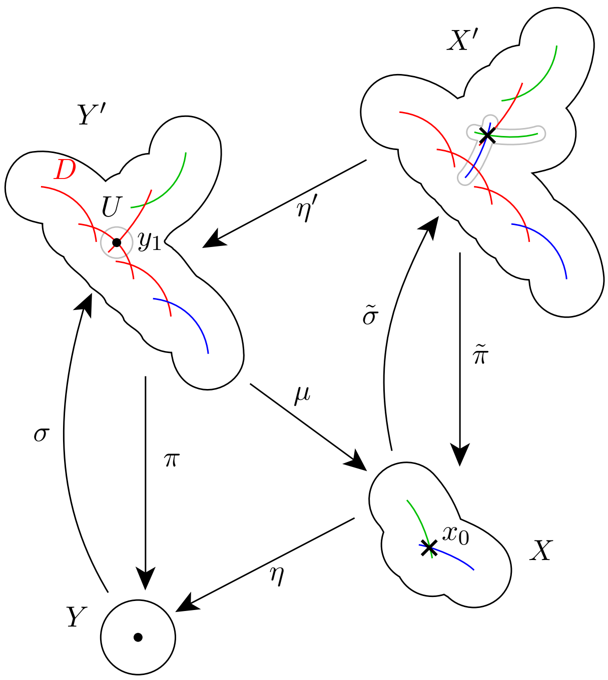


Figure 3.10: Construction of a Kato datum on (X, x_0).

Remark 3.4.7. The construction portrayed here can be reversed: from a Kato datum (π, σ) on a sandwiched singularity (X, x_0) we can build a Kato datum $(\hat{\pi}, \hat{\sigma})$ on $(\mathbb{C}^2, 0)$, having the property that the associated Kato surfaces $S = S(\pi, \sigma)$ and $\hat{S} = (\hat{\pi}, \hat{\sigma})$ are isomorphic. This gives a more concrete feeling to Kato's classification [Kat79] in the case of Kato surfaces.

3.4.3 Composition of Kato data

Let (π_1, σ_1) and (π_2, σ_2) be two Kato data, both over the same singularity (X, x_0). If we denote by $f_j = \pi_j \circ \sigma_j$ the Kato germs associated to these Kato data, we would like to define a Kato datum (π, σ) over (X, x_0) whose associated Kato germ $f = \pi \circ \sigma$ is the composition $f_1 \circ f_2$.

In order to do so, we lift the Kato datum (π_2, σ_2) to X'_1 via σ_1, as follows.

- We construct the space X'', that will be the total space of the Kato datum over (π, σ), as:
$$X'' := X'_1 \setminus \sigma_1(X) \sqcup X'_2 \Big/ \sigma_1 \circ \pi_2 \colon \partial X'_2 \to \sigma_1(\partial X).$$
- We denote by $\sigma'_2 \colon \sigma_1(X) \subseteq X'_1 \to X''$ the map $\sigma'_2 := \sigma_2 \circ \sigma_1^{-1}$ (up to the natural identification between X'_2 and its copy inside X'').
- We define the map $\pi'_2 \colon X'' \to X'_1$ as:
$$\pi'_2 \equiv \begin{cases} \mathrm{id} & \text{su } X'_1 \setminus \sigma_1(X), \\ \sigma_1 \circ \pi_2 & \text{su } X'_2, \end{cases}$$
which is a bimeromorphic map, that is a local isomorphism outside the exceptional divisor of π_2 in (the copy of X'_2).

The composition $\sigma := \sigma'_2 \circ \sigma_1$ is a well defined local isomorphism. Analogously, the map $\pi := \pi_1 \circ \pi'_2$ is a bimeromorphic map from X'' to X. By construction, we also have $\pi(\sigma(x_0)) = x_0$), and the pair (π, σ) is a Kato datum over (X, x_0) (which is trivial if and only if both π_1 and π_2 are).

The situation is summarised by the diagram:

$$\begin{tikzcd}
& X'' \arrow[dl, "\pi'_2"] \arrow[dd, "\pi"] & \\
X'_1 \arrow[ur, bend left, "\sigma'_2"] \arrow[dr, "\pi_1"] & & X'_2 \arrow[dl, "\pi_2"'] \\
& (X, x_0) \arrow[ul, bend left, "\sigma_1"] \arrow[uu, bend left, "\sigma"] \arrow[ur, bend right, "\sigma_2"'] &
\end{tikzcd}$$

Moreover, one checks directly that:

$$f := \pi \circ \sigma = \pi_1 \circ \pi'_2 \circ \sigma'_2 \circ \sigma_1 = \pi_1 \circ \sigma_1 \circ \pi_2 \circ \sigma_2 \circ \sigma_1^{-1} \circ \sigma_1 = f_1 \circ f_2,$$

as desired.

3.4.4 A valuative criterion for selfsimilarity

Let now (X, x_0) be a selfsimilar normal surface singularity: it admits a kato datum (π, σ) with $\pi\colon X' \to (X, x_0)$ a non-trivial modification.

Let $\mu\colon Z \to (X, x_0)$ be the minimal good resolution of (X, x_0) (or in fact any good resolution). While $\mu^{-1} : X \dashrightarrow Z$ has clearly an indeterminacy point at x_0, the lift $\Phi := \mu^{-1} \cdot \pi : X' \dashrightarrow Z$ might be regular at x_1. If this is the case, then we are done, since $(X, x_0) \cong (X', x_1)$ would dominate, via Φ, the smooth model Z.

While this does not happen in general, it is natural to replace this situation with the one associated to an iterate $(\pi^{(n)}, \sigma^{(n)})$ of the Kato datum (π, σ). In fact, the modification $\pi^{(n)}$ strictly dominate one another, and we have more and more chances that the map $\Phi_n := \mu^{-1} \circ \pi^{(n)}\colon X^{(n)} \to Z$ is regular at $x_n := \sigma^{(n)}(x_0)$.

We can summarize our situation with the following diagram:

$$\begin{tikzcd}
(X^{(n)}, x_n) \arrow[d, shift left, "\pi_n"] \arrow[ddddr, red, "\Phi_n\ ?"] & \\
\vdots \arrow[u, bend left, "\sigma_n"] \arrow[d, shift left, "\pi_3"] & \\
(X'', x_2) \arrow[u, bend left, "\sigma_3"] \arrow[d, shift left, "\pi_2"] \arrow[ddr, dashed, "\Phi_2"] & \\
(X', x_1) \arrow[u, bend left, "\sigma_2"] \arrow[dd, "\pi"] \arrow[dr, dashed, "\Phi"] & \\
 & Z \arrow[dl, "\mu"] \\
(X, x_0) \arrow[uu, bend left, "\sigma"] &
\end{tikzcd}$$

where $\sigma^{(n)} := \sigma_n \circ \cdots \circ \sigma_2 \circ \sigma$ and $\pi^{(n)} := \pi \circ \pi_2 \circ \cdots \circ \pi_n$.

We claim that there exists $n \gg 0$ so that the map Φ_n (in red in the diagram) is regular at x_n.

In order to check the regularity of Φ_n at x_n, we apply the valuative regularity criterion Proposition 2.1.3, which states in this case that Φ_n is regular at x_n if and only if $U_{\pi^{(n)}}(x_n)$ is contained in a connected component of $\mathcal{V}_X \setminus \mathcal{S}^*_\mu$, or equivalently, if and only if

$$U_{\pi^{(n)}}(x_n) \cap \mathcal{S}^\star_\mu = \emptyset,$$

being $U_{\pi^{(n)}}(x_n)$ connected. Notice that valuations in $U_{\pi^{(n)}}(x_n)$ are exactly the ones of the form $\pi^{(n)}_\bullet \mu$ with μ a valuation at $(X^{(n)}, x_n)$. Being $\sigma^{(n)}$ a local isomorphism, these valuations are all of the form $\sigma^{(n)}_\bullet \lambda$, with $\lambda \in \mathcal{V}_X$. To sum up, we get that $\nu \in U_{\pi^{(n)}}(x_n)$ if and only if

$$\nu = \pi^{(n)}_\bullet \mu = \pi^{(n)}_\bullet \sigma^{(n)}_\bullet \lambda.$$

Notice that $\pi^{(n)} \circ \sigma^{(n)} \equiv f_\bullet^n$ on $\mathcal{V}_X \setminus \mathcal{C}_{f^n}$, while curve valuations contracted by f^n are sent exactly to the points at the boundary of $U_{\pi^{(n)}}(x_n)$. We deduce that

$$f_\bullet^n(\mathcal{V}_X) = \overline{U_{\pi^{(n)}}(x_n)}.$$

Our goal in the next sections is to show that, in the setting described above, there exists $n \gg 0$ so that $U_{\pi^{(n)}}(x_n) \cap \mathcal{S}_\mu^* = \emptyset$.

3.4.5 Lift of Kato data to resolutions

We apply Theorem 2.2.1 to the Kato germ $f : (X, x_0) \circlearrowleft$: in this case, there exists a unique eigenvaluation $\nu_\star$, which attracts any divisorial (and in fact quasimonomial, or of finite skewness) valuation: $f_\bullet^n \nu \to \nu_\star$ for $n \gg 0$ (in the weak topology, and in the strong topology whenever $\alpha(\nu_\star) < +\infty$).

We claim that $\nu_\star$ cannot be divisorial. This will be proved in the next section. For now, we see how to conclude from here, or from the apparently weakly assumption $\nu_\star \notin \mathcal{S}_\mu^*$.

Proposition 3.4.8. *Let $f : (X, x_0)$ be a Kato germ, and suppose that its eigenvaluation $\nu_\star$ does not belong ot $\mathcal{S}_\mu^*$. Then there exists $n \gg 0$ such that $f_\bullet^n(\mathcal{V}_X) \cap \mathcal{S}_\mu^* = \emptyset$.*

In order to prove Proposition 3.4.8, we need to resolve the singularities of the Kato data $(\pi^{(n)}, \sigma^{(n)})$, i.e., to lift such Kato data from (X, x_0) via its resolution $\mu\colon Z \to (X, x_0)$. We start from $n = 1$, and we proceed as follows.

- We construct the space Z', By gluing together $X' \setminus \sigma(X)$ and a copy of Z via the natural identification $\sigma \circ \mu$:

$$Z' := X' \setminus \sigma(X) \sqcup Z \big/ \sigma \circ \mu \colon \partial Z \to \sigma(\partial X).$$

- We denote by $\hat{\sigma}\colon Z \to Z'$ the natural identification between Z and its copy inside Z'.

- We define the map $\hat{\pi}\colon Z' \to Z$ as:

$$\hat{\pi} \equiv \begin{cases} \mathrm{id} & \text{on } X' \setminus \sigma_(X), \\ \sigma \circ \mu & \text{on } Z. \end{cases}$$

Clearly this construction generalizes to any iterate of (π, σ), providing a tower of modifications $\hat{\pi}_n$ and of isomorphisms (with their respective images) $\hat{\sigma}_n$.

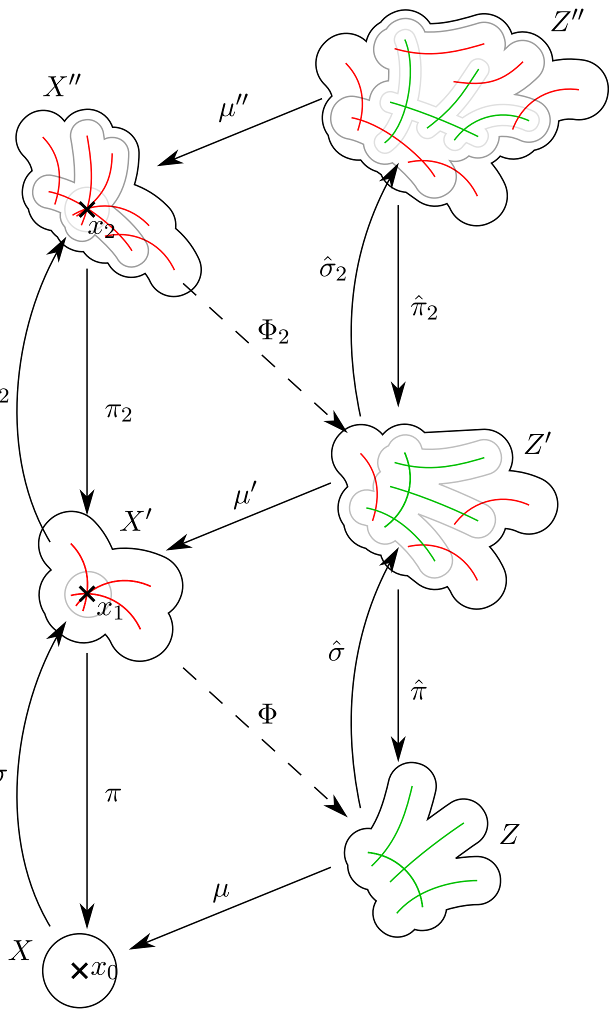


Figure 3.11: Lift of a Kato datum (π, σ) and its second iterate to a good resolution μ.

Proof of Proposition 3.4.8. Let (π, σ) be a Kato datum over (X, x_0), and let $\mu \colon Z \to (X, x_0)$ be a good resolution as in the statement. Let U be the connected component of $\mathcal{V}_X \setminus \mathcal{S}^*_\mu$ containing $\nu_\star$. By Theorem **??**, there exists $n \gg 1$ such that $f^n_\bullet(\nu_E) \in U$ for any $E \in \Gamma^*_\mu$.

Up to replacing (π, σ, f) by $(\pi^{(n)}, \sigma^{(n)}, f^n)$, we may assume $n = 1$. We lift the Kato datum (π, σ) to a pair $(\hat{\pi}, \hat{\sigma})$ as described above. Notice that $\hat{\sigma}$ is an isomorphism, while $\hat{\pi}$ is a proper bimeromorphic map. In particular, $\mu \circ \hat{\pi}$ dominates μ. In terms of skeleta, this condition is equivalent to asking that $\mathcal{S}^*_\mu \subseteq \mathcal{S}^*_{\mu\circ\hat{\pi}}$. Set $S := f_\bullet \mathcal{S}^*_\mu$. But then, $f_\bullet(\mathcal{V}_X)$ is exactly the closure of the connected component of $\mathcal{V}_X \setminus \big(\mathcal{S}^*_{\mu\circ\hat{\pi}} \setminus S\big)$ containing S. Since by assumption $S \cap \mathcal{S}^*_\mu = \emptyset$, we deduce that $U_\pi(x_1) \subseteq \mathcal{V}_X \setminus \mathcal{S}^*_\mu$, and we are done. $\square$

3.4.6 The eigenvaluation is not divisorial

We conclude the proof of Theorem 3.4.2 by showing that Kato germs cannot have divisorial eigenvaluations.

Proposition 3.4.9. *Let $f : (X, x_0) \circlearrowleft$ be a Kato germ. Then the unique eigenvaluation $\nu_\star$ for f cannot be divisorial.*

Remark 3.4.10. Before proceeding with the actual proof, we give a more abstract argument: by the Jacobian formula Equation (**??**), we get that

$$c(f, \nu_\star) A(\nu_\star) = A(\nu_\star) + \nu(R_f).$$

If $\nu_\star = \nu_{E_\star}$ is divisorial, we have that $c(f, \nu_\star) = k_E E = 1$, since it is bounded by the topological degree of f, which is 1. We deduce that $A(\nu_\star) = A(\nu_\star) + \nu(R_f)$.

From here, we would deduce a contradiction, as long as $\nu(R_f) \neq 0$. While when working on $(\mathbb{C}^2, 0)$, we have that R_f is effective, and $\nu(R_f) > 0$ for any f and $\nu \in \hat{\mathcal{V}}^*$, this is not anymore the case in the singular setting.

Proof. We proceed by contradiction, and suppose that $\nu_\star = \nu_{E_\star}$ is divisorial. We take a good resolution $\mu \colon Z \to (X, x_0)$ so that $E_\star \in \mathit{\Gamma}^*_\mu$. By repeating the construction of the previous section, for any $n \geq 1$ we get the lift $(\hat{\pi}^{(n)}, \hat{\sigma}^{(n)})$ of the iterate Kato data $(\pi^{(n)}, \sigma^{(n)})$ of (π, σ) to the good resolution μ. Set $\Gamma := \mathit{\Gamma}_\mu$, and more generally, $\Gamma^{(n)} := \mathit{\Gamma}_{\mu \circ \pi^{(n)}}$ for any $n \geq 0$ (with $\Gamma^{(0)} = \Gamma$).

The datum $(\pi^{(n)}, \sigma^{(n)})$ induce two maps at the level of dual graphs:

- The embedding $i^{(n)} \colon \Gamma \to \Gamma^{(n)}$, induced by the isomorphism $\hat{\sigma}^{(n)} = \hat{\sigma}_n \circ \cdots \circ \hat{\sigma}_1$. Notice that $i^{(n)}$ is an embedding in the sense of decorated graphs: it sends vertices to vertices, edges to edges, and preserves both the self-intersection e and the genus g of each vertex.

- The map $\tau^{(n)} \colon V(\Gamma) \to V(\Gamma^{(n)})$, defined only at the level of vertices, via strict transform by $\pi^{(n)}$; $\tau^{(n)}(E) = E_{\pi^{(n)}}$ for any $E \in V(\Gamma) = \mathit{\Gamma}^*_\mu$. In this case, $\tau^{(n)}$ preserves the genus, and can only decrease (or preserve) the self-intersection.

By unraveling the definitions, we get that $f^n_\bullet \nu_E = \nu_E$ for some $E \in \Gamma$ if and only if $i^{(n)}(E) = \tau^{(n)}(E)$. By contradiction assumption, this happens for $E = E_\star$.

Denote by d the graph distance in Γ, and for any $D \in \Gamma$ and $k \in \mathbb{N}^*$, set

$$L_{\leq k}(D) = \{E \in \Gamma \mid d(E, D) \leq k\}, \qquad L_k(D) = \{E \in \Gamma \mid d(E, D) = k\}.$$

Lemma 3.4.11. *For any $k \in \mathbb{N}$ there exists $n = n(k) \gg 0$ such that $i^{(n)} \equiv \tau^{(n)}$ on $L_{\leq k}(E_\star)$.*

Proof. We proceed the induction, with the case $k = 0$ being exactly our contradiction assumption.

Assume we have proved the statement for a given $k \in \mathbb{N}$, and set $n = n(k)$. Let $E \in L_{k+1}(E_\star)$ be any vertex at distance $k + 1$ from $E_\star$, and let $D \in L_k(E_\star)$ be any vertex such that $d(E, D) = 1$. By assumption, $i^{(n)}(D) = \tau^{(n)}(D) =: D'$. In particular, $e(D) = e(D')$, from which we deduce that the modification $\hat{\pi}^{(n)}$ does not blow-up any point in D. But this means that $\tau^{(n)}$ defines a bijection from $L_{\leq 1}(D)$ to $L_{\leq 1}(D')$, and that the map $\left(i^{(n)}\right)^{-1} \circ \tau^{(n)} : L_{\leq 1}(D) \circlearrowleft$ is a bijection. If m_D is the order of this bijection, then we have that $i^{(nm_D)} \equiv \tau^{(nm_D)}$ on $L_{\leq 1}(D)$.

But then, by taking $m = \operatorname{lcm}\{m_D \mid D \in L_k(E_\star)\}$, we get that $n(k+1) := nm$ satisfies $i^{(nm)} \equiv \tau^{(nm)}$ on $L_{\leq k+1}(E_\star)$. □

We can finally conclude, by applying Lemma 3.4.11 with $k = \operatorname{diam}(\Gamma)$. For $n = n(k)$, we get that $i^{(nm)} \equiv \tau^{(nm)}$ on Γ, which implies that $\hat{\pi}$ is a trivial modification, against the assumption of (π, σ) being a non-trivial Kato datum.

□

Chapter 4

Dynamics of parabolic germs

Introduction

[MR21] S.Mongodi and M.Ruggiero: "Birational properties of tangent to the identity germs without non-degenerate singular directions". Journal of the London Mathematical Society, pp. 1–55, 2023. https://onlinelibrary.wiley.com/.

Our goal is to understand the dynamics of holomorphic germs $f : (\mathbb{C}^d, 0) \circlearrowleft$ that are tangent to the identity, i.e., the differential of f at 0 is the identity. We exclude right away the case of $f = \mathrm{id}$, and in this case, we can write the map as

$$f(z) = z + f^{(h)}(z) + o(\|z\|^h),$$

where $f^{(h)}$ is a non-zero d-uple of homogeneous polynomials of degree $h \geq 2$, called the *multiplicity* of f.

The dynamics of such maps is completely understood in dimension 1. In particular, one can describe precisely the structure of the basin of attraction $\mathcal{B} = \{z \mid f^n(z) \to 0\}$ via the classical Leau-Fatou flower theorem [Lea97, Fat19]: it is the union of $h - 1$ basins $\mathcal{B}_\zeta = \{z \mid f^n(z) \xrightarrow{\zeta} 0\}$ of orbits converging to 0 asymptotically to some real directions ζ.

Our goal in higher dimensions is to describe the basin of attraction $\mathcal{B}_\star$, where $\star$ describes the asymptotic behavior of the orbits we are considering: for us, $\star$ will be a real direction, a complex direction, or a formal irreducible curve (we will always avoid points z whose orbit terminates exactly at 0).

The candidate complex directions v and irreducible curves C along which orbits can converge to 0 can be detected algebraically: v must be a *characteristic direction* for f, i.e., $f^{(h)}(v) = \lambda v$ for some $\lambda \in \mathbb{C}$, while C must be f-invariant. Notice that curves C must be tangent to characteristic directions v, and real directions ζ along which we can converge to zero must be contained in a complex line generated by a characteristic direction.

The basins $\mathcal{B}_\star$ can be described as the space of orbits of f-invariant manifolds, containing 0 at their boundary, and called *$\star$-parabolic manifolds*.

These topics are presented in Section 4.1.

The local dynamics of tangent to the identity germs is strongly related to the continuous dynamics of their associated *infinitesimal generators*, which are (possibly non-saturated, non-convergent) vector fields whose time-1 flow gives the original germ. In Section 4.2 we introduce infinitesimal generators, and we introduce a dictionnary to go from discrete to continuous dynamics, and back. We then review the known constructions of parabolic manifolds, along *non-degenerate* characteristic directions ($\lambda \neq 0$ in the definition above) by [Hak98], and along formal curves by [LRSV22]. The proofs of these results are related to reduction of singularities and local uniformization of the associated infinitesimal generators: we sketch some ideas, and conclude the section with a short survey on the known results in dimension 2.

In the last Section 4.3 we present some of the difficulties to the study of tangent to the identity germs in dimension 3 or higher. We start with presenting Mcquillan and Panazzolo's reduction of singularities of vector fields in dimension 3 (see [MP13]: this result is more technical than the 2-dimensional counterpart, and needs to allow weighted blow-ups, i.e., cyclic quotient singularities on the ambient varieties. We then review in details the construction of parabolic manifolds attached to curves in [LRSV22], and in particular how to compute the dimension of these parabolic manifolds from the associated Ramis-Sibuya normal forms. We then introduce a family of 3-dimensional tangent to the identity germs with only degenerate characteristic directions, and study the reduced singularities of the infinitesimal generator. We detect several family of reduced singularities (and their associated saturation divisor), compute the Ramis-Sibuya normal forms for these; we deduce the existence and dimension of parabolic manifolds attached to the parabolic manifolds associated to the degenerate characteristic directions in our example. We conclude the section with remarks on the existence of parabolic manifolds coming from reduced singularities with a 2-dimensional central manifold.

4.1 Tangent to the identity germs

4.1.1 One-dimensional germs

We start by recalling the dynamical picture of tangent to the identity germs in dimension 1. Let $f : (\mathbb{C}, 0) \circlearrowleft$ be a 1-dimensional tangent to the identity germ, that, up to a linear change of coordinates, we can write under the form

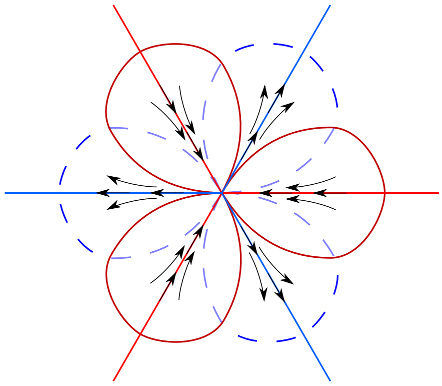
$$f(z) = z\big(1 - z^r + o(z^r)\big).$$

By studying the behavior of f and its truncation at order $h = r + 1$, one can show that the orbits $z_n = f^n(z)$ converging at 0 do tangentially to real directions ζ_k (the r-th roots of unity), i.e., $\frac{z_n}{\|z_n\|} \to \zeta_k$. We denote this asymptotic convergence by $z_n \overset{\zeta_k}{\to} 0$.

The basin of attraction $\mathcal{B}_{\zeta_k} = \{z \mid f^n(z) \overset{\zeta_k}{\to} 0\}$ can be described as the union of preimages by f of connected and simply connected open domains Δ_k, called *attracting petals*, where the map f is holomorphically conjugate to the translation $z \mapsto z + 1$ (the conjugacy on Δ_k realizing this normal form is called *Fatou coordinate*).

If we take the collection of attracting petals, together with the repelling petals (i.e., the attracting petals for f^{-1}), we obtain a pointed neighborhood of the origin.

This result, known as Leau-Fatou's flower theorem, allows to describe quite precisely the dynamics of a tangent to the identity germ around its fixed point.



Leau-Fatou's flower theorem is the main ingredient to the topological classification of tangent to the identity germs by Camacho [Cam78]: f is topologically conjugated to its truncation at order $r+1$, or equivalently, to the time-1 flow of the non-saturated vector field $-z^{r+1}\partial_z$. The formal classification has an extra term of order $2r+1$, whose coefficient β (called *index*) can be computed as

$$\beta = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{dz}{z - f(z)},$$

where γ is a small positive simple loop around 0. The pair (r, β) is a complete invariant for the formal classification of tangent to the identity germs. The analytic classification, due to Ecalle [Éca85, Section 5.1] and Voronin [Vor81], is much subtler; it can be stated in terms of the transition maps of the Fatou coordinates of adjiacent attracting and repelling petals.

4.1.2 Orbits asymptotic to directions and curves

Écalle's approach to the analytic classification of tangent to the identity germs is based on his resurgence theory [Éca81a, Éca81b], which is in itself a very deep and rich work. This approach extends to any dimension, and gives a (rather abstract) formal and analytic classification of tangent to the identity germs in any dimension (see [Éca85, Chapître 5]). Nevertheless, the classification does not allow a complete understanding of the dynamics of tangent to the identity germs on a whole neighborhood of the origin, and a general Leau-Fatou-like result is still lacking in higher dimensions (even though some recent progress have been done in dimension $d = 2$, see [LR22]).

We have seen how in dimension 1 we can decompose the basin of attraction at the origin into basins where the orbits are asymptotic to real directions. In higher dimensions, we can split our study into two steps, by first identifying orbits that converge towards the origin asymtotically to a complex direction $v \in \mathbb{P}^{d-1}_{\mathbb{C}}$, and then looking for attracting real directions ζ that lie inside the complex line $\mathbb{C}v$.

Definition 4.1.1. We say that the orbit of a point $z_0 \in \mathbb{C}^d$ converges *asymptotically* to a complex direction $v \in \mathbb{P}^{d-1}_{\mathbb{C}}$ if $[z_n] \to v$, where $[z_n]$ is the class of $z_n \in \mathbb{C}^d \setminus \{0\}$ in $\mathbb{P}^{d-1}_{\mathbb{C}}$. We denote this convergence by $z_n \xrightarrow{v} 0$.

Recall that we can naturally identify $\mathbb{P}^{d-1}_{\mathbb{C}}$ with the projectivization of the tangent space of $\mathbb{C}^d$ at 0, i.e., with the exceptional divisor $E_0 = \pi_0^{-1}(0)$ of the blow-up $\pi_0 \colon X_{\pi_0} \to (\mathbb{C}^d, 0)$ at 0. In this setting, the condition $z_n \xrightarrow{v} 0$ can be expressed in terms of the lift $f_{\pi_0} \colon X_{\pi_0} \circlearrowleft$ by asking that $\pi_0^{-1}(z_n) \to v$ in X_{π_0}.

With this in mind, we see how prescribing a vector v to which an orbit is asymtotic corresponds to giving a order 1 tangency to the complex direction v, or equivalently to the complex line $\mathbb{C}v$.

One can generalize this concept, and ask for higher orders of tangency to a given (formal, irreducible) curve C.

Definition 4.1.2. We say that the orbit of a point $z_0 \in \mathbb{C}^d$ converges *asymptotically* to a a formal irreducible curve in $(\mathbb{C}^d, 0)$ if for any (point) modification $\pi \colon X_\pi \to (\mathbb{C}^d, 0)$ over the origin, we have

$$\pi^{-1}(z_n) \to p_\pi,$$

where p_π is the intersection between the strict transform C_π of C and the exceptional divisor $\pi^{-1}(0)$. We denote this convergence by $z_n \xrightarrow{C} 0$.

Notice that the point p_π is unique because Γ is irreducible. In practice, one needs to check the property only for the blow-up $\pi_0 \colon X_{\pi_0} \to (\mathbb{C}^d, 0)$, then for the blow-up $\pi_1 \colon X_{\pi_1} \to (\mathbb{C}^d, 0)$ obtained by further blowing-up p_{π_0}, and so on, creating the sequence p_{π_m} of infinitely near points, sometimes called *iterated tangents*. Notice that this sequence of blow-ups solves the curve after finitely many steps, see [CC05, Section 3.2].

As one would expect, directions and curves toward which we can have asymptotic orbits are rather special, and in some sense fixed by the dynamics.

Proposition 4.1.3 ([Hak98, Proposition 2.3],[LRSV22, Section 2])**.** *If there exists $z_0 \in \mathbb{C}^d$ so that:*

- $z_n \xrightarrow{v} 0$*, then v is a* characteristic direction*, i.e., $f^{(h)}(v) = \lambda v$ for some $\lambda \in \mathbb{C}$.*
- $z_n \xrightarrow{C} 0$*, then C is f-invariant.*

Notice that characteristic directions can be described in terms of the action $g : \mathbb{P}^{d-1}_{\mathbb{C}} \circlearrowleft$ induced by $f^{(h)}$. In fact, if the λ in the definition above is in $\mathbb{C}^*$, then $v \in \mathrm{Fix}(g)$ is a (regular) fixed point; if $\lambda = 0$, in this case we have that $v \in \mathrm{Ind}\,(g)$. The former case is referred to as *non-degenerate*, while the latter case is *degenerate*.

In the terminology of [Iva11], the first kind is a *holomorphic fixed point*, while the second is a *meromorphic fixed point* whenever $g(v) \ni v$.

We finally recall that a curve C is invariant by f if $\phi \circ f \in C$ for any $\phi \in C$, where we see a formal curve C as a prime ideal of $\mathbb{C}[\![z]\!]$ such that $\mathbb{C}[\![z]\!]\big/C$ has dimension 1. We will distinguish the case where $f|_C \neq \mathrm{id}_C$, in which case we say that C is a *separatrix.* for f.

4.1.3 Parabolic manifolds

Our goal is now to describe the basin of attraction $\mathcal{B}_\star(f) := \mathcal{B}_\star = \{z \in \mathbb{C}^d \mid f^n(z) \xrightarrow{\star} 0\}$ at 0 associated to the $\star$-asymptotic behavior, where $\star$ may stand for a real direction $\zeta \in \mathbb{S}^{2d-1}$, a complex direction $v \in \mathbb{P}^{d-1}_{\mathbb{C}}$, an irreducible curve C, or nothing at all (in which case we mean the whole basin of attraction, and we omit the symbol $\star$). Notice that one could be interested in other asymptotic behaviors, for example when the orbit of the complex tangent direction to z_n follows some closed real paths, see the recent works by Buff-Raissy.

The role of petals is played by the so called parabolic manifolds.

Definition 4.1.4. Let $f : (\mathbb{C}^d, 0) \circlearrowleft$ be a tangent to the identity germ. A $\star$-*parabolic manifold* for f is a (connected) complex submanifold Δ of $(\mathbb{C}^d, 0)$ such that $0 \in \partial\Delta$, $f(\Delta) \subset \Delta$, and $f^n(z) \xrightarrow{\star} 0$. If $\dim\Delta = 1$ we say that Δ is a parabolic curve, while if $\dim\Delta = d$ we say that Δ is a parabolic domain.

Often, a parabolic manifold is also required to be simply connected, as in most cases it is parametrized by a map $\varphi\colon D \to (\mathbb{C}^d, 0)$, where D is a pluridisk having 0 at its boundary. In this case, we also ask that the map φ extends continuously at 0 and that $\varphi(0) = 0$. This in particular implies that $0 \in \partial\Delta$.

In general, we do not ask that the $\star$-basin of attraction $\mathcal{B}_\star$ is described completely by a unique parabolic manifold Δ. For example, it might happen that a parabolic manifold of dimension 2 is foliated by parabolic curves.

Definition 4.1.5. Let $f : (\mathbb{C}^d, 0) \circlearrowleft$ be a tangent to the identity germ, and $\star$ be as above. We say that a family $(\Delta_\alpha)_\alpha$ of $\star$-parabolic manifolds is *complete* if

$$\mathcal{B}_\star = \bigcup_{\alpha} \bigcup_{n \in \mathbb{N}} f^{-n}(\Delta_\alpha),$$

or in other terms, if for any $z \in \mathcal{B}_\star$, then $f^n(z)$ belongs to Δ_α for a certain α and for $n \gg 0$.

With this terminology, Leau-Fatou's theorem states that for any tangent to the identity germ $f : (\mathbb{C}, 0) \circlearrowleft$, we can cover the punctured neighborhood of the origin with parabolic curves $\Delta_1^+, \ldots, \Delta_r^+$ and $\Delta_1^-, \ldots, \Delta_r^-$ for f and f^{-1} respectively. Each parabolic curve Δ_j^+ gives a complete family for the basin of attraction $\mathcal{B}_{\zeta_j}(f)$ tangent to an attracting direction ζ_j, and similarly Δ_j^- gives a complete family for the basin of attraction $\mathcal{B}_{\zeta_j^-}(f^{-1})$ tangent to a repelling direction ζ_j^-.

Putting attracting petals together, we get a complete family $(\Delta_j^+)_j$, for the basin $\mathcal{B}(f)$, while putting repelling petals together we get a complete family $(\Delta_j^-)_j$, for the basin $\mathcal{B}(f^{-1})$.

4.2 Infinitesimal generator and resolutions

4.2.1 Infinitesimal generator

The theory of tangent to the identity germs is intimately related to the one of vector fields. In fact, to any tangent to the identity germ $f\colon (\mathbb{C}^d, 0) \circlearrowleft$ is associated a unique (formal, possibly non-convergent) vector field χ, that has multiplicity $h \geq 2$ at 0, and satisfying $f = \exp \chi$ (see e.g. [Éca85, Section 5.2], [BCL08],[LRSV22]). The vector field χ is called the *infinitesimal generator* of f, and denoted by $\chi = \log f$.

If ϕ is a local coordinate (hence defining a germ of smooth hypersurface $\{\phi = 0\}$) at 0, we have

$$\phi \circ \exp \chi = \sum_{n=0}^{\infty} \frac{\chi^n(\phi)}{n!}, \tag{4.1}$$

where χ^n denotes the derivation χ applied n times. In particular, we notice that if χ has order h, then the expression in coordinates of χ and $f - \mathrm{id}$ coincide up to order $2(h-1)$.

To go from properties of the infinitesimal generator χ to properties of its associated tangent to the identity germ f, two obstacles come in our way.

- The first obstacle is more analytical in nature: the infinitesimal generator χ is a formal vector field, that may be not convergent. It does however satisfy the summability condition $\frac{1}{r}$-Gevrey, with $h = r+1$ being the order of χ at 0 (see [BL09]): if we write $\chi = \sum_{n \geq h} \chi^{(n)} \partial_z$, with $\chi^{(n)} \partial_z := \sum_{j=1}^{d} \chi^{(n),j} \partial_{z_j}$ is the homogeneous part of χ of degree n, then we have that $\sum_{n \geq h} \frac{\chi^{(n)}}{(n!)^{\frac{1}{r}}}$ is convergent. This property is the first to check when applying Ecalle's resurgence techniques.

- The second obstacle is more geometrical in nature, and is related to having to deal with non-saturated vector fields. In fact, the set $\mathrm{Fix}(f)$ of points fixed by f coincides with the singularities $\mathrm{Sing}(\chi)$ of the associated infinitesimal generator, which is saturated if and only if $\mathrm{codim}\big(\mathrm{Sing}(\chi)\big) \geq 2$.

  Even assuming that $\mathrm{Fix}(f) = \{0\}$, in the resolution of singularities of χ we start by blowing up the origin: $\pi_0 \colon X_{\pi_0} \to (\mathbb{C}^d, 0)$. In this case, the pullback $\chi_{\pi_0} := \pi_0^* \chi$ has singularities along all the exceptional divisor $\pi_0^{-1}(0)$, and χ_{π_0} is not saturated. The saturation $\hat{\chi}_{\pi_0}$ of χ_{π_0} defines a (formal) foliation $\mathcal{F}_{\pi_0}$, the lift of the foliation $\mathcal{F}$ induced by χ.

  To iterate the process and work with germs tangent to the identity, we consider sequences of blow-ups $\pi \colon X_\pi \to (\mathbb{C}^d, 0)$ along centers that are made by singular points for the saturated infinitesimal generator (we say that π is *adapted* to f, or to χ).

  The infinitesimal generators behave functorially with respect to modifications adapted to f, meaning that the operations of taking lifts and taking infinitesimal generators commute. When we saturate lift χ_π to $\hat{\chi}_\pi$, we lose this functoriality. This corresponds to the fact that the foliation $\mathcal{F}_\pi$, while retaining some geometrical informations of the dynamics of f (for example orbits are contained to leaves when $\mathcal{F}$ is convergent), it does not give much information about the speed at which we run across the leaves of $\mathcal{F}_\pi$.

4.2.2 Tangent to the identity germ - infintesimal generator dictionary

Most of the concept introduced for tangent to the identity germs $f\colon (\mathbb{C}^d, 0) \circlearrowleft$ can be translated in terms of the associated infinitesimal generator χ.

We denote by $\pi_0\colon X_{\pi_0} \to (\mathbb{C}^d, 0)$ the blow-up of the origin, by $\chi_{\pi_0} := \pi_0^*\chi$ the lift of χ to X_π, by $\overline{\chi}_{\pi_0}$ the saturation of χ_{π_0} with respect to the exceptional divisor $E_0 := \pi_0^{-1}(0)$ (i.e., if $E_0 \underset{\text{loc}}{=} \{x = 0\}$ is a local equation of the exceptional divisor, then we only simplify the common x factor to all the coordinates of χ_{π_0}), and by $\hat{\chi}_{\pi_0}$ the saturation of $\hat{\chi}_{\pi_0}$.

tangent to the identity germ $f = \exp\chi$	infinitesimal generator $\chi = \log f$
$f^{(h)}$ initial part of $f - \mathrm{id}$	$f^{(h)}\partial_z$ initial part of χ
$\mathrm{Fix}(f)$ fixed points of f	$\mathrm{Sing}(\chi)$ singularities of χ
C is f-invariant	C is a separatrix for χ
$\mathrm{id}_C \equiv f_C\colon C \circlearrowleft$	$C \subseteq \mathrm{Sing}(\chi)$
v characteristic direction	$v \in \mathrm{Sing}(\overline{\chi}_{\pi_0})$
v *singular* direction	$v \in \mathrm{Sing}(\hat{\chi}_{\pi_0})$
v is characteristic (resp., singular) non-degenerate / degenerate	the eigenvalue of $\overline{\chi}_{\pi_0}^{(1)}$ (resp., $\hat{\chi}_{\pi_0}^{(1)}$) transversal to E_0 is $\neq 0$ / $= 0$

The first property is a direct consequence of Equation (4.1), as we have already mentioned. The second, third and fourth properties are a direct consequence of the definition; notice that if $d = 2$, then C is f-invariant and $f|_C \not\equiv \mathrm{id}_C$ if and only if C is a separatrix for $\hat{\chi}$, or equivalently for the corresponding foliation $\mathcal{F}$. For these reasons, we say that an invariant curve C satisfying $f_C \not\equiv \mathrm{id}_C$ is a *separatrix* of f.

The fifth property is a matter of direct computation, while the sixth property is the definition of a singular direction. Notice that in general the blown-up infinitesimal generator χ_π is not saturated, and all directions that are tangent to the exceptional divisor (and that we call *exceptional*) are automatically characteristic; singular directions detect which ones are really special: they are the singularities of the foliation $\mathcal{F}_{\pi_0}$.

Finally, the last characterization is also a direct computation, which holds both for characteristic and singular directions. Notice that for the saturated infinitesimal generator, the exact value of an eigenvalue of $\hat{\chi}_{\pi_0}^{(1)}$ is not well defined, but its vanishing is.

4.2.3 Construction of parabolic manifolds

As a first consequence of this correspondence between discrete and continuous (formal) dynamics, we gather in the next statement two very general results of existence of parabolic manifolds, attached respectively to (non-degenerate) characteristic directions and to separatrices.

Theorem 4.2.1. *Let $f : (\mathbb{C}^d, 0) \circlearrowleft$ be a tangent to the identity germ.*

(a) [Hak98] v-parabolic curves exist for any non-degenerate *characteristic direction v.*

(b) [LRRS21, LRSV22] C-parabolic manifolds exist for any separatrix C.

In both cases, the parabolic manifolds constructed are in fact tangent to suitable real directions, contained in the line $\mathbb{C}v$ generated by v in the first case, and by the tangent vector to C at 0 in the second case.

Notice that the v-parabolic curves of Theorem 4.2.1 are not necessarily associated to a separatrix C tangent to v. The ones that do are sometimes called *robust*, alluding to the fact that they survive to blow-ups. At the same time, the separtrices C might be tangent to degenerate characteristic directions. In the second statement, we can also determine the dimensions of the parabolic manifolds (which depend on the real direction they are tangent to), starting from some normal forms up to weighted blowups, and called *Ramis-Sibuya normal forms.*

The proof of Theorem 4.2.1.(a) relies on the construction of a (possibly transcendental) separatrix C, while the proof of Theorem 4.2.1.(b) relies instead in a local uniformization result of the infinitesimal generator χ along C. We will give more details in the next section.

4.2.4 Parabolic manifolds in dimension 2

We describe here briefly the state of the art for v-parabolic curves and domains in dimension 2. Let $f\colon (\mathbb{C}^2, 0) \circlearrowleft$ be a planar tangent to the identity germ, and χ its associated infinitesimal generator. Seidenberg's theorem [Sei68] guarantees the existence of a regular modification $\pi\colon X_\pi \to (\mathbb{C}^2, 0)$, adapted to f, with the property that the saturated lifted infinitesimal generator $\hat{\chi}_\pi$ has only *elementary* singularities, i.e., singularities whose linear part has at least a non-zero eigenvalue. At any singularity $p \in \pi^{-1}(0)$, we can consider the *index* α, defined as the ratio of the two eigenvalues (up to inversion $\alpha \mapsto \alpha^{-1}$)), which is an invariant of the singularity of the singularity of $\hat{\chi}_\pi$, i.e., of the foliation $\mathcal{F}_\pi$.

One can blow-up furthermore, and avoid *resonances*, i.e., the case $\alpha \in \mathbb{N}^*$, and in fact we can assume $\alpha \notin \mathbb{Q}_{>0}$ (up to weighted blow-up). In the algebraic geometry language of [MP13], elementary singularities are *log-canonical*, reduced singularities are *canonical*, and elementary singularities of index in $\mathbb{N}^*$ are called *radial*. There are mainly three classes of reduced singularities.

- *Poincaré* singularities, where $\alpha \in \mathbb{C} \setminus (\mathbb{R}_{\leq 0} \cup \mathbb{Q}_{>0})$: in this case, $\hat{\chi}_\pi$ is locally linearizable, and has exactly two (convergent) separatrices, that can be taken as the coordinate axes.

- *Siegel* (or *saddle*) singularities, where $\alpha \in \mathbb{R}_{<0}$: in this case the analytic classification is more involved, but we still have two (convergent) separatrices, that can be taken as the coordinate axes.

- *saddle-node* singularities, for which $\alpha = 0$. In this case we have again two separatrices: one, convergent, tangent to the eigenspace associated to the non-zero eigenvalue, and called *strong separatrix*; and another, not convergent in general, corresponding to a central manifold, and called *weak separatrix*.

These classes of singularities are classified, from the classical works by Poincaré and Dulac, to the classifications of [MR83, PY94] in the Siegel case, and [MR82] in the saddle-node case (see also [Sau09] for an approach via resurgence theory). We also refer to [IY08, Lor21] for general references about singularities of complex analytic foliations.

By keeping track of the type of reduced singularities that can appear in a resolution (via a generalization of the index to non-elementary singularities), [CS82] prove the existence of a (convergent) separatrix for any (saturated) vector field χ. We also mention the approach in [Can93], based on Newton-Puiseux polygons, which carries over in the Gevrey category.

When v is a non-degenerate characteristic direction for f, Theorem 4.2.1.(a) gives v-parabolic manifolds. This is done by either finding a formal separatrix C tangent to v, or by constructiong a "transcendental separatrix" C. The latter case happens when we find a resonant elementary singularity with index $h \in \mathbb{N}^*$, in which case C is described in suitable coordinates by an expression of the form $y = \psi(x, x^h \log x)$, where $\psi \in \mathbb{C}[\![x, u]\!]$ is a formal power series. Hakim also shows (see [Hak98, Theorem 5.1]) how, under suitable conditions, it is possible to enlarge the parabolic curve into a parabolic domain.

When f has an isolated fixed point at 0, then the infinitesimal generator χ has an isolated singularity, and Camacho-Sad result garantees the existence of a separatrix C tangent to v, around which one can build parabolic curves (see [Aba01]). In terms of discrete dynamics, one shows that in the reduction of singularities of χ we have at least a point with a non-degenerate characteristic direction, to which we can apply Hakim's result.

From here, focus has been given to the study of degenerate characteristic directions. We mention for example [Mol09, Viv12, Lap16], where special care is given to singularities in given resolution that present resonant Poincaré singularities, and outside of these cases, one can build parabolic curves and domains. In [LS18] (see also [LRRS21]), the authors prove the existence of C-parabolic manifolds attached to separatrices, result later generalized to higher dimension into Theorem 4.2.1.(b).

In the recent work [LR20], one can find a complete the picture for v-parabolic manifolds in dimension 2: in particular, for any degenerate characteristic direction v, we either have a convergent curve tangent to v and pointwise fixed by f, or there is a separatrix (which gives parabolic manifolds), or the resolution of the saturated infinitesimal generator $\hat{\chi}_\pi$ has a saddle-node above v, and we can construct parabolic curves by Hakim's [Hak98].

Notice that, while under some genericity assumption, parabolic manifolds are enough to describe the dynamics of a tangent to the identity germ $f : (\mathbb{C}^2, 0) \circlearrowleft$ around the origin, and even to obtain (together with the parabolic manifolds for f^{-1}) a flower theorem in dimension 2 (see [LR22]), in general one may find parabolic manifolds that are not tangent to any direction: orbits may converge to 0 while the tangent direction may spiral around one or more characteristic directions (see recent examples by Buff-Raissy).

For further details in dimension 2, we also suggest the survey [Aba15].

4.3 Tangent to the identity germs in dimension 3

4.3.1 Resolution of singularities of vector fields

Let us move to dimension 3, and see what difficulties arise here.

Resolution of singularities of vector fields is still available in dimension 3 (but not anymore starting from dimension 4), but much more intricate, due essentially to the fact that the blow-up loci are not anymore points, and that we need necessarily to consider weighted blow-ups (see [Pan06, Section 1.4]); weighted blow-ups introduce some mild singularities (namely, cyclic quotient) on the models X_π to be considered, but this can be dealt with by using orbifolds, or similarly by encoding this additional data with stacks.

In [Pan06], the author provides an algorithm to resolve singularities for analytic vector fields locally defined at the origin of $\mathbb{R}^3$: up to a finite sequence of weighted blow-ups, any real analytic vector field can be assumed to have *elementary* singularities. Up to further blow-ups, one can get even better final normal forms, called *strongly elementary*.

In [MP13, Theorem p.281], these results have been transported to the complex-analytic case. In this case the singularities are classified, following the minimal model problem for algebraic varieties, according to positivity properties of the canonical bundle of the associated foliation. Elementary singularities are called here *log-canonical* (see [MP13, I.ii.1 Definition]), while strongly elementary ones correspond to *canonical* singularities.

For what follows, we will only need smooth models: we recall here the definition of log-canonical singularities in this setting.

Definition 4.3.1. Let X be a smooth 3-fold, D a simple normal crossings (SNC) divisor on X, and χ a vector field locally defined at a point $p \in D$. Then χ is called *log-canonical* if its saturation (with respect to D) is tangent to D, and either regular, or singular at p with a non-nilpotent linear part.

In general, when working with a cyclic quotient singularity (X, p), we can see it as the quotient of $(\mathbb{C}^3, 0)$ by the action of some finite group Γ. Then a log-canonical foliation on (X, p) is induced by a log-canonical Γ-invariant foliation on $(\mathbb{C}^3, 0)$ (see [MP13, I.ii.5 Fact/Definition]).

We also need to recall the definition of (isolated) canonical singularities, (see [MP13, III.i.2 Definition and III.i.3 Fact]).

Definition 4.3.2. Let X be a smooth 3-fold, D a SNC divisor on X, and χ a saturated vector field at X with an isolated singularity at $p \in D$. Then χ is called (D-)*radial* if it is tangent to D and its linear part has eigenvalues $(\lambda_1, \lambda_2, \lambda_3) \in \lambda(\mathbb{N}^*)^3$ for some $\lambda \neq 0$.

A vector field χ as above is called (D-)*canonical* if it is (D-)log-canonical, but not (D-)radial.

In what follows, both log-canonical and canonical singularities are considered (without further mention) with respect to the exceptional divisor D whose support is $\pi^{-1}(0)$. The reduction of singularities for vector fields can be then stated as follows.

Theorem 4.3.3 ([MP13, Theorem p.281 and III.ii.2 Resolution])**.** *Let $(X, \mathcal{F})$ be a holomorphic foliation by curves on a 3-manifold X. Then there exists a sequence of weighted blow-ups $\pi : (X_\pi, D_\pi, \mathcal{F}_\pi) \to (X, \mathcal{F})$ so that $\mathcal{F}_\pi$ has only canonical singularities.*

4.3.2 Parabolic manifolds tangent to separatrices

We have seen with Theorem 4.2.1 two general techniques to construct parabolic manifolds. We have commented at length case (a) of non-degenerate characteristic directions, we review here case (b) of separatrices.

As mentioned above, the proof of Theorem 4.2.1.(b) is based on a local uniformization theorem along complex separatrices of vector fields in any dimension, which reduces χ into a vector field in so called Ramis-Sibuya normal form.

Definition 4.3.4 ([LRSV22, Definition 5.1]). Let χ be a (formal) vector field on $(\mathbb{C}^d, 0)$, and let C be a separatrix of χ. We say that the couple (χ, C) is in *Ramis-Sibuya normal form* if there exists local coordinates $(\mathbf{x}, z)$ at the origin (with $\mathbf{x} = (x_1, \dots, x_{d-1})$), such that C is transverse to $\{z = 0\}$, and f takes the form

$$\chi = z^{r+1}\big(-1 + \beta z^r + \langle z^{r+1}\rangle\big)\partial_z + \Big(\big(D(z) + z^r D_r\big)\mathbf{x} + \langle z^{r+1}\rangle\Big)\partial_{\mathbf{x}}, \tag{4.2}$$

where $r \geq 1$, $\beta \in \mathbb{C}$, $D = \mathrm{Diag}(d_1, \dots, d_{d-1})$ is a diagonal matrix of polynomials of degree at most $r-1$ on z vanishing at the origin, D_r is a constant matrix commuting with D, and $D(z) + z^r D_r \not\equiv 0$.

The local uniformization result can be stated as follows.

Theorem 4.3.5 ([LRSV22, Theorem 5.5]). *Let χ be a vector field on $(\mathbb{C}^d, 0)$, admitting a separatrix C (not contained in the singular locus of χ). Then there exists a sequence of weighted blow-ups $\pi\colon X_\pi \to (\mathbb{C}^d, 0)$ so that $(\hat{\chi}_\pi, C_\pi)$ is in Ramis-Sibuya normal form, where $\hat{\chi}_\pi$ denotes the saturated lift of χ, and C_π is the strict transform of C in X_π.*

The reduction process consists in three steps: first consider an embedded resolution of C, then apply the resolution of singularities Theorem 4.3.3, to obtain in the end the desired normal forms by further (weighted) blow-ups, following results on ODE's by Turrittin [Tur55].

Suppose now we start with a pair (f, C) with $f : (\mathbb{C}^d, 0) \circlearrowleft$ a tangent to the identity germ and C a separatrix of f. We can take its infinitesimal generator χ, and apply Theorem 4.3.5 to obtain a pair (χ_π, C_π) in Ramis-Sibuya normal form. By taking the time-1 flow of χ_π, we get a pair (f_π, C_π), that is also said to be in Ramis-Sibuya normal form. From this normal form, one can describe explicitly the number and dimensions of the parabolic manifolds provided by Theorem 4.2.1.(b).

With the notations of Equation (4.2), write d_j for $j = 1, \dots, d-1$ as

$$d_j(z) = \sum_{k=1}^{r-1} d_k^{(j)} z^k.$$

Given an attracting direction ζ for $f|_\Gamma$ (i.e., any complex r-th root of 1), we set

$$R_j(\zeta) = \big(\mathrm{Re}(d_1^{(j)}\zeta), \dots, \mathrm{Re}(d_{r-1}^{(j)}\zeta^{r-1})\big). \tag{4.3}$$

Definition 4.3.6. We say that ζ is a *node direction* for the variable x (resp., y) if $R_1(\zeta) < 0$ (resp., $R_2(\zeta) < 0$), and a *saddle direction* otherwise, where $<$ denotes the lexicographic order.

Roughly speaking, the coefficient of ∂_z in Equation (4.2) describes the dynamics of $f|_C$, while the matrix D is the leading term of the dynamics of f transverse to C: the attracting and repelling behavior in these j-th coordinate on the petal of $f|_C$ tangent to ζ is described exactly by $R_j(\zeta)$. This is the euristic that leads the description of parabolic manifolds for germs in Ramis-Sibuya normal forms.

Theorem 4.3.7 ([LRSV22, Theorem 6.1]). *Let $f : (\mathbb{C}^d, 0) \circlearrowleft$ be a tangent to the identity germ, and let C be separatrix. Suppose that (f, C) is in Ramis-Sibuya normal form. For any attracting direction ζ for $f|_C$, let $s = s(\zeta) \in \{0, \dots, d-1\}$ be the number of variables for which ζ is a node direction. Then there exists a parabolic manifold Δ_ζ asymptotic to C, of dimension $s(\zeta)+1$, which is connected, simply connected, and which is a fundamental domain for the set of points whose orbit converges to 0 asymptotic to C and tangent to ζ.*

4.3.3 Vector fields without complex separatrices

Recall that one of the ingredients to prove the existence of parabolic curves in dimension 2 is Camacho-Sad's construction of a separatrix for any planar vector fields. The analogous result does not hold in higher dimensions, as showed by Gomez-Mont and Luengo in [GL92]. In fact, the authors are able to construct families of vector fields χ on $(\mathbb{C}^3, 0)$ having only three characteristic directions v_1, v_2, v_3, and so that the (reduced) lift $\hat{\chi}_{\pi_0}$ at the blow-up $\pi_0 \colon X_{\pi_0} \to (\mathbb{C}^3, 0)$ of the origin has singularities at points p_1, p_2, p_3 corresponding to these directions, of a special type: their linear part have a non-trivial Jordan block associated to a non-zero eigenvalue, and the eigenvalues are all exceptional (tangent to the exceptional divisor). The authors then show that these singularities have no separatrices outside of the exceptional divisor, and this is done by blowing-up the points p_1, p_2, p_3, and showing that the only singularities that appear are *simple corners* (see below). The family of simple corners is stable by blow-ups, and in this case it is possible to show that simple corners do not support separatrices (outside of the exceptional divisor).

The flow of these vector fields has been studied by Abate and Tovena in [AT03]. In this case we do not have any parabolic manifold asymptotic to formal curves; however, the non-trivial Jordan block tells us that the points p_1, p_2, p_3 correspond to non-degenerate characteristic directions for f, and we get parabolic curves by Theorem 4.2.1.(a).

4.3.4 Example with no non-degenerate characteristic directions in dimension 3

In a project in collaboration with Samuele Mongodi, we started looking for examples of tangent to the identity germs in dimension 3 for which the known results do not guarantee the existence of parabolic manifolds. First of all, we exclude non-degenerate characteristic directions, because of Hakim's result. This condition corresponds to considering germs f

for which the homogeneous part of smallest degree (after the identity), acting on $\mathbb{P}^2_{\mathbb{C}}$, has no holomorphic fixed points. Examples of families of such maps can be found in [Iva11].

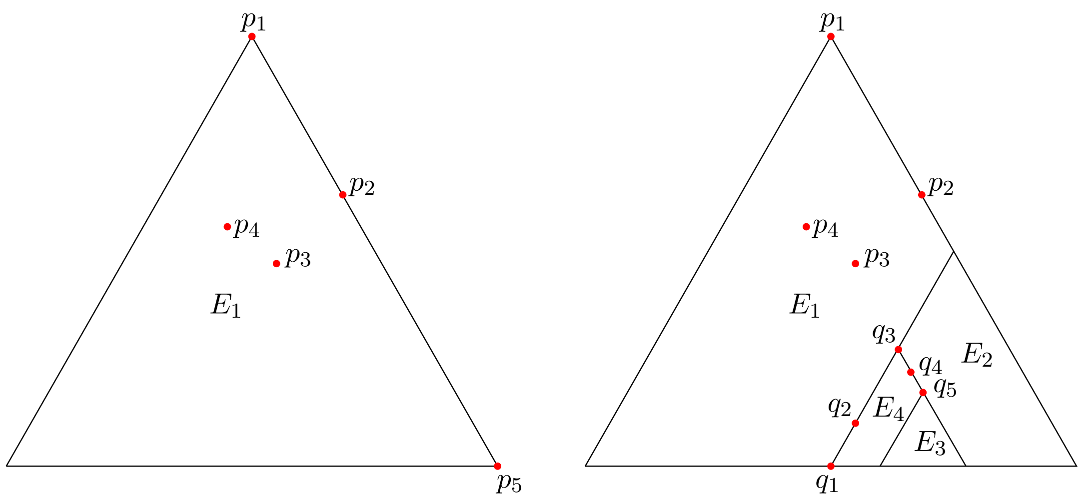
By following Camacho-Sad's construction of separatrices, and Abate's transposition to germs, one could try to find parabolic manifolds by looking for non-degenerate characteristic directions at some infinitely-near point $p \in \pi^{-1}(0)$, where $\pi\colon X_\pi \to (\mathbb{C}^3, 0)$ is a modification (adapted to f).

In [MR21], we exhibit a family of (isolated) tangent to the identity germs $f\colon (\mathbb{C}^3, 0) \circlearrowleft$ having only degenerate characteristic directions, but not admitting non-degenerate characteristic direction on any (reasonable) modification $\pi\colon X_\pi \to (\mathbb{C}^3, 0)$.

The family is given by:

$$f(x, y, z) = (x + yz(y - z) + P, y + x(x^2 - z^2) + Q, z + xz(y - z) + R), \tag{4.4}$$

where P, Q, R are formal power series of order 4 or higher. The elements of this family share the same five characteristic directions $v_1, \dots, v_5$, all degenerate. We denote by $p_1, \dots, p_5$ the corresponding points in $E_1 = \pi_1^{-1}(0)$ the exceptional divisor of the blow-up $\pi_1\colon X_1 \to (\mathbb{C}^3, 0)$ of the origin. By further blow-up three times, one can find a resolution $\pi_0\colon X_{\pi_0} \to (\mathbb{C}^3, 0)$ of the singularities of the infinitesimal generator χ of f. In this model we have 9 singularities, that we denote by $p_1, \dots, p_4, q_1, \dots, q_5$:



By studying these singularities, we show the following result.

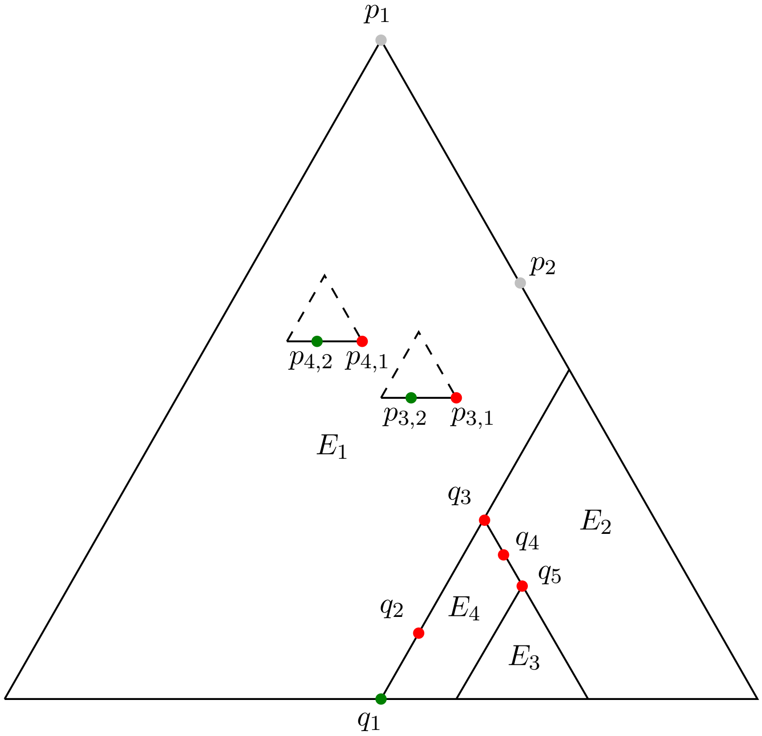
Theorem 4.3.8. *Let $f\colon (\mathbb{C}^3, 0) \circlearrowleft$ be a tangent to the identity germ of the form (4.4). For a Zariski-generic choice of the parameters P, Q, R, the germ f satisfies the following property:*

> *For any regular modification $\pi\colon X \to (\mathbb{C}^3, 0)$ strongly adapted to f and dominating π_0, and for any point $p \in \pi^{-1}(0)$ in the exceptional divisor, the lift $f_\pi\colon X_\pi \circlearrowleft$ of f at p has only degenerate characteristic directions.*

This result suggests that the strategy of looking for non-degenerate characteristic directions on some model will not work in general in dimension 3 or higher.

4.3.5 Special families of resolved singularities

Let $\widetilde{\pi}_0 \colon X_{\widetilde{\pi}_0} \to (\mathbb{C}^3, 0)$ be the modification obtained from X_{π_0} by blowing-up the points p_3 and p_4. In this model we have a total of 11 singularities, where we denote by $p_{j,1}$ and $p_{j,2}$ the ones appearing in the exceptional divisors of p_j, with $j = 3, 4$:



In order to describe the dynamics of f (and its lifts) around these singularties, we need to understand both the singularities of the saturated infinitesimal generator, and the saturation factor, which is supported on the exceptional divisor in our case (we may assume that 0 is an isolated fixed point for f).

In our study, we identify three classes of tangent to the identity germs:

- *Simple corners*, depicted in red, which in suitable coordinates are of the form:

$$f(x, y, z) = \begin{pmatrix} x + (x^a y^b z^c) x \big(\lambda + P\big) \\ y + (x^a y^b z^c) y \big(\mu + Q\big) \\ z + (x^a y^b z^c) R \end{pmatrix}, \tag{4.5}$$

  with $a, b \in \mathbb{N}^*$, $c \in \mathbb{N}$, $\lambda \in \mathbb{C}^*$, $\mu \in \mathbb{C} \setminus (\lambda \mathbb{Q}_{>0})$, $P, Q, R \in \mathfrak{m}$, and $z | R$ if $c > 0$.

  These singularities are the ones introduced in [GL92, AT03], and are known to support no separatrices (outside of the exceptional divisor). The infinitesimal generator here has at least a non-zero eigenvalue, satisfies a non-resonance condition, and is tangent to the exceptional divisor, that has either 2 or 3 irreducible components at the singularity.

- *Degenerate spikes*, depicted in grey, that can be written under the form

$$f(x,y,z) = \begin{pmatrix} x + z^c\big(\lambda x + P\big) \\ y + z^c\big(\mu y + Q\big) \\ z + z^{c+1} R \end{pmatrix}, \tag{4.6}$$

  with $c \in \mathbb{N}^*$, $\lambda \in \mathbb{C}^*$, $\mu \in \lambda\mathbb{R}_{<0}$, $P, Q \in \mathfrak{m}^2$ and $R \in \mathfrak{m}$. In this case, the infinitesimal generator has a central manifold of dimension 1 transverse to the exceptional divisor. This center manifold translates into a transverse separatrix, which gives parabolic manifolds thanks to Theorem 4.2.1.(b). By studying explicitly the reduction to Ramis-Sibuya normal forms of degenerate spikes, we can descrie precisely the number of parabolic manifolds, and their dimensio: in generic situations, they have dimension 1 or 2 (depending on the real direction associated to the parabolic manifold).

- *Spinning corners*, depicted in green, of the form

$$f(x,y,z) = \begin{pmatrix} x + y^b z^c (x + P) \\ y + y^{b+1} z^c Q \\ z + y^b z^{c+1} R \end{pmatrix} \tag{4.7}$$

  with $b,\ c \in \mathbb{N}^*$, $Q,\ R \in \mathfrak{m}$ and $P \in \mathfrak{m}^2$.

  Here the singular point is at the intersection of two irreducible components of the exceptional divisor, and the infinitesimal generator has a central manifold of dimension 2 transverse to the exceptional divisor.

Spinning corners are harder to study than simple corners and degenerate spikes: in [MR21] we give conditions under which we can find separatrices transverse to both components of the exceptional divisor, which again guarantee the existence of parabolic manifolds by Theorem 4.2.1.(b). To achieve the birational study of these singularities, we need to introduce another family of singularities (called *half corners*); by studying the reduction to Ramis-Sibuya normal forms of these families, we are able again to compute the number and dimension of the associated parabolic manifolds, which generically are again of dimension 1 or 2.

These computations describe completely the structure of the parabolic manifolds tangent v_j with $j \leq 4$.

4.3.6 Two-dimensional central manifolds

The case of direction v_5 is subtler. Its study leads us to study the problem of existence of parabolic manifolds in dimension 3 in the case where the (saturated) infinitesimal generator $\hat{\chi}_\pi$ at a point p in the exceptional divisor E_π of some model X_π has a reduced singularity, and a two-dimensional central manifold, that we denote by S.

In an ongoing project in common with André Belotto da Silva and Samuele Mongodi, we tackle this problem when the exceptional divisor E_π is irreducible at p. By studying the resolution of singularities of the saturation ξ of the restriction of $\hat{\chi}_\pi$ to S, we aim at proving that we are in one of the following situations: either there exists a separatrix, or we can find a saddle-node in a blown-up model. Once this is done, we will focus on extending the construction of parabolic manifolds [Hak98, Mol09, LR20, LRRS21] to our setting. As a direct application, we complete the study of f of the form Equation (4.4). Indeed, by resolution of vector fields in dimension 3, we can work with a birational model $\pi\colon X_\pi \to (\mathbb{C}^3, 0)$ dominating the blow-up of the origin, obtained by a weighted blow-up μ of p_5. In this model, we find on the exceptional divisor $\mu^{-1}(p_5)$, which is a weighted projective plane, two reduced singularities for the infinitesimal generator, one satisfying the condition above, and the other not supporting parabolic manifolds.

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