(42) 1989

Cyclic Homology and Characteristic Classes of Bundles with Additional Structures.

An Informal Report

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This short paper is just a report on some work relating cyclic homology and characteristic classes. The proofs and more details may be found in the references at the end of this present redaction. I hope that the informal style adopted here will help a better understanding.

O. Dictionary. The functor which associates to any C^∞ vector bundle V over a manifold X its space of sections $E = \Gamma(V)$ is in fact an equivalence of categories

$$\mathcal{E}(X) \longrightarrow \mathcal{P}(A)$$

Here $\mathcal{E}(X)$ is the category of vector bundles over X and $\mathcal{P}(A)$ is the category of finitely generated projective A-modules with $A = C^\infty(X)$ (Serre-Swan's theorem). To a large extent, the classical Chern-Weil theory on $\mathcal{E}(X)$ can be extended in a Chern-Weil theory on $\mathcal{P}(A)$ for any k-algebra A where k is an arbitrary commutative ring(*) (but A not necessarily commutative). One of the purposes of cyclic homology is to accomplish this goal [C] [K1]. In particular, the generalization of the classical Chern character

$$K(X) \longrightarrow H^{\text{even}}(X)$$

will be

$$K_0(A) \longrightarrow H_{\text{even}}(A)$$

where H_{even} is essentially cyclic homology. The following dictionary gives more examples of translating a geometrical concept into an algebraic one.

Space X	Ring A
Locally compact space X	C^*-algebra
Manifold $X+$ ordinary differential calculus in $\Omega^* X$	Ring $A+$ differential graded algebra $\Omega_*(A)$ with $A = \Omega_0 A$ and A dense subalgebra of a C^*-algebra.
Integration of smooth forms $\int$	Graded trace on $\Omega_*(A)$
Differential forms $\Omega^* X$	Hochschild homology $H_*(A, A)$
De Rham cohomology of forms $H^*(X)$	Cyclic homology $H_*(A)$ or $HC_*(A)$
Vector bundle V	Finitely generated projective module E
Connexion $D : \Gamma(V) \to \Gamma(V \otimes T^* X)$	Connexion $D : E \to E \bigotimes_A \Omega_1 A$
Matrix $\Gamma_{ij}^k = < D_{e_i} e_j,\ dx^k >$	Matrix $\Gamma_{ij} \in M_n(\Omega_1 A)$
Curvature $R = D^2 = d\Gamma + \frac{1}{2}[\Gamma, \Gamma]$ $= d\Gamma + \Gamma^2$	same

(*) We are assuming $\mathbb{Q} \subset k$ for sake of simplicity—see [K3] in general.

Topological K-theory $K^0(X) = K(X)$	Algebraic K-theory $K_0(A) = K(A)$
Topological K-theory $K^{-n}(X) = \widetilde{K}(S^n X^+)$	Topological K-theory $K_n^{\mathrm{top}}(A)$ $(= \pi_{n-1}(GL(A)))$ if A is a Fréchet algebra
Bundles with additional structures	Modules with additional structures
Multiplicative K-theory $MK(X)^{(*)}$	Multiplicative K-theory $MK(A)$

1. The Chern character for the group K_0

Let A be a ring with a unit. We want to define a "Chern character"

$$K_0(A) \longrightarrow H_*(A)$$

where $H_*(A)$ is some kind of "homology theory" associated to A of a De Rham type. We start with a differential graded algebra $\Omega_*(A)$

$$0 \to A = \Omega_0 A \xrightarrow{d} \Omega_1 A \xrightarrow{d} \Omega_2 A \to \cdots$$

(think of $\Omega_* A = \Omega^* X$, usual De Rham complex, if $A = C^\infty X$). If $w_n \in \Omega_n A$, $w_p \in \Omega_p A$, define their graded commutator $[w_n, w_p] = w_n w_p - (-1)^{np} w_p w_n$. Denote by $\overline{\Omega}_* A$ the quotient of $\Omega_* A$ by the k-module generated by graded commutators. The *non commutative De Rham homology* of A (or rather $\Omega_* A$) is the homology of the complex (cf. [K1])

$$0 \to \overline{\Omega}_0 A \xrightarrow{d} \overline{\Omega}_1 A \xrightarrow{d} \overline{\Omega}_2 A \xrightarrow{d} \cdots$$

which we shall denote by $H_*(A)$. Now the Chern character

$$\mathrm{ch} : K_0(A) \longrightarrow H_{2\cdot}(A)$$

can be defined in two ways:

a) If $E = \mathrm{Im}(e)$ with $e^2 = e \in M_r(A)$, then we put $\mathrm{ch}_n(E) = \frac{1}{n!}\mathrm{Trace}(e(de)^{2n}) \in H_{2n}(A)$. This is the simplest definition ([K1], [K3], [C]).

b) A *connexion* on E is a k-linear map

$$D : E \otimes_A \Omega_* A \longrightarrow E \otimes_A \Omega_{*+1}(A)$$

with the Leibnitz rule

$$D(s.w) = D(s).w + (-1)^{\deg(s)} s.dw$$

Its *curvature* $R = D^2$ is A-linear and we put $\mathrm{ch}_n(E)$ (also denoted by $\mathrm{ch}_n(D)$) $= \frac{1}{n!}\mathrm{Trace}(R^n) \in H_{2n}(A)$ because the homology class is independent of the choice

(*) The letter M stands for "multiplicative". We give up the notation $\mathcal{H}$ used in [K3] for typographical reasons.

of D. In this algebraic setting there is a *universal* example $\Omega_0 A = A$, $\Omega_1 A = \operatorname{Ker}(A \otimes A \to A)$ which is a A-bimodule, $\Omega_n A = \Omega_1 A \bigotimes_A \Omega_1 A \bigotimes_A \cdots \bigotimes_A \Omega_1 A$ (n factors); $d : A \to \Omega_1 A$ defined by $d(x) = 1 \otimes x - x \otimes 1$, etc....

Theorem: $H_n A = \operatorname{Ker}\left(HC_n(A) \xrightarrow{B} H_{n+1}(A, A)\right)$ *where* $HC_n(A)$ *is the cyclic homology of A. Connes and* $H_n(A, A)$ *is the Hochschild homology*(*).

2. The Chern character for the groups K_1 and K_i.

If $\alpha \in GL_r(A)$, define

$$\operatorname{ch}_n(\alpha) = \frac{(n-1)!}{(2n-1)!} \operatorname{Trace}(\alpha^{-1} d\,\alpha)^{2n-1}$$

This induces a character ($l = n - 1$)

$$K_1(A) \longrightarrow H_{1+2l}(A) \subset HC_{1+2l}(A)$$

More generally, one can define character maps (for $l \geq 0$)

$$K_i(A) \longrightarrow H_{i+2l}(A)$$

where K_i are the Quillen K-groups. This connects the homology of the group $GL(A)$ and the Lie algebra homology of $gl(A)$. In fact, one has $\operatorname{Prim}(H_*(gl(A))) = HC_{*-1}(A)$ (Loday-Quillen-Feigan-Tsygan theorem: cf. [LQ]).

If A is a Banach (or Fréchet) algebra, these higher Chern characters can be extended to topological K-theory through commutative diagrams

$$\begin{array}{ccc} K_i(A) & \longrightarrow & H_{i+2l}(A) \\ \downarrow & \nearrow & \\ K_i^{\text{top}}(A) & & \end{array} \qquad \begin{array}{ccc} K_i^{\text{top}}(A) & \longrightarrow & H_{i+2l}(A) \\ \beta \downarrow & & \downarrow S \\ K_{i+2}^{\text{top}}(A) & \longrightarrow & H_{i+2l-2}(A) \end{array}$$

where β is Bott periodicity and S is the periodicity map of cyclic homology (coming from the periodicity of the homology of finite cyclic groups).

3. "Relative K-theory"—Borel regulators.

Although the characters defined in §2 detect some part of algebraic or topological K-theory, this is not the full story!

Examples.

a) The simplest case of failure is the determinant map $K_1(\mathbf{C}) \to \mathbf{C}^*$. This is not covered completely by §2.

(*) Strictly speaking, one has to consider *reduced* cyclic homology in the statement of this theorem (cf. [C], [K3]).

b) If $A = \mathbf{Z}$, all characters in $H_*(A \bigotimes_{\mathbf{Z}} \mathbf{Q})$ are trivial. We know (by Borel and Quillen) that $K_5(\mathbf{Z}) = (\mathbf{Z} \bigoplus$ finite group) for example. This is not covered by §2 again.

c) If $A = C^\infty(X)$, A-modules correspond to C^∞-vector bundles. One would like to construct characteristic classes for bundles with additional structures: bundles which are flat, or foliated (X a foliated manifold), or analytic (X a complex analytic manifolds), or algebraic (X a complex algebraic manifolds).

1st approach. If A is a Fréchet algebra, denote by $\mathcal{F}(A)$ the homotopy fiber of the obvious map $BGL(A)^+ \to BGL(A)^{\text{top}}$ and define $K_n^{\text{rel}}(A) = \pi_n(\mathcal{F}(A))$

Theorem. (cf. [K3] and [CK]) *One has a commutative diagram*

$$\begin{array}{ccccccccc} K_{n+1}(A) & \to & K_{n+1}^{\text{top}}(A) & \to & K_n^{\text{rel}}(A) & \to & K_n(A) & \to & K_n^{\text{top}}(A) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_{n+1}(A,A) & \to & HC_{n+1}(A) & \to & HC_{n-1}(A) & \to & H_n(A,A) & \to & HC_n(A) \end{array}$$

where 1st row = homotopy exact sequence of the fibration, 2nd row = fundamental sequence of A. Connes, columns = various Chern characters; the map $K_n(A) \to H_n(A,A)$ is the Dennis trace map.

Examples.

a) $A = \mathbf{C}$—Diagram chasing gives an homomorphism $\alpha: K_{2i+1}(\mathbf{C}) \to \mathbf{C}^*$ which is an isomorphism on the torsion part (Suslin). The composition $K_{4i+1}(\mathbf{Z}) \to K_{4i+1}(\mathbf{C}) \to (\mathbf{C})^* \xrightarrow{\log|\,|} \mathbf{R}$ is the Borel regulator, therefore not trivial.

b) $A = C^\infty(S^1)$. One gets an homomorphism

$$K_2(C^\infty(S^1)) \longrightarrow \mathbf{C}^*$$

It is associated to the Kac-Moody extension of $SL(C^\infty(S^1))$ by $\mathbf{C}^*$ (cf. [CK]).

4. Multiplicative K-Theory.

The simplest case of such a "theory" is when we are considering flat complex vector bundles. Denote by $BGL_n(\mathbf{C})$(resp. $BGL_n(\mathbf{C})^\delta$) the classifying space of $GL_n(\mathbf{C})$ whith the usual (resp. discrete) topology. The Chern character is interpreted as a map of spaces $BGL_n(\mathbf{C}) \to \Pi K(\mathbf{C}, 2i)$ (product of Eilenberg-Mac Lane spaces) which homotopy fiber we denote by $\mathcal{F}_n$.

Theorem. *There is a canonical map $BGL_n(\mathbf{C})^\delta \to \mathcal{F}_n$ included in a commutative* *diagram*

$$\begin{array}{ccc} & & \mathcal{F}_n \\ & \nearrow & \downarrow \\ BGL_n(\mathbf{C})^\delta & \longrightarrow & BGL_n(\mathbf{C}) \\ & \overset{0}{\searrow} & \downarrow \\ & & \Pi K(\mathbf{C}, 2i) \end{array}$$

Making $n = \infty$, $\mathcal{F}_\infty$ can be interpreted as a classifying space of "multiplicative K-theory" $[X, \mathcal{F}_\infty] = K^{-1\text{top}}(X;\ \mathbf{C}^*)$: it fits into an exact sequence

$$K^{-1\text{top}}(X;\ \mathbf{Z}) \to K^{-1\text{top}}(X;\ \mathbf{C}) \to K^{-1\text{top}}(X;\ \mathbf{C}^*) \xrightarrow{\partial} K^{\text{top}}(X;\ \mathbf{Z}) \to$$

where $K^{*\text{top}}(X;\ \mathbf{Z})$ is topological K-theory and $K^{*\text{top}}(X;\ \mathbf{C})$ is De Rham cohomology (this is of course the origin of the terminology). If we put $K_{\mathbf{C}}(X) = [X,\ BGL(\mathbf{C})^+]$ (algebraic K-theory of Quillen of the space X), one has a characteristic map

$$K_{\mathbf{C}}(X) \longrightarrow K^{-1\text{top}}(X;\ \mathbf{C}^*)$$

Composed with the usual "Chern classes"

$$K^{-1\text{top}}(X;\ \mathbf{C}^*) \xrightarrow{c_k} H^{\text{odd}}(X;\ \mathbf{C}^*)$$

it gives essentially the Chern-Cheeger-Simons characteristic classes for flat complex vector bundles.

5. General setup for multiplicative K-theory.

It seems to be the following in the algebraic situation of §1: we give ourselves an extra data which is a decreasing filtration F^r of the differential graded algebra $\Omega_* A$ with $F^0 = \Omega_* A$, $F^r F^s \subset F^{r+s}$ and $d(F^r) \subset F^r$.

Examples from geometry ($A = C^\infty(X)$)

1. X complex analytic manifold. The Hodge filtration $F^r\Omega^*(X) = \bigoplus_{p \geq r} \Omega^{p,q}(X)$. If X is compact Kãeler, Hodge theory tells us that $H^*(F^r)$ injects in $H^*(X)$ (as $\bigoplus_{p\geq r} H^{p,q}(X)$).

2. X complex algebraic manifold. There is a refinement of the Hodge filtration called the Hodge-Deligne filtration F^r; $H^*(F^r)$ injects also in $H^*(X)$.

3. X foliated manifold; F^r is then generated by products of differentials of r functions transversal to the leaves, we have $F^r = 0$ for r larger than the codimension of the leaves.

4. X any manifold and put $F^r = 0$ for $r > 0$: this is the *trivial* filtration. It is of interest for characteristic classes of flat bundles (as in §4).

5. X any manifold and put $F^r\Omega^n = \Omega^n$ if $r \leq n$, and 0 otherwise. This is the filtration "bête" (Deligne terminology). It will be of interest in §6.

Grothendieck K-theory

Define the following category $\mathcal{P}^F(A)$:

Objects are finitely generated projective A-modules provided with a connexion $D: E \otimes \Omega_* \to E \otimes \Omega_{*+1}$ such that the curvature R factorizes through $E \otimes F^1\Omega_{*+2}$:

$$\begin{array}{ccc} E \otimes \Omega_* & \xrightarrow{R} & E \otimes \Omega_{*+2} \\ & \searrow \quad \nearrow & \\ & E \otimes F^1\Omega_{*+2} & \end{array}$$

Morphisms $\alpha: (E, D) \to (E', D')$ are A-module maps making the following diagram commutative

$$\begin{array}{ccc} E \otimes \Omega_* & \to & E' \otimes \Omega_* \\ \downarrow & & \downarrow \\ E \otimes \Omega_{*+1} & \to & E' \otimes \Omega_{*+1}/E' \otimes F^1\Omega_{*+1} \end{array}$$

(They commute with the connexions mod F^1.) This category is an *exact* category for the obvious definition of exact sequences. Call $K^F(A)$ its Grothendieck group.

Comment. We are really coming back to the origin of K-theory. If X is a complex projective algebraic manifold, $\Omega_*(A) = \Omega^*(X)$, F^r the Hodge filtration, this is the original definition of Grothendieck (use the Newlander-Nirenberg theorem and the paper of Atiyah-Hitchin-Singer). We call it $K(X)$ as Grothendieck did.

But, as we know, $K(X)$ is difficult to compute in general, much more than its topological counterpart $K^{\text{top}}(X)$ due to Atiyah and Hirzebruch (which is $K(A)$ in this context). So we need an intermediary group $MK^F(A)$ which can be inserted between $K^F(A)$ and $K(A)$

$$K^F(A) \longrightarrow MK^F(A) \longrightarrow K(A)$$

with two properties:

It is computable;

It is keeping part of the geometry.

How to define it?

The category $\mathcal{P}^F(A)$ is too rigid: asking the curvature R to be in F^1 is very strong. Instead, we may assume Trace $(R^r) \in F^r$. Precisely, we take triples (E, D, w) where D is a connexion on E and $ch_r(D) \equiv dw_r \bmod F^r$ with $w = \Sigma w_r$. $MK^F(A)$ is the Grothendieck group built out of these triples.

Theorem: *There is an exact sequence*

$$K_1(A) \to \oplus H_{2r-1}(\Omega_* A/F^r) \to MK^F(A) \to K(A) \to \oplus H_{2r}(\Omega_* A/F^r).$$

Moreover, the map $K^F(A) \to K(A)$ factorizes through $MK^F(A)$.

Example 1. X complex projective algebraic manifold with the Hodge filtration on $\Omega_* A = \Omega^*(X)$. Then the group $MK^F(A)$ denoted by $MK(X)$ can be computed using Hodge theory. We have an exact sequence

$$0 \longrightarrow T^\beta \longrightarrow MK(X) \longrightarrow G \longrightarrow 0$$

where T^β is a compact torus of dimension $\beta = \mathrm{rank}(H^{\mathrm{odd}}(X))$ and G a finitely generated abelian group(*). The map $K(X) \to MK(X)$ is far to be trivial.

Example 2. X foliated manifold with the canonical filtration on its De Rham complex. The normal bundle of the foliation has a natural class in $MK^F(A)$. All the known characteristic classes of the foliation (like Godbillon-Vey) can be simply deduced from this class using the fact that $F^r = 0$ for $r >$ codimension of the foliation.

6. Higher and lower multiplicative K-theory.

If A is a (real or complex) Banach algebra one can define groups $MK_n^F(A)$ (with $MK_0^F(A) = MK^F(A)$) with a natural periodicity map $MK_n^F(A) \to MK_{n-8}^F(A)$: they fit into an exact sequence

$$K_{n+1}^{\mathrm{top}}(A) \to \bigoplus H_{2r-n-1}(\Omega_* A/F^r) \to MK_n^F(A) \to K_n^{\mathrm{top}}(A) \to \bigoplus H_{2r-n}(\Omega_* A/F^r)$$

Example 1. With a good choice of the filtration, we have a nice group $MK_n(A)$ (suggested by G. Segal) in the diagram:

$$\begin{array}{ccccccccc} K_{n+1}^{\mathrm{top}}(A) & \to & K_n^{\mathrm{rel}}(A) & \to & K_n(A) & \to & K_n^{\mathrm{top}}(A) & \to & K_{n-1}^{\mathrm{rel}}(A) \\ \| & & \downarrow & & \downarrow & & \| & & \downarrow \\ K_{n+1}^{\mathrm{top}}(A) & \to & HC_{n-1}(A) & \to & MK_n(A) & \to & K_n^{\mathrm{top}}(A) & \to & HC_{n-2}(A) \end{array}$$

(Compare with §3.)

Example 2: Higher multiplicative K-theory seems to be the good set up for regulators in algebraic geometry. With the Hodge filtration, one gets exact sequences of the type

$$K_{n+1}^{\mathrm{top}}(X) \to \bigoplus_{\substack{p+q=2r-n-1 \\ p<r}} H^{p,q}(X) \to MK_n(X) \to K_n^{\mathrm{top}}(X) \to \bigoplus_{\substack{p+q=2r-n \\ p<r}} H^{p,q}(X)$$

(*) we have $\mathrm{rank}(G) \leq \sum_p \dim H^{p,p}(X)$

with a "regulator map" $K_n(X) \to MK_n(X)$. The groups $K_n(X)$ are related with Deligne cohomology and Beilinson work. Some work has also been done by Soulé in this direction.

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