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Algèbre/*Algebra*

Periodic cyclic cohomology of group rings

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Abstract. We generalize previous results ([2], [3], [4], etc.) relative to the cyclic homology and cohomology of the group algebra of G. In many cases, we express them in terms of the (co)homology of the discrete groups $\overline{\mathrm{Z}}(u) = \mathrm{Z}(u)/\mathrm{C}(u)$, where $< u >$ runs through the set of conjugacy classes of G and where $\mathrm{Z}(u)$ (resp. $\mathrm{C}(u)$) denotes the centralizer of u (resp. the cyclic group generated by u).

Cohomologie cyclique périodique d'algèbres de groupes

Résumé. *Nous généralisons les résultats de différents auteurs* ([2], [3], [4], *etc.*) *sur l'homologie et la cohomologie cycliques de l'algèbre d'un groupe G. Dans de nombreux cas, nous les exprimons en termes de (co)homologie des groupes discrets $\overline{\mathrm{Z}}(u) = \mathrm{Z}(u)/\mathrm{C}(u)$, où $< u >$ parcourt l'ensemble des classes de conjugaison de G, $\mathrm{Z}(u)$ (resp. $\mathrm{C}(u)$) désignant le centralisateur de u (resp. le groupe cyclique engendré par u).*

Version française abrégée

Soient R un anneau commutatif et $R[G]$ l'algèbre du groupe G. Plusieurs auteurs ([2], [3], [4], etc.) ont étudié l'homologie et la cohomologie cycliques de $R[G]$. D'après [2] et [4] par exemple, on a l'isomorphisme $\mathrm{HC}_*(R[G]) \cong \sum_{<u>} \mathrm{H}_*(\mathrm{B}G(u))$, où $< u >$ parcourt l'ensemble des classes de conjugaison de G et où $G(u) = \mathbf{R} \times_{\mathbf{Z}} \mathrm{Z}(u)$. Dans cette formule, $\mathrm{Z}(u)$ désigne le centralisateur de u et $\mathbf{Z}$ opère sur $\mathbf{R}$ et $\mathrm{Z}(u)$ via la translation par 1 et u respectivement; l'homologie est prise à coefficients dans R. L'homologie cyclique se décompose ainsi en $\mathrm{HC}_{\mathrm{ell}}(R[G]) \times \mathrm{HC}_{\mathrm{hyp}}(R[G])$, où la composante $\mathrm{HC}_{\mathrm{ell}}$ (resp. $\mathrm{HC}_{\mathrm{hyp}}$) correspond aux éléments u d'ordre fini (resp. infini).

La composante $\mathrm{HC}_{\mathrm{hyp}}$ a été déterminée essentiellement dans [2] et le but de cette Note est de calculer la composante $\mathrm{HC}_{\mathrm{ell}}$ en fonction de l'homologie d'Eilenberg–Mac Lane des groupes $\overline{\mathrm{Z}}(u) = \mathrm{Z}(u)/\mathrm{C}(u)$, $\mathrm{C}(u)$ désignant le groupe cyclique (fini) engendré par u. Cette analyse se fait

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plus commodément en *cohomologie* cyclique par l'intermédiaire de la suite spectrale associée à la fibration $BS^1 \to BG(u) \to B\overline{Z}(u)$. La première différentielle non triviale d_3, de cette suite spectrale s'insère dans la suite exacte

$$H^2(B\overline{Z}(u)) \longrightarrow H^2(BG(u)) \longrightarrow H^2(BS^1) \xrightarrow{d_3} H^3(B\overline{Z}(u)) \longrightarrow H^3(BG(u))$$

Soient θ le générateur canonique de $H^2(BS^1; \mathbf{Z})$, $\beta : H^2(B\overline{Z}(u); Z/n) \to H^3(B\overline{Z}(u); Z)$ l'homomorphisme de Bockstein et e la classe dans $H^2(B\overline{Z}(u); \mathbf{Z}/n)$ de l'extension centrale

$$1 \to \mathbf{Z}/n = C(u) \to Z(u) \to \overline{Z}(u) \to 1$$

Nous montrons alors que $d_3(\theta)$ est l'image $\beta_R(e)$ de $\beta(e)$ par le morphisme naturel $H^2(B\overline{Z}(u); \mathbf{Z}) \to H^3(B\overline{Z}(u); R)$. Nous en déduisons le théorème suivant (qui contient comme cas particulier le théorème principal de [4]) :

THÉORÈME. – *Supposons que* $\beta_R(e) = 0$ *pour une classe de conjugaison* $< u >$. *La cohomologie* $H^*(BG(u))$ *est alors isomorphe à l'algèbre tensorielle* $H^*(BS^1) \otimes H^*(B\overline{Z}(u)) \cong H^*(B\overline{Z}(u))[t]$, *où* t *est de degré* 2. *Dans le cas où cette propriété est vraie pour toute classe de conjugaison elliptique* $< u >$ *de* G, *on a donc*

$$HC^*_{ell}(R[G]) \cong \prod_{<u>ell} H^*(BS^1) \otimes H^*(B\overline{Z}(u)) \cong \prod_{<u>ell} H^*(B\overline{Z}(u))[t]$$

Soit $\{A_\alpha\}$ une famille de groupes abéliens. Nous définissons le « produit réduit » $A = \prod'_\alpha A_\alpha[t, t^{-1}]$ comme la limite inductive du système dénombrable (B_n), où $B_n = \prod_\alpha A_\alpha[t]$, l'application $B_n \to B_{n+1}$ étant définie via la multiplication par t. Un élément de A peut être identifié formellement à une famille de polynômes laurentiens $a_\alpha(t) \in A_\alpha[t, t^{-1}]$, où les exposants négatifs de t sont uniformément minorés. Le calcul suivant de la cohomologie cyclique périodique de $R[G]$ est une conséquence directe des observations précédentes.

THÉORÈME. – *Pour chaque classe de conjugaison* $< u >$, *où* u *est d'ordre fini dans* G, *supposons que* $\beta_R(e) = 0$. *Alors* $HP^*_{ell}(R[G]) \cong \prod'_{<u>ell} H^*(B\overline{Z}(u))[t, t^{-1}]$.

Remarques. – 1. Avec toujours la même hypothèse $\beta_R(e) = 0$ pour tout $< u >$, nous pouvons démontrer un résultat analogue pour l'*homologie* cyclique périodique. Si nous supposons G fini (pour éviter des problèmes de limite projective), nous avons alors les isomorphismes suivants :

$$HP_0(R[G]) \cong \prod_{<u>} \prod_k H_{2k}(B\overline{Z}(u)) \quad \text{et} \quad HP_1(R[G]) \cong \prod_{<u>} \prod_k H_{2k+1}(B\overline{Z}(u))$$

2. Si $\beta_R(e) \neq 0$, les calculs précédents ne sont pas valables en général. Comme exemple, nous pouvons choisir $G = \tilde{A}_5$, le revêtement à deux feuillets du groupe alterné A_5 et $R = \mathbf{Z}/2$. Soit u l'élément central de G d'ordre 2. Alors $H^*(BG(u)) \cong \mathbf{Z}/2[v_4] \otimes E(z_3)[\theta]$, où θ est défini ci-dessus. Cette cohomologie est différente de celle de $BS^1 \times BA_5$. De ce calcul, nous déduisons aussi l'isomorphisme suivant : $HP^*(BG(u)) \cong \mathbf{Z}/2[v_4] \otimes E(z_3)[\theta, \theta^{-1}]$.

1. Introduction and notations

1.1. Let R be any commutative ring. Cyclic (co)homology of the group algebra $R[G]$ has been extensively studied by various authors (*see* [2], [3], [4], etc.). According to [2] and [4] for instance, the cyclic homology can be expressed as the following direct sum

$$\mathrm{HC}_*(R[G]) \cong \sum_{<u>} \mathrm{H}_*(\mathrm{B}G(u))$$

Here $< u >$ runs through the set of all conjugacy classes of elements u in G; $\mathrm{Z}(u)$ is the centralizer of u and $G(u)$ the topological group $\mathbf{R} \times_{\mathbf{Z}} \mathrm{Z}(u)$, where $\mathbf{Z}$ acts via the translation by u (resp. 1) in the group $\mathrm{Z}(u)$ (resp. $\mathbf{R}$). In this formula, $\mathrm{B}\Gamma$ is a general notation for the classifying space of the topological group Γ and H_* is the homology with coefficients in R. An analogous formula holds for cyclic cohomology (using products instead of sums and the cohomology of $\mathrm{B}G(u)$ instead of its homology).

1.2. More precisely, the space whose (co)homology groups are the cyclic (co)homology groups can be expressed as the disjoint union of the $\mathrm{B}G(u)$. The "elliptic" (resp. "hyperbolic") case, according to the terminology of [2], corresponds to the case where u is of finite (resp. infinite) order. Therefore, we can split the cyclic (co)homology as the product

$$\mathrm{HC}_{\mathrm{ell}}(R[G]) \times \mathrm{HC}_{\mathrm{hyp}}(R[G])$$

according to conjugacy classes of elliptic and hyperbolic elements respectively. The periodic cyclic cohomology $\mathrm{HP}^0(R[G])$ (resp. $\mathrm{HP}^1(R[G])$) is the direct limit of the groups $\mathrm{HC}^{\mathrm{even}}(R[G])$ (resp. $\mathrm{HC}^{\mathrm{odd}}(R[G])$) by the S-map of Connes. It also splits as the product of an hyperbolic and an elliptic part.

1.3. The hyperbolic summand is easy to determine according to [2]: the space $\mathrm{B}G(u)$ is homotopically equivalent to $\mathrm{B}\overline{\mathrm{Z}}(u)$, where $\overline{\mathrm{Z}}(u)$ is the quotient group $\mathrm{Z}(u)/\mathrm{C}(u)$ with the $\mathrm{C}(u) =$ the cyclic group generated by u. Let us consider now the following fibration

$$\mathrm{BZ}(u) \longrightarrow \mathrm{B}\overline{\mathrm{Z}}(u) \xrightarrow{f} \mathrm{BS}^1 = \mathrm{CP}^\infty$$

If χ_u denotes $f^*(\theta)$, where θ is the canonical generator of $\mathrm{H}^2(\mathrm{CP}^\infty)$, then $\mathrm{HP}^*(\mathrm{B}G(u))$ is isomorphic to $\mathrm{H}^*(\mathrm{B}\overline{\mathrm{Z}}(u))(\chi_u^{-1})$ and $\mathrm{HP}^*_{\mathrm{hyp}}(R[G]) \cong \prod_{<u>\mathrm{hyp}} \mathrm{H}^*(\mathrm{B}\overline{\mathrm{Z}}(u))(\chi_u^{-1})$, at least if the number of conjugacy classes $< u >$ is finite.

2. The elliptic case

2.1. Let us consider now the case where u is of finite order n, which is more difficult to handle. In that case, we have the isomorphism $G(u) = \mathbf{R} \times_{\mathbf{Z}} \mathrm{Z}(u) \cong \mathrm{S}^1 \times_{\mathbf{Z}/n} \mathrm{Z}(u)$, where $\mathbf{Z}/n$ is embedded in S^1 as roots of the unity and embedded in $\mathrm{Z}(u)$ as $\mathrm{C}(u)$. We consider now the cohomology spectral sequence associated to the fibration

$$\mathrm{BS}^1 \to \mathrm{B}G(u) \to \mathrm{B}\overline{\mathrm{Z}}(u)$$

where $\overline{Z}(u)$ is again the quotient group $Z(u)/C(u)$. It implies the exact sequence

$$H^2(B\overline{Z}(u)) \longrightarrow H^2(BG(u)) \longrightarrow H^2(BS^1) \xrightarrow{d_3} H^3(B\overline{Z}(u)) \longrightarrow H^3(BG(u))$$

where d_3 is the first non-trivial differential.

2.2. Lemma. – *Let θ be the canonical generator of $H^2(BS^1)$. Then $d_3(\theta)$ is the image $\beta_R(e)$ of $\beta(e)$ by the homomorphism $H^2(B\overline{Z}(u); \mathbf{Z}) \to H^3(B\overline{Z}(u); R)$. Here $\beta : H^2(B\overline{Z}(u); Z/n) \to H^3(B\overline{Z}(u); Z)$ is the Bockstein homomorphism and e is the class in $H^2(B\overline{Z}(u); \mathbf{Z}/n)$ of the central extension*

$$1 \to \mathbf{Z}/n = C(u) \to Z(u) \to \overline{Z}(u) \to 1$$

Sketch of the proof. – It is enough to prove the theorem when $R = \mathbf{Z}$. The class e can be thought of as the image of the tautological element in $H^1(BC(u)); C(u)) \cong H^2(B^2C(u); C(u))$ by the homomorphism $H^2(B^2C(u); C(u)) \to H^2(B\overline{Z}(u); C(u))$, induced by the classifying map of the principal fibration $BZ(u) \to B\overline{Z}(u)$. On the other hand, let us consider the map $d_3 : H^2(BC(u)) \to H^3(B^3C(u))$, associated to the cohomology spectral sequence of the fibration $BC(u) \to EBC(u) \to B^2C(u)$. It is an isomorphism and we have the commutative diagram

$$\begin{array}{ccc} H^2(BC(u)) & \longrightarrow & H^3(B\overline{Z}(u)) \\ \| & & \| \\ H^2(BC(u)) & \xrightarrow{\cong} & H^3(B^2C(u)) \end{array}$$

Since the generator of $H^2(BC(u); \mathbf{Z}) \cong \mathbf{Z}/n$ is the Bockstein of the canonical generator of $H^1(BC(u)); C(u)) \cong \mathbf{Z}/n$, the lemma follows.

2.3. Theorem. – *Let R any ring of coefficients and let us assume that $\beta_R(e) = 0$ for a specific conjugacy class $< u >$. Then the cohomology $H^*(BG(u))$ is isomorphic to the tensor algebra $H^*(BS^1) \otimes H^*(B\overline{Z}(u)) \cong H^*(B\overline{Z}(u))[t]$, where t is of degree 2. If this happens to any elliptic conjugacy class $< u >$, we have*

$$HC^*_{ell}(R[G]) \cong \prod_{<u>ell} H^*(BS^1) \otimes H^*(B\overline{Z}(u)) \cong \prod_{<u>ell} H^*(B\overline{Z}(u))[t]$$

Proof. – The hypothesis implies that the restriction map $H^2(BG(u)) \to H^2(BS^1)$ is onto. Therefore, the fiber of the fibration $BS^1 \to BG(u) \to B\overline{Z}(u)$ is totally homologous to 0 and the spectral sequence degenerates: we have $H^*(BG(u)) \cong H^*(B\overline{Z}(u))$ as cohomology algebras.

2.4. *Remark.* – If $n = 0$ in R, we have automatically $\beta_R(e) = 0$.

2.5. If $Z(u)$ is Abelian, we have also $\beta_R(e) = 0$. This is due to the fact that $G(u) \cong S^1 \times \overline{Z}(u)$ as a group over $\overline{Z}(u)$. The isomorphism $S^1 \times_{C(u)} Z(u) \to S^1 \times_{C(u)} \overline{Z}(u)$ is defined by $(z, s) \mapsto (z\varepsilon(s)^{-1}, \overline{s})$, where $s \in Z(u)$, $\overline{s}$ is its class in $\overline{Z}(u)$ and $\varepsilon : Z(u) \to S^1$ is any extension of the inclusion $C(u) \subset S^1$ sending u to $e^{\frac{2i\pi}{n}}$ ($C(u)$ acts trivially on $\overline{Z}(u)$).

However, our hypothesis in 2.3 is more general than the one formulated in [4], where this argument is used, as we can see by the following example. Let Q_8 be the quaternion group of order eight and let G be the central extension

$$1 \longrightarrow \mathbf{Z}/2 \longrightarrow G \longrightarrow Q_8 \longrightarrow 1$$

corresponding to a non-trivial element in $H^2(Q_8; \mathbf{Z}/2) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2$ (*see* [1], p. 132). Clearly, the canonical inclusion $C(u) \subset S^1$ cannot be extended to $Z(u)$. However, we have $H^3(B\overline{Z}(u); \mathbf{Z}) = 0$ in this case (*see* again [1], p. 170).

2.6. *Remark.* – If $B_{\mathbf{Z}}(e) = 0$, we can in fact describe completely the homotopy type of $\mathrm{B}G(u)$. In the following sequence of homotopy fibrations:

$$\mathrm{BS}^1 \longrightarrow \mathrm{B}G(u) \longrightarrow \mathrm{B}\overline{\mathrm{Z}}(u) \xrightarrow{\lambda} \mathrm{B}^2\mathrm{S}^1 = K(\mathbf{Z}, 3)$$

the map λ is defined by the Bockstein $\beta_{\mathbf{Z}}(e)$ of the cohomology class e of the extension considered in 2.2. Since $\lambda = 0$, $\mathrm{B}G(u)$ has the homotopy type of $\mathrm{B}\overline{\mathrm{Z}}(u) \times \Omega(\mathrm{B}^2\mathrm{S}^1) \cong \mathrm{B}\overline{\mathrm{Z}}(u) \times \mathrm{BS}^1$.

3. Computation of periodic cyclic cohomology

3.1. Let $\{A_\alpha\}$ be a family of Abelian groups. We define the "reduced product" $A = \prod'_\alpha A_\alpha [t, t^{-1}]$ as the direct limit of the numerable system (B_n), where $B_n = \prod_\alpha A_\alpha [t]$, the map $B_n \xrightarrow{t} B_{n+1}$ being defined as the multiplication by t. An element of A may be identified formally with a family of Laurent polynomials $a_\alpha(t) \in A_\alpha [t, t^{-1}]$, where the negative exponents of t are uniformally bounded from below. The following complete computation of the periodic cyclic cohomology of $R[G]$ is a direct consequence of the previous observations.

3.2. Theorem. – *For each conjugacy class $< u >$, where u is of finite order in G, let us assume that $\beta_R(e) = 0$. Then* $\mathrm{HP}^*_{\mathrm{ell}}(R[G]) \cong \prod'_{<u>\mathrm{ell}} \mathrm{H}^*(\mathrm{B}\overline{\mathrm{Z}}(u))[t, t^{-1}]$.

3.3. *Remark.* – Under the same hypothesis $\beta_R(e) = 0$ for each $< u >$, we can prove an analogous statement for periodic cyclic *homology*. If G is finite (in order to avoid inverse limit problems), one has

$$\mathrm{HP}_0(R[G]) \cong \prod_{<u>} \prod_k \mathrm{H}_{2k}(\mathrm{B}\overline{\mathrm{Z}}(u)) \quad \text{and} \quad \mathrm{HP}_1(R[G]) \cong \prod_{<u>} \prod_k \mathrm{H}_{2k+1}(\mathrm{B}\overline{\mathrm{Z}}(u))$$

3.4. *Remark.* – If $\beta_R(e) \neq 0$, the previous results are not true in general. An example can be provided with $G = \tilde{\mathrm{A}}_5$, the double cover of the alternating group A_5 and $R = \mathbf{Z}/2$. Let u be the central element of G of order 2. Then $\mathrm{H}^*(\mathrm{B}G(u)) \cong \mathbf{Z}/2[v_4] \otimes E(z_3)[\theta]$, where θ is as before the image of the generator of $\mathrm{H}^*(\mathrm{BS}^1)$. This is very different from the cohomology of $\mathrm{BS}^1 \times \mathrm{BA}_5$. From this computation, we obtain the following isomorphism

$$\mathrm{HP}^*(\mathrm{B}G(u)) \cong \mathbf{Z}/2[v_4] \otimes E(z_3)[\theta, \theta^{-1}]$$

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