Master of Science in Mathematics *Michel Waldschmidt* Master Training Program
Royal University of Phnom Penh RUPP URPP - Université Royale de Phnom Penh
Centre International de Mathématiques Pures et Appliquées CIMPA
Coopération Mathématique Interuniversitaire Cambodge France

Linear Algebra

First Assignment, October 1, 2010

Exercise 1. Consider the following extended arrays,

$$A = \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right) \qquad B = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 0 \end{array}\right) \qquad C = \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 1 \\ 1 & 0 & 2 & 2 \end{array}\right)$$

$$D = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \qquad E = \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad F = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \end{array}\right)$$

$$G = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 3 & 4 & 0 \\ 3 & 4 & 5 & 0 \end{array}\right) \qquad H = \left(\begin{array}{ccc|c} 2 & -3 & 5 & 1 \\ 1 & 1 & 1 & 0 \\ -2 & -2 & -2 & 1 \end{array}\right) \qquad J = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 1 & -1 & 1 & 5 \end{array}\right)$$

For each of them,
a) say whether it is on row echelon form or not.
b) say whether it is on reduced row echelon form or not?
c) write the corresponding system of three linear equations in three variables, and say whether it has no solution, or exactly one solution, or finitely many solutions, or infinitely many solutions.

Exercise 2. Can you write an array corresponding to a linear system of three linear equations in three variables having exactly three solutions?

Exercise 3. Can you write an array corresponding to a homogeneous linear system of two linear equations in three variables having exactly one solution?

Exercise 4. Can you write an array corresponding to a linear system of three linear equations in four variables having no solution?

Exercise 5. Using row elementary transformations, compute the inverse of the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$$

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First Assignement, October 1, 2010 — solutions

Solution exercise 1
A: not in row echelon form (the first row is zero), hence not in reduced row echelon form. A row reduced echelon form is

$$\left(\begin{array}{ccc|c} 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The associated system is

$$\left\{\begin{array}{rcl} 0x_1 + 0x_2 + 0x_3 & = & 0 \\ x_2 + 2x_3 & = & 1 \\ x_3 & = & 2 \end{array}\right.$$

The variable x_1 is free, hence there are infinitely many solutions.
B: in row echelon form, but not in reduced row echelon form. There is a unique solution.
C: not in row echelon form, hence not in reduced row echelon form. There is no solution.
D: in reduced row echelon form. There is no solution.
E: in reduced row echelon form. There are infinitely many solutions.
F: not in row echelon form, hence not in reduced row echelon form. There is a unique solution.
G: not in row echelon form, hence not in reduced row echelon form. There are infinitely many solutions.
H:not in row echelon form, hence not in reduced row echelon form. There is no solution.
J: not in row echelon form, hence not in reduced row echelon form. There are infinitely many solutions.

Solution exercise 2
No: either there is no solution, or a single solution, or infinitely many solutions.

Solution exercise 3
No: when the number of variables of a homogeneous linear system is bigger than the number of equations, there are infinitely many solutions.

Solution exercise 4
For instance

$$\left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

Solution exercise 5
Use the elementary matrices

$$E_1 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

write

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$E_1 A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, \quad E_1 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix},$$

$$E_2 E_1 A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_2 E_1 = \begin{pmatrix} -5 & 2 \\ -3 & 1 \end{pmatrix},$$

$$E_3 E_2 E_1 A = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_3 E_2 E_1 = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}.$$

and deduce

$$A^{-1} = E_3 E_2 E_1 = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}.$$

Remark. The original version of Exercise 3 was:
Can you write an array corresponding to a homogeneous linear system of three linear equations in two variables having exactly one solution?
The answer is yes, one of many examples is

$$\begin{cases} x_1 & & & = & 0 \\ & & x_2 & = & 0 \\ x_1 & + & x_2 & = & 0 \end{cases}$$