Master of Science in Mathematics *Michel Waldschmidt* Master Training Program
Royal University of Phnom Penh RUPP URPP - Université Royale de Phnom Penh
Centre International de Mathématiques Pures et Appliquées CIMPA
Coopération Mathématique Interuniversitaire Cambodge France

Real Analysis

First Assignment, October 1, 2010

Exercise 1. Can you give an example
a) of a set of real numbers which is bounded above and not bounded below?
b) of a set of real numbers which is bounded below and not bounded above?
c) of a finite bounded set of real numbers which is not bounded above?
d) of a finite bounded set of real numbers which is not bounded below?
e) of an infinite set of real numbers which is bounded ?

Exercise 2. Let S be a set of real numbers. Define

$$S' = \{-x\,;x\in S\}.$$

a) Assume S is bounded above. Can you deduce that S' is bounded above? Can you deduce that S' is bounded below?
b) Assume S is bounded. Define

$$M=\sup S \quad\text{and}\quad m=\inf S.$$

Show that S' is bounded and compute

$$M'=\sup S' \quad\text{and}\quad m'=\inf S'.$$

c) Assume S is open. Can you deduce that S' is open? Can you deduce that S' is closed?
d) Denote by S^0 the interior of S, by $\partial(S)$ the boundary of S, by $\overline{S}$ the closure of S and by $(S')^0$ the interior of S', by $\partial(S')$ the boundary of S', by $\overline{S'}$ the closure of S'. What are the relations between these sets?

Exercise 3. Let S_1 and S_2 be two sets of real numbers. Assume S_1 is bounded above and S_2 is bounded below. Define

$$M_1=\sup S_1 \quad\text{and}\quad m_2=\inf S_2.$$

Assume $S_1\cap S_2\neq\emptyset$.
a) Prove $m_2\le M_1$.
b) Give an example where $m_2=M_1$.
c) Assume $m_2=M_1$. What can you say of $S_1\cap S_2$?

Exercise 4. Let n be a positive integer and $t_1,\dots,t_n$ be real numbers. Prove by induction

$$|t_1+t_2+\cdots+t_n|\le|t_1|+|t_2|+\cdots+|t_n|.$$

Exercise 5.
a) Let t be a positive irrational number. Show that $\sqrt{t}$ is irrational.
b) Show that any non zero real number is the product of two irrational numbers.

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First Assignement, October 1, 2010 — solutions

Solution exercise 1
a) An example is $(-\infty, 0)$.
b) An example is $(0, +\infty)$.
c) and d): No. A finite set is bounded.
e) An example is $(0, 1)$.

Solution exercise 2
a) We cannot deduce that S is bounded above: for instance when $S = (-\infty, 0)$ we have $S' = (0, +\infty)$. But we can deduce that S' is bounded below and $\inf S' = -\sup S$.
b) $m' = -M$ and $M' = -m$.
c) We have $(S')^0 = (S^0)'$, $\partial(S') = (\partial(S))'$ and $\overline{S'} = (\overline{S})'$ where, for a set T, we denote by T' the set of $-x$ for $x \in T$.

Solution exercise 3
a) Since $S_1 \cap S_2 \neq \emptyset$, there exists $x \in S_1 \cap S_2$. Since $x \in S_1$ we have $x \leq M_1$ Since $x \in S_2$ we have $x \geq m_2$. Hence $m_2 \leq M_1$.
b) For $S_1 = [1, 2]$ and $S_2 = [2, 3]$ we have $m_2 = M_1 = 2$.
c) If $m_2 = M_1$ then $S_1 \cap S_2$ is the singleton $\{m_2\} = \{M_1\}$. Indeed we have seen that any $x \in S_1 \cap S_2$ has $x \leq M_1$ and $x \geq m_2$, hence $x = m_2 = M_1$.

Solution exercise 4
Let $t_1, \ldots, t_n$ be real numbers. The inequality

$$|t_1 + t_2 + \cdots + t_n| \leq |t_1| + |t_2| + \cdots + |t_n|$$

is true when $n = 1$ (trivial) and when $n = 2$ (triangle inequality). Assume that $n \geq 2$ is an integer and that this inequality is valid for $n - 1$. Then

$$|t_1 + t_2 + \cdots + t_{n-1} + t_n| \leq |t_1 + t_2 + \cdots + t_{n-1}| + |t_n| \leq |t_1| + |t_2| + \cdots + |t_{n-1}| + |t_n|$$

by the triangle inequality and the induction hypothesis.

Solution exercise 5
a) If $\sqrt{t} = a/b$ is rational then $t = a^2/b^2$ is also rational.
b) Any irrational number t is the product $(\sqrt{t})(\sqrt{t})$ of two irrational numbers. Further, if c is a non zero rational number and t an irrational number, then c is the product $(c/t)t$. Since t is irrational and $c \neq 0$, then c/t is also irrational.