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Linear Algebra

Second Assignment, October 11, 2010 1 hour

Exercise 1.
Determine the balanced chemical reaction when reactants are C_2H_6 and O_2, while products are CO_2 and H_2O.

Exercise 2. Let u and v be two real numbers. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & u & v \end{pmatrix}.$$

a) Compute the determinant of A.
b) Write a necessary and sufficient condition on u and v for the matrix A to be invertible.
c) Compute the adjoint A' of A.
d) Compute the determinant of A'.
e) Compute AA'.
f) Assume the matrix A is invertible. Write the matrix A^{-1}, and use Cramer's rule for solving the system of 3 linear equations in 3 variables x, y, z:

$$\begin{cases} x & + & & & z & = & 0 \\ x & + & y & & & = & 0 \\ & & uy & + & vz & = & 1. \end{cases}$$

Exercise 3. Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be three points in the plane $\mathbf{R}^2$.
a) Using a determinant, write a necessary and sufficient condition for the existence of a line L passing through these three points.
b) Using a determinant, write a necessary and sufficient condition for the existence of a unique circle C passing through these three points.
c) Assume that there is a unique circle C passing through these three points. Using a determinant, write a necessary and sufficient condition for the circle C to pass through 0.

Exercise 4. Consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 5 & 2 & 1 & 4 \end{pmatrix} \in S_6$$

a) Decompose σ into a product of disjoint cycles.
b) Decompose σ into a product of transpositions.
c) Deduce the signature $\epsilon(\sigma)$ of σ.

Exercise 5. Compute the distance of the point $(2, -3)$ to the line of equation $3x + 4y + 1 = 0$.

Linear Algebra

Second Assignement, October 11, 2010 — solutions

Solution exercise 1
The chemical equation[1]

$$xC_2H_6 + yO_2 \longrightarrow zCO_2 + tH_2O$$

gives the system of 3 homogeneous linear equations (one for each of the atoms of carbon C, hydrogen H and oxygen O) in 4 variables x, y, z, t:

$$\begin{cases} 2x = z \\ 6x = 2t \\ 2y = 2z + t \end{cases}$$

with associated array

$$\begin{pmatrix} 2 & 0 & -1 & 0 \\ 6 & 0 & 0 & -2 \\ 0 & 2 & -2 & -1 \end{pmatrix}.$$

The smallest solution in positive integers is $(x, y, z, t) = (2, 7, 4, 6)$, which gives rise to the balanced chemical equation

$$2C_2H_6 + 7O_2 \longrightarrow 4CO_2 + 6H_2O;$$

all other solutions are $(x, y, z, t) = (2k, 7k, 4k, 6k)$ where k is a positive integer.

Solution exercise 2
a) The determinant of A is $u + v$.
b) A necessary and sufficient condition on u and v for the matrix A to be invertible is $u + v \neq 0$.
c) The adjoint A' of A is

$$A' = \begin{pmatrix} v & u & -1 \\ -v & v & 1 \\ u & -u & 1 \end{pmatrix}.$$

d)[2] The determinant of A' is $(u + v)^2$.
e) The product AA' is the diagonal matrix

$$\det(A)I_3 = \begin{pmatrix} u+v & 0 & 0 \\ 0 & u+v & 0 \\ 0 & 0 & u+v \end{pmatrix}.$$

[1] C_2H_6 is the formula for the molecule of Ethane, while O_2 is the Oxygen, CO_2 the Carbon dioxide and H_2O the Water.

[2] In general, for a $n \times n$ matrix, the determinant of the adjoint A' is $\det(A)^{n-1}$, because the product AA' is $\det(A)I_n$, which is a diagonal matrix with determinant $\det(A)^n$.

f) If A is invertible, the matrix A^{-1} is

$$\frac{1}{\det(A)}A' = \begin{pmatrix} \frac{v}{u+v} & \frac{u}{u+v} & \frac{-1}{u+v} \\ \frac{-v}{u+v} & \frac{v}{u+v} & \frac{1}{u+v} \\ \frac{u}{u+v} & \frac{-u}{u+v} & \frac{1}{u+v} \end{pmatrix}.$$

Cramer's rule for solving the associated system of 3 linear equations in 3 variables x, y, z gives

$$(u+v)x = \det\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & u & v \end{pmatrix} = -1, \quad (u+v)y = \det\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & v \end{pmatrix} = 1, \quad (u+v)z = \det\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & u & 1 \end{pmatrix} = 1.$$

The unique solution is given by $x = -y = -z = -1/(u+v)$.

Solution exercise 3

a) A line L with equation $ax + by + c = 0$ passes through the three points (x_1, y_1), (x_2, y_2), (x_3, y_3) if and only if

$$\begin{cases} ax_1 + by_1 + c &= 0, \\ ax_2 + by_2 + c &= 0, \\ ax_3 + by_3 + c &= 0. \end{cases}$$

The existence of such a line L is equivalent to the existence of a non–trivial solution (a, b, c) the this system of three homogeneous linear equations in three variables (a, b, c), hence it is equivalent to

$$\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} = 0.$$

b) A necessary and sufficient condition for the existence of a unique circle C passing through these three points is that they are not on a line, hence this condition can be written

$$\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} \neq 0.$$

c) Let an equation of the unique circle C passing through these three points be

$$a(x^2 + y^2) + bx + cy + d = 0.$$

Then C passes through 0 if and only if $d = 0$, which means that the system of 3 homogeneous linear equations in 3 variables (a, b, c)

$$\begin{cases} a(x_1 + y_1^2) & + & bx_1 & + & cy_1 & = & 0 \\ a(x_2 + y_2^2) & + & bx_2 & + & cy_2 & = & 0 \\ a(x_3 + y_3^2) & + & bx_3 & + & cy_3 & = & 0 \end{cases}$$

has a non–trivial solution. Hence the answer is

$$\det\begin{pmatrix} x_1^2+y_1^2 & x_1 & y_1\\ x_2^2+y_2^2 & x_2 & y_2\\ x_3^2+y_3^2 & x_3 & y_3\end{pmatrix}=0.$$

Solution exercise 4

$$\sigma=\begin{pmatrix}1&2&3&4&5&6\\3&6&5&2&1&4\end{pmatrix}=(1\;3\;5)(2\;6\;4)=(1\;3)(1\;5)(2\;6)(2\;4).$$

The number of transpositions in the product in the right hand side is even, hence the signature $\epsilon(\sigma)$ is $+1$. Also the number of cycles of even length is 0, an even number.

Solution exercise 5

The distance of a point P in the plane $\mathbf{R}^2$ of coordinates (x_0, y_0) to a line L of equation $ax+by+c=0$ is given by

$$\frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}\cdot$$

Here $x_0=2$, $y_0=-3$, $a=3$, $b=4$, $c=1$, hence the distance of the point P of coordinates $(2,-3)$ to the line L of equation $3x+4y+1=0$ is

$$\frac{|6-12+1|}{\sqrt{9+16}}=\frac{|-5|}{5}=1.$$

Let us check that the orthogonal projection H of P on L has coordinates $(13/5,-11/5)$: this point is on L since $3\cdot 13/5+4\cdot(-11/5)+1=0$, the vector $\overrightarrow{PH}$ is

$$(13/5,-11/5)-(2,-3)=(3/5,4/5),$$

hence is parallel to $(a,b)=(3,4)$, and therefore perpendicular to L. The length of $\overrightarrow{PH}$ is $\sqrt{(3/5)^2+(4/5)^2}=1$.