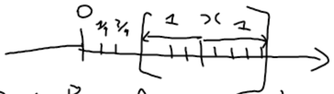
Approximation

$$\frac{p_n}{q_n} \xrightarrow[n\to+\infty]{} x \implies q_n \xrightarrow[n\to+\infty]{} +\infty$$

$\forall n \quad \frac{p_n}{q_n} \neq x$

$q_n > 0$

dem



$\cdot q \quad \left\{ p \,/\, \frac{p}{q} \in [x-1\,;\, x+1] \right\}$ fini

$\left\{ \frac{p}{q}\,;\ q \le H\,;\ \dots \right\}$ fini

$$\varepsilon = \min_{\substack{\frac{p}{q} \in [x-1,\, x+1] \\ \frac{p}{q} \neq x;\ q \le H}} \left| x - \frac{p}{q} \right| > 0$$

$\frac{\varepsilon}{2} \qquad \frac{p_n}{q_n} \to x \qquad \exists N \ \forall n \geq N$

$\left| \frac{p_n}{q_n} - x \right| \le \frac{\varepsilon}{2} \Rightarrow q_n \geq H$

$\square$

$\frac{1}{q^\alpha} \qquad \left| \frac{p}{q} - x \right| \le \frac{K}{q^\alpha} \qquad K, \alpha$ constante

* $x = \frac{a}{b} \in \mathbb{Q}$

$$\left| \frac{p}{q} - \frac{a}{b} \right| = \frac{|pb - aq|}{bq} \geq \frac{1}{bq}$$

$\frac{p}{q} \neq \frac{a}{b}$

⚠ $n \in \mathbb{Z},\ n \neq 0 \Rightarrow |n| \geq 1$

* **Prop:** $x \notin \mathbb{Q}$, $x \in \mathbb{R}$ $\iff \exists\, \infty$té $\frac{p}{q}$ $\quad \left| x - \frac{p}{q} \right| \le \frac{1}{q^2}$

Def: Pour $x \in \mathbb{R}$, on note $\gamma(x)$ la borne sup des γ tq $\left\{ \frac{p}{q} \in \mathbb{Q}\,;\ \left| x - \frac{p}{q} \right| \le \frac{1}{\gamma q} \right\}$ est infini

Prop: (Hurwitz) (i) $\forall x \in \mathbb{R}\setminus\mathbb{Q} \quad \gamma(x) \geq \sqrt{5}$

(ii) $\exists x \in \mathbb{R}\setminus\mathbb{Q}$ tq $\gamma(x) = \sqrt{5}$

dem suites de Farey d'indice n

on écrit dans l'ordre croissant tous les rationnels $\frac{p}{q} \in [0,1]$ et $q \le n$

ex: $n = 6 \quad 0, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, 1$

$\frac{p}{q} < \frac{r}{s}$ 2 fractions consécutives alors $qr - ps = 1$

$$\frac{p}{q} < \frac{u}{v} < \frac{r}{s}$$

$$uq - vp = 1 \Longrightarrow u = (p+r)\lambda$$
$$vr - us = 1 \qquad v = (q+s)\lambda$$
$$u(q+s) = v(p+r)$$

(i) Lemme | $\exists$ cste $\frac{p}{q} < x < \frac{r}{s}$ avec $qr - ps = 1$

dem: $\forall H$; $\min\{|x - \frac{p}{q}|;\ q \le H\}$ est réalisé en $\frac{a}{b}$

1) supposons $\frac{a}{b} < x$ $\qquad \frac{p}{q} = \frac{a}{b}$

$p \wedge q = 1$

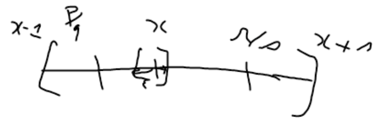
Bezout $\exists r, s \quad rq - ps = 1$

$$0 \le s < q \le H$$

$$|x - \frac{p}{q}| \le |x - \frac{r}{s}|$$

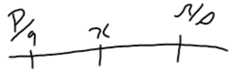
$\frac{p}{q} < x$

$\frac{p}{q} < \frac{r}{s}$



on prend $\varepsilon = \min\{|x - \frac{p}{q}|;\ q \le H\}$

$\exists H' \quad q \le H' \quad |x - \frac{p}{q}| \le \frac{\varepsilon}{2} \Longrightarrow q \ge H \rightsquigarrow$ cste



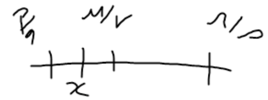
$$\delta = \min\left\{q^2 \left|\frac{p}{q} - x\right|,\ s^2 \left|\frac{r}{s} - x\right|\right\}$$

Lemme: $\delta \le \frac{1}{2}$

dem:
$$\frac{\delta}{q^2} \le x - \frac{p}{q}$$
$$\frac{\delta}{s^2} \le \frac{r}{s} - x$$
$$\delta\left(\frac{1}{q^2} + \frac{1}{s^2}\right) \le \frac{r}{s} - \frac{p}{q} = \frac{1}{sq}$$
$$\frac{s}{q} + \frac{q}{s} \le \frac{1}{\delta}$$

$$2 \le \frac{q}{s} + \frac{s}{q} \le \frac{1}{\delta}$$

$t + \frac{1}{t}$ prend son min en 1 min = 2



$\delta \le \frac{1}{2}$

Lemme: $\delta = \min\left\{q^2|x - \frac{p}{q}|;\ s^2|x - \frac{r}{s}|;\ v^2|x - \frac{u}{v}|\right\}$

$u = p + r$

$v = q + s$ $\qquad$ alors $\delta \le \frac{1}{\sqrt{5}}$

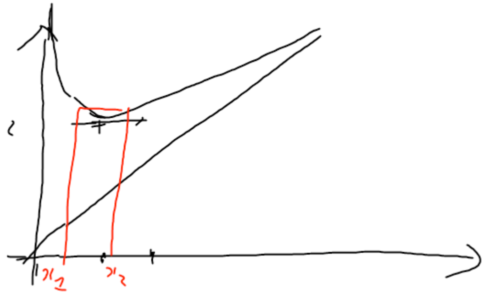
On applique le lemme précédent:

· $\frac{p}{q} < x < \frac{u}{v}$

$$uq - pv = (p+r)q - p(q+s)$$
$$= qr - ps = 1$$

$$\frac{q}{v} + \frac{v}{q} \leqslant \frac{1}{\delta}$$

· $\frac{p}{q} < x < \frac{r}{s}$ $\qquad \frac{q}{s} + \frac{s}{q} \leqslant \frac{1}{\delta}$



Soit x_i tq $x_i + \frac{1}{x_i} = \frac{1}{\delta}$ $\qquad X^2 - \frac{1}{\delta}X + 1 = 0$

$i = 1, 2$ $\qquad \Delta = \left(\frac{1}{\delta}\right)^2 - 4$

$\frac{s}{q}, \frac{v}{q}$ $\quad \frac{q}{s}, \frac{q}{v} \in [x_1, x_2]$

$$\frac{s}{q} - \frac{v}{q} = \frac{s-q-s}{q} = -1$$

$$|x_2 - x_1| = \sqrt{\frac{1}{\delta^2} - 4} \geqslant 1$$

$\frac{1}{\delta^2} \geqslant 5 \iff \delta \leqslant \frac{1}{\sqrt{5}} \iff \gamma = \frac{1}{\delta} \geqslant \sqrt{5}$

(ii) $\quad X^2 - X - 1 = (X - \phi)(X + \phi^{-1})$

$\phi = \frac{1+\sqrt{5}}{2}$

$-\phi^{-1}$

Lemme: $\forall q \geqslant 1$
$$\left|\phi - \frac{p}{q}\right| > \frac{1}{\sqrt{5}\,q^2 + 1/\sqrt{5}}$$

dem: Il suffit de prendre les $\frac{p}{q}$ tq $\left|\phi - \frac{p}{q}\right| \leqslant \frac{1}{\sqrt{5}\,q^2}$

$$X^2 - X - 1 = (X - \phi)(X + \phi^{-1})$$
$$p^2 - qp - q^2 = q^2\left(\frac{p}{q} - \phi\right)\left(\frac{p}{q} + \phi^{-1}\right)$$

$\left|\frac{p}{q} - \phi\right| \leqslant \frac{1}{\sqrt{5}\,q^2} \implies |p^2 - qp - q^2| \geqslant 1$

$p \leqslant q\phi + \frac{1}{\sqrt{5}\,q}$ $\qquad \left|\frac{p}{q} - \phi\right| \geqslant \frac{1}{q^2}\,\frac{1}{\sqrt{5} + \frac{1}{\sqrt{5}\,q^2}} = \frac{1}{q^2\sqrt{5} + 1/\sqrt{5}}$

$$\frac{p}{q} + \phi^{-1} \leqslant \underbrace{\phi + \phi^{-1}}_{=\sqrt{5}} + \frac{1}{\sqrt{5}\,q^2}$$

$\square$

$F_0 = 0 \quad F_1 = 1 \quad F_n = F_{n-1} + F_{n-2}$

$$F_n = \frac{1}{\sqrt{5}}\left(\phi^n + (-1)^{n-1}\phi^{-n}\right)$$

Lemm: $F_{n-1}^2\left(\phi - \frac{F_n}{F_{n-1}}\right) \longrightarrow \frac{1}{\sqrt{5}}$

dem: . $F_n = \left[\frac{\phi^n}{\sqrt{5}}\right]$

$$u_n = \frac{F_n}{F_{n-1}} \to \phi$$

$$F_n^2 - F_{n-1}F_n - F_{n-1}^2 = (-1)^{n-1} = F_{n-1}^2 (u_n - \phi)(u_n + \phi^{-1})$$

$$u_n + \phi^{-1} \longrightarrow \phi + \phi^{-1} = \sqrt{5} \quad \text{OK} \quad \square$$

rem $\gamma(x) = \gamma(x+1)$

$$\left|x - \frac{p}{q}\right| < \frac{1}{\gamma(x) q^2}$$

$$\left|x+1 - \frac{p+q}{q}\right| < \frac{1}{\gamma(x) q^2}$$

$\gamma(x+1) \geq \gamma(x)$, $\gamma(x-1) \geq \gamma(x)$ $\Rightarrow$ égaux

$$\gamma(x) = \gamma(-x)$$

$$\gamma\left(\frac{1}{x}\right) = \gamma(x)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}) \quad x \longmapsto \frac{ax+b}{cx+d}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\frac{ax+b}{cx+d} - \frac{ap+bq}{cp+dq} = \pm \frac{qx-p}{(cx+d)(cp+dq)}$$

$$\gamma(\phi) = \sqrt{5}$$

$$\gamma\left(\frac{a\phi+b}{c\phi+d}\right) = \sqrt{5}$$

théorème: $x \notin \bigcup_{GL_2(\mathbb{Z})} (\phi)$

$$\gamma(x) \geq \sqrt{8}$$

$$\gamma(\sqrt{2}) = \sqrt{8}$$

$\rightsquigarrow$ suite $\lambda_m = \sqrt{9 - \frac{4}{m^2}}$

où la suites des m est celle de Markoff

$f_1(x) = x^2 - x - 1 \rightsquigarrow \gamma = \lambda_1 = \sqrt{5} \qquad \overline{1}$

$f_2(x) = 2x^2 + 4x - 2 \rightsquigarrow \gamma = \lambda_2 = \sqrt{8} \qquad \overline{2}$

$f_5(x) = 5x^2 + 11x - 5 \rightsquigarrow \gamma = \lambda_5 = \qquad \overline{2211}$

$\overline{221111}$

$f(X,Y) = aX^2 + bXY + cY^2$

$a, b, c \in \mathbb{R}$

$\Delta(f) = b^2 - 4ac \neq 0$

$m(f) = \min\left\{ |f(x,y)| \; ; \; (x,y) \in \mathbb{Z}^2 \setminus \{(0,0)\} \right\}$

$$C(f) = \frac{m(f)}{\sqrt{|\Delta(f)|}}$$

· $\underline{\Delta < 0}$ (Lagrange-Hermite). $C(f) \leq \frac{1}{\sqrt{3}}$

égalité $f = X^2 + XY + Y^2$

. $\forall \rho \in \left]0, \frac{1}{\sqrt{3}}\right] \; \exists f$ tq $C(f) = \rho$

$\underline{\Delta > 0}$

$C(f) \leq \frac{1}{\sqrt{5}}$ $\quad$ = té $f_0 = X^2 - XY - Y^2$

sinon $f \not\sim f_0 \Rightarrow C(f) \leq \frac{1}{\sqrt{8}}$

$\underline{\text{Markoff}}$: $\exists \; f_i$ suite infinie

$C(f_i) \longrightarrow \frac{1}{3}$

f $\Delta(f)$ fixé $\rightsquigarrow \left\{ \frac{1}{C(f)} \right\} = \left\{ \lambda_n \right\}$

$\underline{\text{Def}}$: $x \in K$ est algébrique s'il est racine d'un polynôme à coefficients dans $\mathbb{Q}$

$\exists \; P(X) = a_n X^n + \cdots + a_0$

tq $a_n x^n + \cdots + a_0 = 0$

$\underline{\text{ex}}$: $\sqrt{2}$ $\quad X^2 - 2$

$\underline{\text{Def}}$: x algébrique $\rightsquigarrow$ son degré = le + petit des degrés d'1 $P \in \mathbb{Q}[X]$ tq $P(x) = 0$

$\exists ! \; P \in \mathbb{Q}[X]$ unitaire de degré $\deg x$ tq $P(x) = 0$

On l'appelle le polynôme minimal de x

Il est irréductible

$\underline{\text{dem}}$: $\underline{\text{irred}}$: $P = QR \Rightarrow Q(x) = 0$ ou $R(x) = 0$

par minimalité de $\deg P$

$\Downarrow$

$P = \lambda Q$ ou $P = \lambda R$

où $\lambda \in \mathbb{Q}^\times$

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unicité : $I_x = \{ Q \in \mathbb{Q}[X] ; Q(x) = 0 \}$

c'est un idéal de $\mathbb{Q}[X]$

$\exists\, \mu_x$ unitaire $\in \mathbb{Q}[X]$

tq $I_x = (\mu_x)$

si $P / P(x) = 0 \Rightarrow \mu_x \mid P$

si deg P est minimal $\Rightarrow$ P irred $\Rightarrow \mu_x = \lambda P$

unitaire $\Rightarrow \lambda = 1$

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Prop : soit x entier alg de degré $d \geq 2$

$\exists\, K_x$ tq $\forall\, \frac{p}{q} \in \mathbb{Q}$

$$\left| x - \frac{p}{q} \right| \geq \frac{K_x}{q^d}$$

ex : $\sum_n 10^{-n!}$ est transcendant

dem : $\left| \mu_x(x) - \mu_x\left(\frac{p}{q}\right) \right| \leq K \left| x - \frac{p}{q} \right|$ $\quad \frac{p}{q} \in [x-1, x+1]$

$K = \sup_{c \in [x-1, x+1]} |\mu'(c)|$

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$$\mu_x\left(\frac{p}{q}\right) = \left(\frac{p}{q}\right)^d + a_{d-1}\left(\frac{p}{q}\right)^{d-1} + \dots + a_0$$

$$q^d \mu_x\left(\frac{p}{q}\right) = p^d + a_{d-1}\, q\, p^{d-1} + \dots + a_0 q^d$$

$$\left| q^d \mu_x\left(\frac{p}{q}\right) \right| \geq 1$$

$$\left| x - \frac{p}{q} \right| \geq \frac{1}{K q^d} \qquad K_x := \frac{1}{K}$$