Master of Science in Mathematics *Michel Waldschmidt* Master Training Program
Royal University of Phnom Penh RUPP URPP - Université Royale de Phnom Penh
Centre International de Mathématiques Pures et Appliquées CIMPA
Coopération Mathématique Interuniversitaire Cambodge France

MMA105: Discrete Mathematics

Second Assignment, May 1, 2009

Exercise 1. Define u_n for $n \geq 0$ by $u_0 = 0$, $u_1 = 1$ and

$$u_{n+1} = 5u_n - 6u_{n-1} \qquad (n \geq 2).$$

a) Give the numerical values of u_2, u_3, u_4 and u_5. Next write a formula for u_n valid for all $n \geq 1$ and prove it.
b) Show that the sequence $(u_{n+1}/u_n)_{n\geq 1}$ has a limit and compute it.
c) Check that u_{100} is a multiple of u_{50}.
d) Write the rational fraction $f(z)$ whose Taylor expansion at the origin is

$$\sum_{n\geq 0} u_n z^n.$$

What is the radius of convergence of this series? What is the value of $f(1/6)$?

Exercise 2. Let $a = 2^6 \times 3^7 \times 5^4$ and $b = 2^2 \times 3^4 \times 5 \times 7^5$ (it is not useful to compute the numerical values).
a) Which are the integers $x \in \mathbf{Z}$ such that x^5 is a multiple of a?
b) What is the decomposition into prime factors of the product ab?
c) What is the decomposition into prime factors of the gcd of a and b?
d) What is the decomposition into prime factors of the lcm of a and b?

Exercise 3.
a) Write the decomposition into prime factors of 318 and 222, and deduce the gcd and the lcm of these two numbers.
b) Compute two integers u and v such that $318u + 222v = \gcd(318, 222)$.

Exercise 4. Denote by $(F_n)_{n\geq 0}$ the Fibonacci sequence. Check for $n \geq 1$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}^n = (-1)^n \begin{pmatrix} F_{n-1} & -F_n \\ -F_n & F_{n+1} \end{pmatrix}.$$

What is the product?

Solution exercice 1
$X^2 - 5X + 6 = (X-2)(X-3)$.
Base: 2^n, 3^n. Solution qui commence par 0, 1: $u_n = 3^n - 2^n$.
La suite en question commence par $0 = 3^0 - 2^0$, $1 = 3 - 2$, $5 = 9 - 4$, $19 = 27 - 8$, $65 = 81 - 16$, $211 = 243 - 32$
$u_{100} = 3^{100} - 2^{100}$ est le produit de $u_{50} = 3^{50} - 2^{50}$ par $3^{50} + 2^{50}$.
Plus généralement u_n divise u_{2n}.

La série génératrice est

$$f(z) = \sum_{n \geq 0} \frac{u_n}{2^n} = \frac{1}{(1-2z)(1-3z)} = \frac{1}{1-3z} - \frac{1}{1-2z}$$

avec rayon de convergence 1/3 et

$$f(1/6) = \frac{1}{1-1/2} - \frac{1}{1-1/3} = \frac{2}{2-1} - \frac{3}{3-1} = 2 - \frac{3}{2} = \frac{1}{2}.$$

Solution exercice 2
$a = 2^6 \times 3^7 \times 5^4$ and $b = 2^2 \times 3^4 \times 5 \times 7^5$
a) Integers $x \in \mathbf{Z}$ such that x^5 is a multiple of a
La réponse est $k\mathbf{Z}$ avec $k = 2^2 3^2 5 = 180$.
b) Decomposition into prime factors of the product ab?
$2^8 \times 3^{11} \times 5^5 \times 7^5$
c) Decomposition into prime factors of the gcd of a and b?
$2^2 \times 3^4 \times 5$
c) Decomposition into prime factors of the lcm of a and b?
$2^6 \times 3^7 \times 5^4 \times 7^5$

Solution exercice 3
a) $318 = 2 \times 3 \times 53$, $222 = 2 \times 3 \times 37$, gcd $= 6$.
b)

$$318 = 222 + 96, \quad 222 = 96 \times 2 + 30, \quad 96 = 30 \times 3 + 6, \quad 30 = 6 \times 5.$$

$$6 = 96 - 3 \times 30 = 96 - 3(222 - 96 \times 2) = -3 \times 222 + 7 \times 96$$

$$6 = -3 \times 222 + 7(318 - 222) = 7 \times 318 - 10 \times 222$$

On vérifie bien $318 \times 7 = 2226 = 222 \times 10 + 6$.

Solution exercice 4
For $n \geq 1$,

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{pmatrix} = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -F_{n-2} & F_{n-1} \\ F_{n-1} & -F_n \end{pmatrix} = \begin{pmatrix} F_{n-1} & -F_n \\ -F_n & F_{n+1} \end{pmatrix}.$$

Pour $n \geq 1$,

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n.$$