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An introduction to irrationality and transcendence methods.

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Lecture 3 [9]

3 Elliptic functions and transcendence

The material in this chapter is taken from a forthcoming paper [6] where references are given.

3.1 Introduction to elliptic functions

3.1.1 The exponential function

The exponential function

$$\begin{array}{rccc} \exp: & \mathbb{C} & \rightarrow & \mathbb{C}^\times \\ & z & \mapsto & e^z \end{array}$$

satisfies both a differential equation and an addition formula:

$$\frac{d}{dz}e^z = e^z, \qquad e^{z_1+z_2} = e^{z_1}e^{z_2}.$$

It is a homomorphism of the additive group $\mathbb{C}$ of complex numbers onto the multiplicative group $\mathbb{C}^\times$ of non-zero complex numbers, with kernel

$$\ker \exp = 2\pi i\mathbb{Z},$$

hence it yields an isomorphism between the quotient additive group $\mathbb{C}/2\pi i\mathbb{Z}$ and the multiplicative group $\mathbb{C}^\times$.

The group $\mathbb{C}^\times$ is the group of complex points of the multiplicative group $\mathbb{G}_m$; $z \mapsto e^z$ is the exponential function of the multiplicative group $\mathbb{G}_m$. We shall replace this algebraic group by an elliptic curve. We could replace it also by other commutative algebraic groups. As a first example, the exponential function of the additive group $\mathbb{G}_a$ is

$$\begin{array}{ccc} \mathbb{C} & \rightarrow & \mathbb{C} \\ z & \mapsto & z \end{array}$$

[9] http://www.math.jussieu.fr/∼miw/articles/pdf/AWSLecture3.pdf

More general examples are commutative linear algebraic groups; over an algebraically closed field, these are nothing else than products of several copies of the additive and multiplicative group. Further examples of algebraic groups are abelian varieties. In full generality, algebraic groups are extensions of abelian varieties by commutative linear algebraic groups.

3.1.2 Basic concepts

An elliptic curve may be defined as

- $y^2 = C(x)$ for a squarefree cubic polynomial $C(x)$,
- a connected compact Lie group of dimension 1,
- a complex torus $\mathbb{C}/\Omega$ where Ω is a lattice in $\mathbb{C}$,
- a Riemann surface of genus 1,
- a non-singular cubic in $\mathbb{P}_2(\mathbb{C})$ (together with a point at infinity),
- an algebraic group of dimension 1, with underlying projective algebraic variety.

We shall use the Weierstraß form

$$E = \bigl\{(t : x : y)\ ;\ y^2 t = 4x^3 - g_2 x t^2 - g_3 t^3\bigr\} \subset \mathbb{P}_2.$$

Here g_2 and g_3 are complex numbers, with the only assumption $g_2^3 \neq 27 g_3^2$, which means that the discriminant of the polynomial $4X^3 - g_2 X - g_3$ does not vanish.

An analytic parametrization of the complex points $E(\mathbb{C})$ of E is given by means of *the Weierstraß elliptic function* $\wp$, which satisfies the differential equation

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3. \tag{3.1}$$

It has a double pole at the origin with principal part $1/z^2$ and also satisfies an addition formula

$$\wp(z_1 + z_2) = -\wp(z_1) - \wp(z_2) + \frac{1}{4} \cdot \left(\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}\right)^2. \tag{3.2}$$

The exponential map of the Lie group $E(\mathbb{C})$ is

$$\begin{array}{rccc} \exp_E : & \mathbb{C} & \to & E(\mathbb{C}) \\ & z & \mapsto & \bigl(1 : \wp(z) : \wp'(z)\bigr). \end{array}$$

The kernel of this map is a *lattice* in $\mathbb{C}$ (that is a discrete rank 2 subgroup),

$$\Omega = \ker \exp_E = \{\omega \in \mathbb{C}\ ;\ \wp(z+\omega) = \wp(z)\} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2.$$

Hence $\exp_E$ induces an isomorphism between the quotient additive group $\mathbb{C}/\Omega$ and $E(\mathbb{C})$ with the law given by (3.2). The elements of Ω are the *periods* of $\wp$. A pair (ω_1, ω_2) of fundamental periods is given by

$$\omega_i = 2\int_{e_i}^{\infty} \frac{dx}{\sqrt{4x^3 - g_2x - g_3}}, \qquad (i = 1, 2),$$

where

$$4x^3 - g_2x - g_3 = 4(x - e_1)(x - e_2)(x - e_3).$$

Indeed, since $\wp'$ is periodic and odd, it vanishes at $\omega_1/2$, $\omega_2/2$ and $(\omega_1 + \omega_2)/2$, hence the values of $\wp$ at these points are the three distinct complex numbers e_1, e_2 and e_3 (recall that the discriminant of $4x^3 - g_2x - g_3$ is not 0).

Conversely, given a lattice Ω, there is a unique Weierstraß elliptic function $\wp_\Omega$ whose period lattice is Ω (see § 3.1.6). We denote its invariants in the differential equation (3.1) by $g_2(\Omega)$ and $g_3(\Omega)$.

We shall be interested mainly (but not only) with elliptic curves which are defined over the field of algebraic numbers: they have a Weierstraß equation with algebraic g_2 and g_3. However we shall also use the Weierstraß elliptic function associated with the lattice $\lambda\Omega$ where $\lambda \in \mathbb{C}^\times$ may be transcendental; the relations are

$$\wp_{\lambda\Omega}(\lambda z) = \lambda^{-2}\wp_\Omega(z), \qquad g_2(\lambda\Omega) = \lambda^{-4}g_2(\Omega), \qquad g_3(\lambda\Omega) = \lambda^{-6}g_3(\Omega). \quad (3.3)$$

The lattice $\Omega = \mathbb{Z}+\mathbb{Z}\tau$, where τ is a complex number with positive imaginary part, satisfies

$$g_2(\mathbb{Z} + \mathbb{Z}\tau) = 60G_2(\tau) \quad \text{and} \quad g_3(\mathbb{Z} + \mathbb{Z}\tau) = 140G_3(\tau),$$

where, for $G_k(\tau)$ (with $k \geq 2$) are the Eisenstein series

$$G_k(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} (m + n\tau)^{-2k}. \qquad (3.4)$$

3.1.3 Morphisms between elliptic curves. The modular invariant

If Ω and Ω' are two lattices in $\mathbb{C}$ and if $f : \mathbb{C}/\Omega \to \mathbb{C}/\Omega'$ is an analytic homomorphism, then the map $\mathbb{C} \to \mathbb{C}/\Omega \to \mathbb{C}/\Omega'$ factors through a homothecy $\mathbb{C} \to \mathbb{C}$ given by some $\lambda \in \mathbb{C}$ such that $\lambda\Omega \subset \Omega'$:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\lambda} & \mathbb{C} \\ \downarrow & & \downarrow \\ \mathbb{C}/\Omega & \xrightarrow[f]{} & \mathbb{C}/\Omega' \end{array}$$

If $f \neq 0$, then $\lambda \in \mathbb{C}^\times$ and f is surjective.

Conversely, if there exists $\lambda \in \mathbb{C}$ such that $\lambda\Omega \subset \Omega'$, then $f_\lambda(x+\Omega) = \lambda x+\Omega'$ defines an analytic surjective homomorphism $f_\lambda : \mathbb{C}/\Omega \to \mathbb{C}/\Omega'$. In this case $\lambda\Omega$ is a subgroup of finite index in Ω', hence the kernel of f_λ is finite and there exists $\mu \in \mathbb{C}^\times$ with $\mu\Omega' \subset \Omega$: the two elliptic curves $\mathbb{C}/\Omega$ and $\mathbb{C}/\Omega'$ are *isogeneous*.

If Ω and Ω^* are two lattices, $\wp$ and $\wp^*$ the associated Weierstraß elliptic functions and g_2, g_3 the invariants of $\wp$, the following statements are equivalent:
(i) There is a 2×2 matrix with rational coefficients which maps a basis of Ω to a basis of Ω^*.
(ii) There exists $\lambda \in \mathbb{Q}^\times$ such that $\lambda\Omega \subset \Omega^*$.
(iii) There exists $\lambda \in \mathbb{Z} \setminus \{0\}$ such that $\lambda\Omega \subset \Omega^*$.
(iv) The two functions $\wp$ and $\wp^*$ are algebraically dependent over the field $\mathbb{Q}(g_2, g_3)$.
(v) The two functions $\wp$ and $\wp^*$ are algebraically dependent over $\mathbb{C}$.

The map f_λ is an isomorphism if and only if $\lambda\Omega = \Omega'$.

The number

$$j = \frac{1728 g_2^3}{g_2^3 - 27 g_3^2}$$

is the *modular invariant* of the elliptic curve E. Two elliptic curves over $\mathbb{C}$ are isomorphic if and only if they have the same modular invariant.

Set $\tau = \omega_2/\omega_1$, $q = e^{2\pi i\tau}$ and $J(e^{2\pi i\tau}) = j(\tau)$. Then

$$\begin{aligned} J(q) &= q^{-1}\left(1 + 240\sum_{m=1}^{\infty} m^3 \frac{q^m}{1-q^m}\right)^3 \prod_{n=1}^{\infty}(1-q^n)^{-24} \\ &= \frac{1}{q} + 744 + 196884\, q + 21493760\, q^2 + \cdots \end{aligned}$$

3.1.4 Endomorphisms of an elliptic curve; complex multiplications

Let Ω be a lattice in $\mathbb{C}$. The set of analytic endomorphisms of $\mathbb{C}/\Omega$ is the subring

$$\mathrm{End}(\mathbb{C}/\Omega) = \bigl\{ f_\lambda \; ; \; \lambda \in \mathbb{C} \;\text{ with }\; \lambda\Omega \subset \Omega \bigr\}$$

of $\mathbb{C}$. We also call it the ring of endomorphisms of the associated elliptic curve, or of the corresponding Weierstraß $\wp$ function and we identify it with the subring

$$\bigl\{ \lambda \in \mathbb{C} \; ; \; \lambda\Omega \subset \Omega \bigr\}$$

of $\mathbb{C}$. The *field of endomorphisms* is the quotient field $\mathrm{End}(\mathbb{C}/\Omega) \otimes_{\mathbb{Z}} \mathbb{Q}$ of this ring.

If $\lambda \in \mathbb{C}$ satisfies $\lambda\Omega \subset \Omega$, then λ is either a rational integer or else an algebraic integer in an imaginary quadratic field. For such a λ, $\wp_\Omega(\lambda z)$ is a rational function of $\wp_\Omega(z)$; the degree of the numerator is λ^2 if $\lambda \in \mathbb{Z}$ and $N(\lambda)$ otherwise (here, N is the norm of the imaginary quadratic field); the degree of the denominator is $\lambda^2 - 1$ if $\lambda \in \mathbb{Z}$ and $N(\lambda) - 1$ otherwise.

Let E be the elliptic curve attached to the Weierstraß $\wp$ function. The ring of endomorphisms $\mathrm{End}(E)$ of E is either $\mathbb{Z}$ or else an order in an imaginary quadratic field k. The latter case arises if and only if the quotient $\tau = \omega_2/\omega_1$ of a pair of fundamental periods is a quadratic number; in this case the field of endomorphisms of E is $k = \mathbb{Q}(\tau)$ and the curve E has *complex multiplications* – this is the so-called *CM case*. This means also that the two functions $\wp(z)$

and $\wp(\tau z)$ are algebraically dependent. In this case, the value $j(\tau)$ of the modular invariant j is an algebraic integer whose degree is the class number of the quadratic field $k = \mathbb{Q}(\tau)$.

Remark. *From Gel'fond-Schneider's Theorem 2.43 one deduces the transcendence of the number*

$$e^{\pi\sqrt{163}} = 262\ 537\ 412\ 640\ 768\ 743.999\ 999\ 999\ 999\ 250\,072\,59\ldots$$

If we set

$$\tau = \frac{1+i\sqrt{163}}{2}, \quad q = e^{2\pi i\tau} = -e^{-\pi\sqrt{163}},$$

then the class number of the imaginary quadratic field $\mathbb{Q}(\tau)$ *is* 1*, we have* $j(\tau) = -(640\ 320)^3$ *and*

$$\left| j(\tau) - \frac{1}{q} - 744 \right| < 10^{-12}.$$

Also

$$\left(e^{\pi\sqrt{163}} - 744\right)^{1/3} = 640\,319.999\,999\,999\,999\,999\,999\,999\,999\,390\,31\ldots$$

Let $\wp$ be a Weierstraß elliptic function with field of endomorphisms k. Hence $k = \mathbb{Q}$ if the associated elliptic curve has no complex multiplication, while in the other case k is an imaginary quadratic field, namely $k = \mathbb{Q}(\tau)$, where τ is the quotient of two linearly independent periods of $\wp$. Let $u_1, \ldots, u_d$ be non-zero complex numbers. Then the functions $\wp(u_1 z), \ldots, \wp(u_d z)$ are algebraically independent (over $\mathbb{C}$ or over $\mathbb{Q}(g_2, g_3)$, this is equivalent) if and only if the numbers $u_1, \ldots, u_d$ are linearly independent over k. This generalizes the fact that $\wp(z)$ and $\wp(\tau z)$ are algebraically dependent if and only if the elliptic curve has complex multiplications. Much more general and deeper results of algebraic independence of functions (exponential and elliptic functions, zeta functions...) were proved by W.D. Brownawell and K.K. Kubota.

If $\wp$ is a Weierstraß elliptic function with algebraic invariants g_2 and g_3, if E is the associated elliptic curve and if k denotes its field of endomorphisms, then the set

$$\mathcal{L}_E = \Omega \cup \{u \in \mathbb{C} \setminus \Omega\ ;\ \wp(u) \in \overline{\mathbb{Q}}\}$$

is a k-vector subspace of $\mathbb{C}$: this is the set of *elliptic logarithms of algebraic points on* E. It plays a role with respect to E similar to the role of $\mathcal{L}$ for the multiplicative group $\mathbb{G}_{\mathrm{m}}$.

Let $k = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field with class number $h(-d) = h$. There are h non-isomorphic elliptic curves $E_1, \ldots, E_h$ with ring of endomorphisms the ring of integers of k. The numbers $j(E_i)$ are conjugate algebraic integers of degree h; each of them generates the Hilbert class field H of k (maximal unramified abelian extension of k). The Galois group of H/k is isomorphic to the ideal class group of k.

Since the group of roots of units of an imaginary quadratic field is $\{-1, +1\}$ except for $\mathbb{Q}(i)$ and $\mathbb{Q}(\varrho)$, where $\varrho = e^{2\pi i/3}$, it follows that there are exactly two elliptic curves over $\mathbb{Q}$ (up to isomorphism) having an automorphism group bigger than $\{-1,+1\}$. They correspond to Weierstraß elliptic functions $\wp$ for which there exists a complex number $\lambda \neq \pm 1$ with $\lambda^2\wp(\lambda z) = \wp(z)$.

The first one has $g_3 = 0$ and $j = 1728$. An explicit value for a pair of fundamental periods of the elliptic curve

$$y^2t = 4x^3 - 4xt^2$$

follows from computations by Legendre using Gauss's lemniscate function

$$\omega_1 = \int_1^\infty \frac{dx}{\sqrt{x^3-x}} = \frac{1}{2}B(1/4,1/2) = \frac{\Gamma(1/4)^2}{2^{3/2}\pi^{1/2}} \quad \text{and} \quad \omega_2 = i\omega_1. \tag{3.5}$$

The lattice $\mathbb{Z}[i]$ has $g_2 = 4\omega_1^4$, thus

$$\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} (m+ni)^{-4} = \frac{\Gamma(1/4)^8}{2^6\cdot 3\cdot 5\cdot \pi^2}\cdot$$

The second one has $g_2 = 0$ and $j = 0$. Again from computations by Legendre one deduces that a pair of fundamental periods of the elliptic curve

$$y^2t = 4x^3 - 4t^3$$

is

$$\omega_1 = \int_1^\infty \frac{dx}{\sqrt{x^3-1}} = \frac{1}{3}B(1/6,1/2) = \frac{\Gamma(1/3)^3}{2^{4/3}\pi} \quad \text{and} \quad \omega_2 = \varrho\omega_1. \tag{3.6}$$

The lattice $\mathbb{Z}[\varrho]$ has $g_3 = 4\omega_1^6$, thus

$$\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} (m+n\varrho)^{-6} = \frac{\Gamma(1/3)^{18}}{2^8\cdot 5\cdot 7\cdot \pi^6}\cdot$$

These two examples involve special values of Euler's Gamma function

$$\Gamma(z) = \int_0^\infty e^{-t}t^z\cdot\frac{dt}{t} = e^{-\gamma z}z^{-1}\prod_{n=1}^\infty\left(1+\frac{z}{n}\right)^{-1}e^{z/n}, \tag{3.7}$$

where

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^n \frac{1}{k} - \log n\right) = 0.577\,215\,664\,901\,532\,860\,606\,512\,09\ldots$$

is Euler's constant, while Euler's Beta function is

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 x^{a-1}(1-x)^{b-1}dx.$$

More generally, the formula of Chowla and Selberg (1966) expresses periods of elliptic curves with complex multiplications as products of Gamma values: *if k is an imaginary quadratic field and $\mathcal{O}$ an order in k, if E is an elliptic curve with complex multiplications by $\mathcal{O}$, then the corresponding lattice Ω determines a vector space $\Omega \otimes_{\mathbb{Z}} \mathbb{Q}$ which is invariant under the action of k and thus has the form $k \cdot \omega$ for some $\omega \in \mathbb{C}^\times$ defined up to elements in $k^\times$. In particular, if $\mathcal{O}$ is the ring of integers $\mathbb{Z}_k$ of k, then*

$$\omega = \alpha\sqrt{\pi} \prod_{\substack{0<a<d \\ (a,d)=1}} \Gamma(a/d)^{w\epsilon(a)/4h},$$

where α is a non-zero algebraic number, w is the number of roots of unity in k, h is the class number of k, ϵ is the Dirichlet character modulo the discriminant d of k.

3.1.5 Standard relations among Gamma values

Euler's Gamma function satisfies the following relations
(Translation)

$$\Gamma(z+1) = z\Gamma(z);$$

(Reflection)

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)};$$

(Multiplication) For any positive integer n,

$$\prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n}\right) = (2\pi)^{(n-1)/2} n^{-nz+(1/2)} \Gamma(nz).$$

D. Rohrlich (see also § 5.25) conjectured that any multiplicative relation among Gamma values is a consequence of these standard relations, while S. Lang was more optimistic

Conjecture 3.8 (D. Rohrlich)**.** *Any multiplicative relation*

$$\pi^{b/2} \prod_{a \in \mathbb{Q}} \Gamma(a)^{m_a} \in \overline{\mathbb{Q}}$$

with b and m_a in $\mathbb{Z}$ is a consequence of the standard relations.

Conjecture 3.9 (S. Lang)**.** *Any algebraic dependence relation with algebraic coefficients among the numbers $(2\pi)^{-1/2}\Gamma(a)$ with $a \in \mathbb{Q}$ is in the ideal generated by the standard relations.*

3.1.6 Quasi-periods of elliptic curves and elliptic integrals of the second kind

Let $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$. The *Weierstraß canonical product* attached to this lattice is the entire function σ_Ω defined by

$$\sigma_\Omega(z) = z \prod_{\omega \in \Omega \setminus \{0\}} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{z^2}{2\omega^2}}.$$

It has a simple zero at any point of Ω.

Hence the Weierstraß sigma function plays, for the lattice Ω, the role which is played by the function

$$z \prod_{n \geq 1} \left(1 - \frac{z}{n}\right) e^{z/n} = -e^{\gamma z} \Gamma(-z)^{-1}$$

for the set of positive integers $\mathbb{N} \setminus \{0\} = \{1, 2, \dots\}$ (see the infinite product (3.7) for Euler's Gamma function), and also by the function

$$\pi^{-1} \sin(\pi z) = z \prod_{n \in \mathbb{Z} \setminus \{0\}} \left(1 - \frac{z}{n}\right) e^{z/n}$$

for the set $\mathbb{Z}$ of rational integers.

The Weierstraß sigma function σ associated with a lattice in $\mathbb{C}$ is an entire function of *order* 2:

$$\limsup_{r \to \infty} \frac{1}{\log r} \cdot \log \log \sup_{|z| = r} |\sigma(z)| = 2;$$

the product $\sigma^2 \wp$ is also an entire function of order 2 (this can be checked by using infinite products, but it is easier to use the quasi-periodicity of σ, see formula (3.10) below).

The logarithmic derivative of the sigma function is *the Weierstraß zeta function* $\zeta = \sigma'/\sigma$ whose Laurent expansion at the origin is

$$\zeta(z) = \frac{1}{z} - \sum_{k \geq 2} s_k z^{2k-1},$$

where, for $k \in \mathbb{Z}$, $k \geq 2$,

$$s_k = s_k(\Omega) = \sum_{\substack{\omega \in \Omega \\ \omega \neq 0}} \omega^{-2k} = \omega_1^{-2k} G_k(\tau)$$

(recall (3.4); also $\tau = \omega_2/\omega_1$).

The derivative of ζ is $-\wp$. From

$$\wp'' = 6\wp^2 - (g_2/2)$$

one deduces that $s_k(\Omega)$ is a homogenous polynomial in $\mathbb{Q}[g_2, g_3]$ of weight $2k$ for the graduation of $\mathbb{Q}[g_2, g_3]$ determined by assigning to g_2 the degree 4 and to g_3 the degree 6.

As a side remark, we notice that for any $u \in \mathbb{C} \setminus \Omega$ we have

$$\mathbb{Q}(g_2, g_3) \subset \mathbb{Q}\big(\wp(u), \wp'(u), \wp''(u)\big).$$

Since its derivative is periodic, the function ζ is *quasi-periodic*: for each $\omega \in \Omega$ there is a complex number $\eta = \eta(\omega)$ such that

$$\zeta(z + \omega) = \zeta(z) + \eta.$$

These numbers η are the *quasi-periods* of the elliptic curve. If (ω_1, ω_2) is a pair of fundamental periods and if we set $\eta_1 = \eta(\omega_1)$ and $\eta_2 = \eta(\omega_2)$, then, for $(a, b) \in \mathbb{Z}^2$,

$$\eta(a\omega_1 + b\omega_2) = a\eta_1 + b\eta_2.$$

Coming back to the sigma function, one deduces that

$$\sigma(z + \omega_i) = -\sigma(z) \exp\Big(\eta_i\big(z + (\omega_i/2)\big)\Big) \qquad (i = 1, 2). \tag{3.10}$$

The zeta function also satisfies an addition formula:

$$\zeta(z_1 + z_2) = \zeta(z_1) + \zeta(z_2) + \frac{1}{2} \cdot \frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}.$$

The Legendre relation relating the periods and the quasi-periods

$$\omega_2\eta_1 - \omega_1\eta_2 = 2\pi i,$$

when ω_2/ω_1 has positive imaginary part, can be obtained by integrating $\zeta(z)$ along the boundary of a fundamental parallelogram.

In the case of complex multiplication, if τ is the quotient of a pair of fundamental periods of $\wp$, then the function $\zeta(\tau z)$ is algebraic over the field $\mathbb{Q}\big(g_2, g_3, z, \wp(z), \zeta(z)\big)$.

Examples For the curve $y^2t = 4x^3 - 4xt^2$ the quasi-periods attached to the above mentioned pair of fundamental periods (3.5) are

$$\eta_1 = \frac{\pi}{\omega_1} = \frac{(2\pi)^{3/2}}{\Gamma(1/4)^2}, \qquad \eta_2 = -i\eta_1; \tag{3.11}$$

it follows that the fields $\mathbb{Q}(\omega_1, \omega_2, \eta_1, \eta_2)$ and $\mathbb{Q}\big(\pi, \Gamma(1/4)\big)$ have the same algebraic closure over $\mathbb{Q}$, hence the same transcendence degree. For the curve $y^2t = 4x^3 - 4t^3$ with periods (3.6) they are

$$\eta_1 = \frac{2\pi}{\sqrt{3}\omega_1} = \frac{2^{7/3}\pi^2}{3^{1/2}\Gamma(1/3)^3}, \qquad \eta_2 = \varrho^2\eta_1. \tag{3.12}$$

In this case the fields $\mathbb{Q}(\omega_1, \omega_2, \eta_1, \eta_2)$ and $\mathbb{Q}\big(\pi, \Gamma(1/3)\big)$ have the same algebraic closure over $\mathbb{Q}$, hence the same transcendence degree.

3.1.7 Elliptic integrals

Let

$$\mathcal{E} = \{(t : x : y) \in \mathbb{P}_2; y^2 t = 4x^3 - g_2 x t^2 - g_3 t^3\}$$

be an elliptic curve. The field of rational (meromorphic) functions on $\mathcal{E}$ over $\mathbb{C}$ is $\mathbb{C}(\mathcal{E}) = \mathbb{C}(\wp, \wp') = \mathbb{C}(x, y)$ where x and y are related by the cubic equation $y^2 = 4x^3 - g_2 x - g_3$. Under the isomorphism $\mathbb{C}/\Omega \to \mathcal{E}(\mathbb{C})$ given by $(1 : \wp : \wp')$, the differential form dz is mapped to dx/y. The holomorphic differential forms on $\mathbb{C}/\Omega$ are λdz with $\lambda \in \mathbb{C}$.

The differential form $d\zeta = \zeta'/\zeta$ is mapped to $-xdx/y$. The differential forms of second kind on $\mathcal{E}(\mathbb{C})$ are $adz + bd\zeta + d\chi$, where a and b are complex numbers and $\chi \in \mathbb{C}(x, y)$ is a meromorphic function on $\mathcal{E}$.

Assume that the elliptic curve $\mathcal{E}$ is defined over $\overline{\mathbb{Q}}$: the invariants g_2 and g_3 are algebraic. We shall be interested with differential forms which are defined over $\overline{\mathbb{Q}}$. Those of second kind are $adz + bd\zeta + d\chi$, where a and b are algebraic numbers and $\chi \in \overline{\mathbb{Q}}(x, y)$.

An elliptic integral is an integral

$$\int R(x, y) dx$$

where R is a rational function of x and y, while y^2 is a polynomial in x of degree 3 or 4 without multiple roots, with the proviso that the integral cannot be integrated by means of elementary functions. One may transform this integral as follows: one reduces it to an integral of $dx/\sqrt{P(x)}$ where P is a polynomial of 3rd or 4th degree; in case P has degree 4 one replaces it with a degree 3 polynomial by sending one root to infinity; finally one reduces it to a Weierstraß equation by means of a birational transformation. The value of the integral is not modified.

For transcendence purposes, if the initial differential form is defined over $\overline{\mathbb{Q}}$, then all these transformations involve only algebraic numbers.

3.2 Transcendence results of numbers related with elliptic functions

3.2.1 Elliptic analog of Lindemann's Theorem on the transcendence of π and of Hermite-Lindemann Theorem on the transcendence of $\log \alpha$.

The first transcendence result on periods of elliptic functions was proved by C.L. Siegel as early as 1932.

Theorem 3.13 (Siegel, 1932). *Let $\wp$ be a Weierstraß elliptic function with period lattice $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. Assume that the invariants g_2 and g_3 of $\wp$ are algebraic. Then at least one of the two numbers ω_1, ω_2 is transcendental.*

One main feature of Siegel's proof is that he used Dirichlet's box principle (the so-called Thue-Siegel Lemma which is included in his 1929 paper) to construct an auxiliary function. This idea turned out to be of fundamental importance for the solution of Hilbert's seventh problem by Gel'fond and Schneider two years later.

In the case of complex multiplication, it follows from Theorem 3.13 that *any non-zero period of $\wp$ is transcendental.*

From formulae (3.5) and (3.6) it follows as a consequence of Siegel's 1932 result that both numbers $\Gamma(1/4)^4/\pi$ and $\Gamma(1/3)^3/\pi$ are transcendental.

Other consequences of Siegel's result concern the transcendence of the length of an arc of an ellipse [4]

$$2\int_{-b}^{b}\sqrt{1+\frac{a^2x^2}{b^4-b^2x^2}}\,dx$$

for algebraic a and b, as well as the transcendence of an arc of the lemniscate $(x^2+y^2)^2 = 2a^2(x^2-y^2)$ with a algebraic.

A further example of application of Siegel's Theorem is the transcendence of values of hypergeometric series related with elliptic integrals

$$\begin{aligned} K(z) &= \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-z^2x^2)}} \\ &= \frac{\pi}{2}\cdot {}_2F_1\left(1/2,\ 1/2\ ;\ 1 \bigm| z^2\right), \end{aligned}$$

where ${}_2F_1$ denotes Gauss hypergeometric series

$${}_2F_1\left(a,\ b\ ;\ c \bigm| z\right) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\cdot\frac{z^n}{n!}$$

with $(a)_n = a(a+1)\cdots(a+n-1)$.

Further results on this topic were obtained by Th. Schneider in 1934 and in a joint work by K. Mahler and J. Popken in 1935 using Siegel's method. These results were superseded by Th. Schneider's work in 1936 where he proved a number of definitive results on the subject, including:

Theorem 3.14 (Schneider, 1936)**.** *Assume that the invariants g_2 and g_3 of $\wp$ are algebraic. Then for any non-zero period ω of $\wp$, the numbers ω and $\eta(\omega)$ are transcendental.*

It follows from Theorem 3.14 that any non-zero period of an elliptic integral of the first or second kind is transcendental:

Corollary 3.15. *Let $\mathcal{E}$ be an elliptic curve over $\overline{\mathbb{Q}}$, p_1 and p_2 two algebraic points on $\mathcal{E}(\overline{\mathbb{Q}})$, w a differential form of first or second kind on $\mathcal{E}$ which is defined over $\overline{\mathbb{Q}}$, holomorphic at p_1 and p_2 and which is not the differential of a rational function. Let γ be a path on $\mathcal{E}$ from p_1 to p_2. In case $p_1 = p_2$ one assumes that γ is not homologous to 0. Then the number*

$$\int_\gamma w$$

is transcendental.

Examples: Using Corollary 3.15 and formulae (3.11) and (3.12), one deduces that the numbers

$$\Gamma(1/4)^4/\pi^3 \quad \text{and} \quad \Gamma(1/3)^3/\pi^2$$

are transcendental.

The main results of Schneider's 1936 paper are as follows (see [4]):

Theorem 3.16 (Schneider, 1936)**. 1.** *Let $\wp$ be a Weierstraß elliptic function with algebraic invariants g_2, g_3. Let β be a non-zero algebraic number. Then β is not a pole of $\wp$ and $\wp(\beta)$ is transcendental.*
More generally, if a and b are two algebraic numbers with $(a, b) \neq (0, 0)$, then for any $u \in \mathbb{C} \setminus \Omega$ at least one of the two numbers $\wp(u)$, $au + b\zeta(u)$ is transcendental.
2. *Let $\wp$ and $\wp^*$ be two algebraically independent elliptic functions with algebraic invariants g_2, g_3, g_2^*, g_3^*. If $t \in \mathbb{C}$ is not a pole of $\wp$ or of $\wp^*$, then at least one of the two numbers $\wp(t)$ and $\wp^*(t)$ is transcendental.*
3. *Let $\wp$ be a Weierstraß elliptic function with algebraic invariants g_2, g_3. Then for any $t \in \mathbb{C} \setminus \Omega$, at least one of the two numbers $\wp(t)$, e^t is transcendental.*

It follows from Theorem 3.16.2 that the quotient of an elliptic integral of the first kind (between algebraic points) by a non-zero period is either in the field of endomorphisms (hence a rational number, or a quadratic number in the field of complex multiplications), or a transcendental number.

Here is another important consequence of Theorem 3.16.2.

Corollary 3.17 (Schneider, 1936)**.** *Let $\tau \in \mathcal{H}$ be a complex number in the upper half plane $\Im m(\tau) > 0$ such that $j(\tau)$ is algebraic. Then τ is algebraic if and only if τ is imaginary quadratic.*

In this connection we quote Schneider's second problem in [4], which is still open (see papers by Wakabayashi quoted in [6].

Conjecture 3.18 (Schneiders' second problem)**.** *Prove Corollary 3.17 without using elliptic functions.*

Sketch of proof of Corollary 3.17 as a consequence of part 2 of Theorem 3.16.

Assume that both $\tau \in \mathcal{H}$ and $j(\tau)$ are algebraic. There exists an elliptic function with algebraic invariants g_2, g_3 and periods ω_1, ω_2 such that

$$\tau = \frac{\omega_2}{\omega_1} \quad \text{and} \quad j(\tau) = \frac{1728 g_2^3}{g_2^3 - 27 g_3^2}\cdot$$

Set $\wp^*(z) = \tau^2 \wp(\tau z)$. Then $\wp^*$ is a Weierstraß function with algebraic invariants g_2^*, g_3^*. For $u = \omega_1/2$ the two numbers $\wp(u)$ and $\wp^*(u)$ are algebraic. Hence the two functions $\wp(z)$ and $\wp^*(z)$ are algebraically dependent. It follows that the corresponding elliptic curve has non-trivial endomorphisms, therefore τ is quadratic. □

A quantitative refinement of Schneider's Theorem on the transcendence of $j(\tau)$ given by A. Faisant and G. Philibert in 1984 became useful 10 years later in connection with Nesterenko's result (see § 3.3).

We will not review the results related with abelian integrals, but only quote the first result on this topic, which involves the Jacobian of a Fermat curve: in 1941 Schneider proved that *for a and b in $\mathbb{Q}$ with a, b and $a+b$ not in $\mathbb{Z}$, the number*

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

is transcendental. We notice that in his 1932 paper, C.L. Siegel had already announced partial results on the values of the Euler Gamma function.

Schneider's above mentioned results deal with elliptic (and abelian) integrals of the first or second kind. His method can be extended to deal with elliptic (and abelian) integrals of the third kind (this is Schneider's third problem in [4]).

As pointed out by J-P. Serre in 1979, it follows from the quasi-periodicity of the Weierstraß sigma function (3.10) that the function

$$F_u(z) = \frac{\sigma(z+u)}{\sigma(z)\sigma(u)} e^{-z\zeta(u)}$$

satisfies

$$F_u(z+\omega_i) = F_u(z) e^{\eta_i u - \omega_i \zeta(u)}.$$

Theorem 3.19. *Let u_1 and u_2 be two non-zero complex numbers. Assume that g_2, g_3, $\wp(u_1)$, $\wp(u_2)$, β are algebraic and $\mathbb{Z}u_1 \cap \Omega = \{0\}$. Then the number*

$$\frac{\sigma(u_1+u_2)}{\sigma(u_1)\sigma(u_2)} e^{\left(\beta - \zeta(u_1)\right)u_2}$$

is transcendental.

From the next corollary, one can deduce that non-zero periods of elliptic integrals of the third kind are transcendental.

Corollary 3.20. *For any non-zero period ω and for any $u \in \mathbb{C} \setminus \Omega$ the number $e^{\omega\zeta(u) - \eta u + \beta\omega}$ is transcendental.*

Further results on elliptic integrals are due to M. Laurent.

Ya. M. Kholyavka wrote several papers devoted to the approximation of transcendental numbers related with elliptic functions.

Quantitative estimates (measures of transcendence) related with the results of this section were derived by N.I. Fel'dman. There are further papers by S. Lang, N.D. Nagaev, N. Hirata, E. Reyssat, M. Laurent , R. Tubbs, G. Diaz, N. Saradha, P. Grinspan.

3.2.2 Elliptic analogs of the Six Exponentials Theorem

Elliptic analogs of the Six Exponentials Theorem 2.46 were considered by S. Lang and K. Ramachandra in the 1960's.

Let d_1, d_2 be non-negative integers and m a positive integer, let $x_1, \dots, x_{d_1}$ be complex numbers which are linearly independent over $\mathbb{Q}$, let $y_1, \dots, y_m$ be complex numbers which are linearly independent over $\mathbb{Q}$ and let $u_1, \dots, u_{d_2}$ be non-zero complex numbers. We consider Weierstraß elliptic functions $\wp_1, \dots, \wp_{d_2}$ and we denote by K_0 the field generated over $\mathbb{Q}$ by their invariants $g_{2,k}$ and $g_{3,k}$ $(1 \le k \le d_2)$. We assume that the d_2 functions $\wp_1(u_1 z), \dots, \wp_{d_2}(u_{d_2} z)$ are algebraically independent. We denote by K_1 the field generated over K_0 by the numbers $\exp(x_i y_j)$, $(1 \le i \le d_1,\ 1 \le j \le m)$ together with the numbers $\wp_k(u_k y_j)$, $(1 \le k \le d_2,\ 1 \le j \le m)$. Next, define

$$K_2 = K_1(y_1, \dots, y_m), \qquad K_3 = K_1\bigl(x_1, \dots, x_{d_1},\ u_1, \dots, u_{d_2}\bigr),$$

and let K_4 be the compositum of K_2 and K_3:

$$K_4 = K_1\bigl(y_1, \dots, y_m,\ x_1, \dots, x_{d_1},\ u_1, \dots, u_{d_2}\bigr).$$

The Theorems of Hermite-Lindemann (Theorem 2.1), Gel'fond-Schneider (Theorem 2.43), the Six Exponentials Theorem 2.46 and their elliptic analogs due to Schneider, Lang and Ramachandra can be stated as follows.

Any one of the four assumptions below will imply $d_1 + d_2 > 0$, the case where d_1 (resp. d_2) vanishes means that one considers only elliptic (resp. exponential) functions.

Theorem 3.21.
1. *Assume* $(d_1 + d_2)m > m + d_1 + 2d_2$. *Then the field* K_1 *has transcendence degree* ≥ 1 *over* $\mathbb{Q}$.
2. *Assume either* $d_1 \ge 1$ *and* $m \ge 2$, *or* $d_2 \ge 1$ *and* $m \ge 3$. *Then* K_2 *has transcendence degree* ≥ 1 *over* $\mathbb{Q}$.
3. *Assume* $d_1 + d_2 \ge 2$. *Then* K_3 *has transcendence degree* ≥ 1 *over* $\mathbb{Q}$.
4. *Assume* $d_1 + d_2 \ge 1$. *Then* K_4 *has transcendence degree* ≥ 1 *over* $\mathbb{Q}$.

Parts 3 and 4 of Theorem 3.21 are consequences of the Schneider-Lang criterion, which deals with meromorphic functions satisfying differential equations, while parts 1 and 2 follow from a criterion which involves no differential equations. Such criteria were given by Schneider, Lang and Ramachandra.

Theorem 3.21 also includes Theorem 3.16 apart from the case $b \ne 0$ in part 1 of that statement. However there are extensions of Theorem 3.21 which include results on Weierstraß zeta functions (and also on Weierstraß sigma functions in connection with elliptic integrals of the third kind).

Here is a corollary of part 1 of Theorem 3.21 (take $d_1 = 0$, $d_2 = 3$, $\wp_1 = \wp_2 = \wp_3 = \wp$, $m = 4$, $y_1 = 1$, $y_2 \in \mathrm{End}(E) \setminus \mathbb{Q}$, $y_3 = v_1/u_1$, $y_4 = y_2 y_3$ — there is an alternative proof with $d_2 = 2$ and $m = 6$).

Corollary 3.22. *Let E be an elliptic curve with algebraic invariants g_2, g_3. Assume E has complex multiplications. Let*

$$M = \begin{pmatrix} u_1 & u_2 & u_3 \\ \\ v_1 & v_2 & v_3 \end{pmatrix}$$

be a 2×3 matrix whose entries are elliptic logarithms of algebraic numbers, i.e. u_i and v_i ($i = 1, 2, 3$) are in $\mathcal{L}_E$. Assume that the three columns are linearly independent over $\mathrm{End}(E)$ *and the two rows are also linearly independent over* $\mathrm{End}(E)$. *Then the matrix M has rank* 2.

In the non-CM case, one deduces from Theorem 3.21 a similar (but weaker) statement according to which such matrices $\bigl(u_{ij}\bigr)$ (where $\wp(u_{ij})$ are algebraic numbers) have rank ≥ 2 if they have size 2×5 (taking $d_1 = 0$, $d_2 = 2$ and $m = 5$) or 3×4 (taking $d_1 = 0$ and either $d_2 = 3$, $m = 4$ or $d_2 = 4$ and $m = 3$) instead of 2×3.

Lower bounds better than 2 for the rank of matrices of larger sizes are known, but we will not discuss this question here. We just mention the fact that higher dimensional considerations are relevant to a problem discussed by B. Mazur on the density of rational points on varieties.

3.2.3 Linear independence of numbers related with elliptic functions

From Schneider's Theorem 3.16 part 1, one deduces the linear independence over the field of algebraic numbers of the three numbers 1, ω and η, when ω is a non-zero period of a Weierstraß elliptic function (with algebraic invariants g_2 and g_3) and $\eta = \eta(\omega)$ is the associated quasi-period of the corresponding Weierstraß zeta function. However, the Gel'fond-Schneider method in one variable alone does not yield strong results of linear independence. Baker's method is better suited for this purpose.

3.2.4 Linear independence of periods and quasi-periods

Baker's method of proof for his Theorem 2.45 on linear independence of logarithms of algebraic numbers was used as early as 1969 and 1970 by A. Baker himself when he proved the transcendence of linear combinations with algebraic coefficients of the numbers ω_1, ω_2, η_1 and η_2 associated with an elliptic curve having algebraic invariants g_2 and g_3. His method is effective: it provides quantitative Diophantine estimates.

In 1971 J. Coates proved the transcendence of linear combinations with algebraic coefficients of ω_1, ω_2, η_1, η_2 and $2\pi i$. Moreover, he proved in that in the non-CM case, the three numbers ω_1, ω_2 and $2\pi i$ are $\overline{\mathbb{Q}}$-linearly independent. Further results including usual logarithms of algebraic numbers are due to T. Harase in 1974 and 1976.

The final result on the question of linear dependence of periods and quasi-periods for a single elliptic function was given by D.W. Masser in 1975.

Theorem 3.23 (Masser, 1975)**.** *Let $\wp$ be a Weierstraß elliptic function with algebraic invariants g_2 and g_3, denote by ζ the corresponding Weierstraß zeta function, let ω_1, ω_2 be a basis of the period lattice of $\wp$ and let η_1, η_2 be the associated quasi-periods of ζ. Then the six numbers 1, ω_1, ω_2, η_1, η_2 and $2\pi i$ span a $\overline{\mathbb{Q}}$-vector space of dimension 6 in the non-CM case, 4 in the CM case:*

$$\dim_{\overline{\mathbb{Q}}}\{1, \omega_1, \omega_2, \eta_1, \eta_2, 2\pi i\} = 2 + 2\dim_{\overline{\mathbb{Q}}}\{\omega_1, \omega_2\}.$$

The fact that the dimension is 4 in the CM case means that there are two independent linear relations among these 6 numbers. One of them is $\omega_2 = \tau\omega_1$ with $\tau \in \overline{\mathbb{Q}}$. The second one can be written

$$C^2\tau\eta_2 - AC\eta_1 + \gamma\omega_1 = 0$$

where $A + BX + CX^2$ is the minimal polynomial of τ over $\mathbb{Z}$ and γ is an element in $\mathbb{Q}(g_2, g_3, \tau)$.

D.W. Masser also produced quantitative estimates (measures of linear independence). In 1976, R. Franklin and D.W. Masser obtained an extension involving a logarithm of an algebraic number.

Further results can be found in papers by P. Bundschuh, S. Lang, D.W. Masser and M. Anderson

3.2.5 Elliptic analog of Baker's Theorem

The elliptic analog of Baker's Theorem 2.45 on linear independence of logarithms was proved by D.W. Masser in 1974 in the CM case.

His proof also yields quantitative estimates (measures of linear independence of elliptic logarithms of algebraic points on an elliptic curve). Such estimates have a number of applications: this was shown by A.O. Gel'fond for usual logarithms of algebraic numbers [2] and further consequences of such lower bounds in the case of elliptic curves for solving Diophantine equations (integer points on elliptic curves) were derived by S. Lang.

Lower bounds for linear combinations of elliptic logarithms in the CM case were obtained by several mathematicians including J. Coates, D.W. Masser, J. Coates and S. Lang, M. Anderson. The work of Yu Kunrui yields similar estimates, but his method is not that of Baker-Masser: instead of using a generalization of Gel'fond's solution to Hilbert's seventh problem, Yu Kunrui uses a generalization in several variables of Schneider's solution to the same problem. Again, this method is restricted to the CM case.

The question of linear independence of elliptic logarithms in the non-CM case was settled only in 1980 by D. Bertrand and D.W. Masser. They found a new proof of Baker's Theorem 2.45 using functions of several variables and they succeeded to extend this argument to the situation of elliptic functions, either with or without complex multiplication. The criterion they use is the one that Schneider established in 1949 for his proof of the transcendence of Beta values. This criterion (revisited by S. Lang) deals with Cartesian products. From the several variables point of view, this is a rather degenerate situation; much deeper results are available, including Bombieri's solution in 1970 of Nagata's Conjecture, which involves Hörmander L^2-estimates for analytic functions of several variables. However Bombieri's Theorem does not seem to yield new transcendence results, so far.

But so far these deeper results do not give further transcendence results in our context.

Theorem 3.24 (D.W. Masser 1974 for the CM case, D. Bertrand and D.W. Masser 1980 for the non-CM case)**.** *Let $\wp$ be a Weierstraß elliptic function with algebraic invariants g_2, g_3 and field of endomorphisms k. Let $u_1, \ldots, u_n$ be k-linearly independent complex numbers. Assume, for $1 \leq i \leq n$, that either $u_i \in \Omega$ or else $\wp(u_i) \in \overline{\mathbb{Q}}$. Then the numbers $1, u_1, \ldots, u_n$ are linearly independent over the field $\overline{\mathbb{Q}}$.*

This means that *for an elliptic curve E which is defined over $\overline{\mathbb{Q}}$, if $u_1, \ldots, u_n$ are elements in $\mathcal{L}_E$ which are linearly independent over the field of endomorphisms of E, then the numbers $1, u_1, \ldots, u_n$ are linearly independent over $\overline{\mathbb{Q}}$.*

The method of Bertrand-Masser yields only weak Diophantine estimates (measures of linear independence of logarithms).

3.2.6 Further results of linear independence

Theorem 3.23 deals only with periods and quasi-periods associated with one lattice, Theorem 3.24 deals only with elliptic logarithms of algebraic points on one elliptic curve. A far reaching generalization of both results was achieved by G. Wüstholz in 1987 when he succeeded in extending Baker's Theorem to abelian varieties and integrals, and, more generally, to commutative algebraic groups. If we restrict his general result to products of a commutative linear group, of copies of elliptic curves as well as of extensions of elliptic curves by the additive or the multiplicative group, the resulting statement settles the questions of linear independence of logarithms of algebraic numbers, of elliptic logarithms of algebraic points, including periods, quasi-periods, elliptic integrals of the first, second or third kind. This is a main step towards an answer to the questions of M. Kontsevich and D. Zagier on periods. [3].

Wüstholz's method can be extended to yield measures of linear independence of logarithms of algebraic points on an algebraic group. The first effective such lower bounds were given in 1989 by Philippon and Waldschmidt. As a special case, they provide the first measures of linear independence for elliptic logarithms which is also valid in the non-CM case. More generally, they give effective lower bounds for any non-vanishing linear combination of logarithms of algebraic points on algebraic groups (including usual logarithms, elliptic logarithms, elliptic integrals of any kind).

Refinements were obtained by N. Hirata Kohno S. David, N. Hirata Kohno and S. David, M. Ably, and É. Gaudron who uses not only Hirata's reduction argument, but also the work of J-B. Bost (slope inequalities) involving Arakelov's Theory. For instance, thanks to the recent work of David and Hirata-Kohno on the one hand, of Gaudron on the other, one knows that the above mentioned non-vanishing linear combinations of logarithms of algebraic points are not Liouville numbers.

In the p-adic case there is a paper of G. Rémond and F. Urfels dealing with two elliptic logarithms.

Further applications to elliptic curves of the Baker-Masser-Wüstholz method were derived by D.W. Masser and G. Wüstholz.

M For instance, J. Wolfart and G. Wüstholz have shown that the only linear dependence relations with algebraic coefficients between the values $B(a, b)$ of the Euler Beta function at points $(a, b) \in \mathbb{Q}^2$ are those which follow from the Deligne-Koblitz-Ogus relations (see further references in [5]).

3.3 Algebraic independence of numbers related with elliptic functions

3.3.1 Small transcendence degree

We keep the notations and assumptions of section 3.2.2.

The following extension of Theorem 3.21 to a result of algebraic independence containing Gel'fond's 1948 results on the exponential function (see § 2.3.5) is a consequence of the works of many a mathematician, including A.O. Gel'fond [2], A.A. Šmelev, W.D. Brownawell, W.D. Brownawell and K.K. Kubota, G. Wüstholz, D.W. Masser and G. Wüstholz.

Theorem 3.25.
1. *Assume* $(d_1 + d_2)m \geq 2(m + d_1 + 2d_2)$. *Then the field* K_1 *has transcendence degree* ≥ 2 *over* $\mathbb{Q}$.
2. *Assume* $(d_1 + d_2)m \geq m + 2(d_1 + 2d_2)$. *Then* K_2 *has transcendence degree* ≥ 2 *over* $\mathbb{Q}$.
3. *Assume* $(d_1 + d_2)m \geq 2m + d_1 + 2d_2$. *Then* K_3 *has transcendence degree* ≥ 2 *over* $\mathbb{Q}$.
4. *Assume* $(d_1 + d_2)m > m + d_1 + 2d_2$. *Then* K_4 *has transcendence degree* ≥ 2 *over* $\mathbb{Q}$.

Quantitative estimates (measures of algebraic independence) are due to R. Tubbs and E.M. Jabbouri.

Further related results are due to N.I. Fel'dman, R. Tubbs, É. Reyssat, M. Toyoda and T. Yasuda. Measures of algebraic independence are given by M. Ably and by S.O. Shestakov.

Again, like for Theorem 3.21, there are extensions of Theorem 3.25 which include results on Weierstraß zeta functions as well as on functions of several variables, and which have a number of consequences related with abelian functions.

3.3.2 Algebraic independence of periods and quasi-periods

In the 1970's G.V. Chudnovsky proved strong results of algebraic independence (small transcendence degree) related with elliptic functions. One of his most spectacular contributions was obtained in 1976:

Theorem 3.26 (G.V. Chudnovsky, 1976)**.** *Let $\wp$ be a Weierstraß elliptic function with invariants g_2, g_3. Let (ω_1, ω_2) be a basis of the lattice period of $\wp$ and $\eta_1 = \eta(\omega_1)$, $\eta_2 = \eta(\omega_2)$ the associated quasi-periods of the associated Weierstraß zeta function. Then at least two of the numbers g_2, g_3, ω_1, ω_2, η_1, η_2 are algebraically independent.*

A more precise result is that, for any non-zero period ω, at least two of the four numbers g_2, g_3, ω/π, η/ω (with $\eta = \eta(\omega)$) are algebraically independent.

In the case where g_2 and g_3 are algebraic one deduces from Theorem 3.26 that two among the four numbers ω_1, ω_2, η_1, η_2 are algebraically independent; this statement is also a consequence of the next result:

Theorem 3.27 (G.V. Chudnovsky, 1981)**.** *Assume that g_2 and g_3 are algebraic. Let ω be a non-zero period of $\wp$, set $\eta = \eta(\omega)$ and let u be a complex number which is not a period such that u and ω are $\mathbb{Q}$-linearly independent: $u \notin \mathbb{Q}\omega \cup \Omega$. Assume $\wp(u) \in \overline{\mathbb{Q}}$. Then the two numbers*

$$\zeta(u) - \frac{\eta}{\omega}u, \quad \frac{\eta}{\omega}$$

are algebraically independent.

From Theorem 3.26 or Theorem 3.27 one deduces:

Corollary 3.28. *Let ω be a non-zero period of $\wp$ and $\eta = \eta(\omega)$. If g_2 and g_3 are algebraic, then the two numbers π/ω and η/ω are algebraically independent.*

The following consequence of Corollary 3.28 shows that in the CM case, Chudnovsky's results are sharp:

Corollary 3.29. *Assume that g_2 and g_3 are algebraic and the elliptic curve has complex multiplications. Let ω be a non-zero period of $\wp$. Then the two numbers ω and π are algebraically independent.*

As a consequence of formulae (3.5) and (3.6), one deduces:

Corollary 3.30. *The numbers π and $\Gamma(1/4)$ are algebraically independent. Also the numbers π and $\Gamma(1/3)$ are algebraically independent.*

In connection with these results let us quote a conjecture of S. Lang from 1971

Conjecture 3.31. *If $j(\tau)$ is algebraic with $j'(\tau) \neq 0$, then $j'(\tau)$ is transcendental.*

According to Siegel's relation

$$j'(\tau) = 18\frac{\omega_1^2}{2\pi i}\cdot\frac{g_3}{g_2}\cdot j(\tau).$$

Conjecture 3.31 amounts to the transcendence of ω^2/π. Hence Corollary 3.29 implies that Conjecture 3.31 is true at least in the CM case.

Corollary 3.32. *If $\tau \in \mathcal{H}$ is quadratic and $j'(\tau) \neq 0$, then π and $j'(\tau)$ are algebraically independent.*

A quantitative refinement (measure of algebraic independence) of Corollary 3.28 due to G. Philibert turns out to be useful in connection with Nesterenko's work in 1996.

A transcendence measure for $\Gamma(1/4)$ was obtained by P. Philippon and S. Bruiltet.

Theorem 3.33. *For $P \in \mathbb{Z}[X,Y]$ with degree d and height H,*

$$\log|P(\pi,\Gamma(1/4)| > -10^{326}\bigl((\log H + d\log(d+1)\bigr)d^2\bigl(\log(d+1)\bigr)^2.$$

Corollary 3.34. *The number $\Gamma(1/4)$ is not a Liouville number:*

$$\left|\Gamma(1/4) - \frac{p}{q}\right| > \frac{1}{q^{10^{330}}}\cdot$$

Further algebraic independence results are due to D. Bertrand, G.V. Chudnovsky, and E. Reyssat.

We conclude this section by the following open problem, which simultaneously generalizes Theorems 3.26 and 3.27 of G.V. Chudnovsky.

Conjecture 3.35. *Let $\wp$ be a Weierstraß elliptic function with invariants g_2, g_3, let ω be a non-zero period of $\wp$, set $\eta = \eta(\omega)$ and let $u \in \mathbb{C}\setminus\{\mathbb{Q}\omega\cup\Omega\}$. Then at least two of the five numbers*

$$g_2, \quad g_3, \quad \wp(u), \quad \zeta(u) - \frac{\eta}{\omega}u, \quad \frac{\eta}{\omega}$$

are algebraically independent.

Further conjectures are considered in § 5.3.

Chudnovsky's method was extended by K.G. Vasil'ev and P. Grinspan who proved that at least two of the three numbers π, $\Gamma(1/5)$ and $\Gamma(2/5)$ are algebraically independent. Their proof involves the Jacobian of the Fermat curve $X^5 + Y^5 = Z^5$, which contains an abelian variety of dimension 2 as a factor.

3.3.3 Large transcendence degree

Another important (and earlier) contribution of G.V. Chudnovsky goes back to 1974 when he worked on extending Gel'fond's method in order to prove results on large transcendence degree.

Chudnovsky proved that three of the numbers

$$\alpha^{\beta}, \alpha^{\beta^2}, \dots, \alpha^{\beta^{d-1}} \tag{3.36}$$

are algebraically independent if α is a non-zero algebraic number, $\log \alpha$ a non-zero logarithm of α and β an algebraic number of degree $d \geq 7$. The same year, with a much more difficult and highly technical proof, he made the first substantial progress towards a proof that there exist at least n algebraically independent numbers in the set (3.36), provided that $d \geq 2^n - 1$. This was a remarkable achievement since no such result providing a lower bound for the transcendence degree was known. Later, thanks to the work of several mathematicians, including P. Philippon and Yu. V. Nesterenko, the proof was completed and the exponential lower bound for d was reduced to a polynomial bound, until G. Diaz obtained the best known results so far: the transcendence degree is at least $[(d+1)/2]$.

During a short time, thanks to the work of Philippon, the elliptic results dealing with large transcendence degree where stronger than the exponential ones.

In 1980, G.V. Chudnovsky proved the Lindemann-Weierstraß Theorem 2.41 for $n = 2$ and $n = 3$ (small transcendence degree) by means of a clever variation of Gel'fond's method. At the same time he obtained the elliptic analog in the CM case of the Lindemann-Weierstraß Theorem for $n = 2$ and $n = 3$. Also in 1980 he announces further results of small transcendence degree (algebraic independence of 4 numbers).

This method was extended to large transcendence degree by P. Philippon and G. Wüstholz who also succeeded in 1982 to prove the elliptic analog of Lindemann-Weierstraß Theorem on the algebraic independence of $e^{\alpha_1}, \dots, e^{\alpha_n}$ in the CM case:

Theorem 3.37. *Let $\wp$ be a Weierstraß elliptic function with algebraic invariants g_2, g_3 and complex multiplications. Let $\alpha_1, \dots, \alpha_m$ be algebraic numbers which are linearly independent over the field of endomorphisms of E . Then the numbers $\wp(\alpha_1), \dots, \wp(\alpha_n)$ are algebraically independent.*

The same conclusion should also hold in the non-CM case – so far only the algebraic independence of at least $n/2$ of these numbers is known.

Further results on large transcendence degree are due to D.W. Masser and G. Wüstholz W.D. Brownawell, W.D. Brownawell and R. Tubbs, and M. Takeuchi.

3.3.4 Modular functions and Ramanujan functions

S. Ramanujan introduced the following functions

$$P(q) = 1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}, \quad Q(q) = 1+240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}, \quad R(q) = 1-504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}.$$

They are special cases of Fourier expansions of Eisenstein series. Recall the Bernoulli numbers B_k defined by:

$$\frac{z}{e^z-1} = 1-\frac{z}{2}+\sum_{k=1}^{\infty}(-1)^{k+1}B_k\frac{z^{2k}}{(2k)!},$$

$$B_1 = 1/6, \quad B_2 = 1/30, \quad B_3 = 1/42.$$

For $k \geq 1$ the normalized Eisenstein series of weight k is

$$E_{2k}(q) = 1+(-1)^k\frac{4k}{B_k}\sum_{n=1}^{\infty}\frac{n^{2k-1}q^n}{1-q^n}.$$

The connection with (3.4) is

$$E_{2k}(q) = \frac{1}{2\zeta(2k)}\cdot G_k(\tau),$$

for $k \geq 2$, where $q = e^{2\pi i\tau}$. In particular

$$G_2(\tau) = \frac{\pi^4}{3^2\cdot 5}\cdot E_4(q), \qquad G_3(\tau) = \frac{2\pi^6}{3^3\cdot 5\cdot 7}\cdot E_6(q).$$

With Ramanujan's notation we have

$$P(q) = E_2(q), \quad Q(q) = E_4(q), \quad R(q) = E_6(q).$$

The discriminant Δ and the modular invariant J are related with these functions by Jacobi's product formula

$$\Delta = \frac{(2\pi)^{12}}{12^3}\cdot(Q^3-R^2) = (2\pi)^{12}q\prod_{n=1}^{\infty}(1-q^n)^{24} \quad \text{and} \quad J = \frac{(2\pi)^{12}Q^3}{\Delta} = \frac{(2^43^25G_2)^3}{\Delta}.$$

Let q be a complex number, $0<|q|<1$. There exists τ in the upper half plane $\mathcal{H}$ such that $q = e^{2\pi i\tau}$. Select any twelfth root ω of $\Delta(q)$. The invariants g_2 and g_3 of the Weierstraß $\wp$ function attached to the lattice $(\mathbb{Z}+\mathbb{Z}\tau)\omega$ satisfy $g_2^3-27g_3^2 = 1$ and

$$P(q) = 3\frac{\omega}{\pi}\cdot\frac{\eta}{\pi}, \quad Q(q) = \frac{3}{4}\left(\frac{\omega}{\pi}\right)^4 g_2, \quad R(q) = \frac{27}{8}\left(\frac{\omega}{\pi}\right)^6 g_3.$$

According to formulae (3.5) and (3.6), here are a few special values

- For $\tau = i$, $q = e^{-2\pi}$,

$$P(e^{-2\pi}) = \frac{3}{\pi}, \quad Q(e^{-2\pi}) = 3\left(\frac{\omega_1}{\pi}\right)^4, \qquad (3.38)$$
$$R(e^{-2\pi}) = 0 \quad \text{and} \quad \Delta(e^{-2\pi}) = 2^6\omega_1^{12},$$

with

$$\omega_1 = \frac{\Gamma(1/4)^2}{\sqrt{8\pi}} = 2.6220575542\dots$$

- For $\tau = \varrho$, $q = -e^{-\pi\sqrt{3}}$,

$$P(-e^{-\pi\sqrt{3}}) = \frac{2\sqrt{3}}{\pi}, \; Q(-e^{-\pi\sqrt{3}}) = 0, \qquad (3.39)$$
$$R(-e^{-\pi\sqrt{3}}) = \frac{27}{2}\left(\frac{\omega_1}{\pi}\right)^6, \; \Delta(-e^{-\pi\sqrt{3}}) = -2^4 3^3\omega_1^{12},$$

with

$$\omega_1 = \frac{\Gamma(1/3)^3}{2^{4/3}\pi} = 2.428650648\dots$$

3.3.5 Mahler-Manin problem on $J(q)$

After Schneider's Theorem (Corollary 3.17) on the transcendence of the values of the modular function $j(\tau)$, the first results on Eisenstein series (cf. § 3.3.6) go back to 1977 with D. Bertrand's work.

The first transcendence proof using modular forms is due to a team from St Étienne (K. Barré-Sirieix, G. Diaz, F. Gramain and G. Philibert) — hence the nickname *théorème stéphanois* for the next result ; see [1] The Theorem 3.40 answers a conjecture of K. Mahler in the complex case and of Yu. V. Manin in the p-adic case. Manin's question on the arithmetic nature of the p-adic number $J(q)$ is motivated by Mazur's theory, but he also asked "an obvious analog" in the complex case – see Conjecture 3.43 below). We state the result only in the complex case — the paper [1] solves both cases.

Theorem 3.40 (K. Barré, G. Diaz, F. Gramain, G. Philibert, 1996)**.** *Let $q \in \mathbb{C}$, $0 < |q| < 1$. If q is algebraic, then $J(q)$ is transcendental.*

The solution of Manin's problem in the p-adic case has several consequences. It is a tool both for R. Greenberg in his study of zeroes of p-adic L functions and for H. Hida, J. Tilouine and É. Urban in their solution of the main Conjecture for the Selmer group of the symmetric square of an elliptic curve with multiplicative reduction at p.

The proof of Theorem 3.40 involves upper bounds for the growth of the coefficients of the modular function $J(q)$. Such estimates were produced first by K. Mahler. A refined estimate, due to N. Brisebarre and G. Philibert, for the coefficients $c_k(m)$ (which are non-negative rational integers) in

$$\big(qJ(q)\big)^k = \sum_{m=0}^{\infty} c_k(m)q^m,$$

is

$$c_k(m) \le e^{4\sqrt{km}}.$$

According to a remark by D. Bertrand, the upper bound

$$|\tilde{c}_{N,k}(m)| \le C^N m^{12N}$$

($0 \le k \le N$, $N \ge 1$, $m \ge 1$, with an absolute constant C) for the coefficients in the Taylor development at the origin of $\Delta^{2N} J^k$:

$$\Delta(q)^{2N} J(q)^k = \sum_{m=1}^{\infty} \tilde{c}_{Nk}(m) q^m$$

is sufficient for the proof of Theorem 3.40 and is an easy consequence of a Theorem of Hecke, together with the fact that Δ^2 and $\Delta^2 J$ are parabolic modular forms of weight 24.

One of the main tools involved in the proof of Theorem 3.40 is an estimate for the degrees and height of $J(q^n)$ in terms of $J(q)$ (which is assumed to be algebraic) and $n \ge 1$. There exists a symmetric polynomial $\Phi_n \in \mathbb{Z}[X,Y]$, of degree

$$\psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right)$$

in each variable, such that $\Phi_n\bigl(J(q), J(q^n)\bigr) = 0$. Again, K. Mahler was the first to investigate the coefficients of the polynomial $\Phi_n(X,Y)$: he proved that its length (sum of the absolute values of the coefficients) satisfies

$$L(\Phi_n) \le e^{cn^{3/2}}$$

with an absolute constant c. In the special case $n = 2^m$ he had an earlier stronger result, namely,

$$L(\Phi_n) \le 2^{57n} n^{36n}$$

and he claimed that if the sharper upper bound

$$L(\Phi_n) \le 2^{Cn}$$

with a positive absolute constant $C > 0$ were true for $n = 2^m$, he could prove Theorem 3.40. However in 1984 P. Cohen produced asymptotic estimates which show that Mahler's expectation was too optimistic:

$$\lim_{\substack{n=2^m \\ m\to\infty}} \frac{1}{n \log n} \log L(\Phi_n) = 9.$$

In fact she proved more precise results, without the condition $n = 2^m$, which imply, for instance, $\log L(\Phi_n) \sim 6\psi(n) \log n$ for $n \to \infty$.

A reformulation of Theorem 3.40 on the transcendence of $J(q)$ is the following mixed analog of the Four Exponentials Conjecture 2.48:

Corollary 3.41. *Let $\log\alpha$ be a logarithm of a non-zero algebraic number. Let $\mathbb{Z}\omega_1+\mathbb{Z}\omega_2$ be a lattice with algebraic invariants g_2, g_3. Then the determinant*

$$\begin{vmatrix}\omega_1 & \log\alpha\\ \\ \omega_2 & 2\pi i\end{vmatrix}$$

does not vanish.

The Four Exponentials Conjecture for the product of an elliptic curve by the multiplicative group is the following more general open problem:

Conjecture 3.42. *Let $\wp$ be a Weierstraß elliptic function with algebraic invariants g_2, g_3. Let E be the corresponding elliptic curve, u_1 and u_2 be two elements in $\mathcal{L}_E$ and $\log\alpha_1$, $\log\alpha_2$ be two logarithms of algebraic numbers. Assume further that the two rows of the matrix*

$$M=\begin{pmatrix}u_1 & \log\alpha_1\\ u_2 & \log\alpha_2\end{pmatrix}$$

are linearly independent over $\mathbb{Q}$. Then the determinant of M does not vanish.

Another special case of Conjecture 3.42, stronger than Corollary 3.41, is the next question of Yu. V. Manin, who asks to determine the nature of the invariant of the complex elliptic curve having periods 1 and a quotient $(\log\alpha_1)/(\log\alpha_2)$ of two logarithms of algebraic numbers:

Conjecture 3.43 (Yu.V. Manin)**.** *Let $\log\alpha_1$ and $\log\alpha_2$ be two non-zero logarithms of algebraic numbers and let $\mathbb{Z}\omega_1+\mathbb{Z}\omega_2$ be a lattice with algebraic invariants g_2 and g_3. Then*

$$\frac{\omega_1}{\omega_2}\neq\frac{\log\alpha_1}{\log\alpha_2}.$$

In this direction let us quote some of the open problems raised by G. Diaz.

Conjecture 3.44 (G. Diaz)**.** .
1. For any $z\in\mathbb{C}$ with $|z|=1$ and $z\neq\pm1$, the number $e^{2\pi iz}$ is transcendental.
2. If q is an algebraic number with $0<|q|<1$ such that $J(q)\in[0,1728]$, then $q\in\mathbb{R}$.
3. The function J is injective on the set of algebraic numbers α with $0<|\alpha|<1$.

Remark (G. Diaz). Part 3 of Conjecture 3.44 implies the other two and also follows from the Four Exponentials Conjecture 2.48. It also follows from the next Conjecture of D. Bertrand.

Conjecture 3.45 (D. Bertrand)**.** *If α_1 and α_2 are two multiplicatively independent algebraic numbers in the domain $\{q\in\mathbb{C}\,;\,0<|q|<1\}$, then the two numbers $J(\alpha_1)$ and $J(\alpha_2)$ are algebraically independent.*

This Conjecture 3.45, where § 5 is devoted to conjectural statements, which were inspired by a Conjecture of Oort and André) implies the special case of the Four Exponentials Conjecture 2.48, where two of the algebraic numbers are roots of unity and the two others have modulus $\neq 1$.

3.3.6 Nesterenko's Theorem

In 1976, D. Bertrand pointed out that Schneider's Theorem 3.16 on the transcendence of ω/π implies:
For any $q \in \mathbb{C}$ with $0 < |q| < 1$, at least one of the two numbers $Q(q)$, $R(q)$ is transcendental.

He also proved the p-adic analog by means of a new version of the Schneider–Lang criterion for meromorphic functions (he allows one essential singularity) which he applied to Jacobi–Tate elliptic functions. Two years later he noticed that G.V. Chudnovsky's Theorem 3.26 yields:
For any $q \in \mathbb{C}$ with $0 < |q| < 1$, at least two of the numbers $P(q)$, $Q(q)$, $R(q)$ are algebraically independent.

The following result of Yu.V. Nesterenko goes one step further:

Theorem 3.46 (Nesterenko, 1996)**.** *For any $q \in \mathbb{C}$ with $0 < |q| < 1$, three of the four numbers q, $P(q)$, $Q(q)$, $R(q)$ are algebraically independent.*

Among the tools used by Nesterenko in his proof is the following result due to K. Mahler:
The functions P, Q, R are algebraically independent over $\mathbb{C}(q)$.

Also he uses the fact that they satisfy a system of differential equations for $D = q\, d/dq$ discovered by S. Ramanujan in 1916:

$$12\frac{DP}{P} = P - \frac{Q}{P}, \quad 3\frac{DQ}{Q} = P - \frac{R}{Q}, \quad 2\frac{DR}{R} = P - \frac{Q^2}{R}.$$

One of the main steps in his original proof is his following zero estimate:

Theorem 3.47 (Nesterenko's zero estimate)**.** *Let L_0 and L be positive integers, $A \in \mathbb{C}[q, X_1, X_2, X_3]$ a non-zero polynomial in four variables of degree $\le L_0$ in q and $\le L$ in each of the three other variables X_1, X_2, X_3. Then the multiplicity at the origin of the analytic function $A\big(q, P(q), Q(q), R(q)\big)$ is at most $2 \cdot 10^{45} L_0 L^3$.*

In the special case where $J(q)$ is algebraic, P. Philippon produced an alternative proof for Nesterenko's result where this zero estimate 3.47 is not used; instead of it, he used Philibert's measure of algebraic independence for ω/π and η/π. However Philibert's proof requires a zero estimate for algebraic groups.

Using (3.38) one deduces from Theorem 3.46

Corollary 3.48. *The three numbers π, e^{π}, $\Gamma(1/4)$ are algebraically independent.*

while using (3.39) one deduces

Corollary 3.49. *The three numbers π, $e^{\pi\sqrt{3}}$, $\Gamma(1/3)$ are algebraically independent.*

Consequences of Corollary 3.48 are the transcendence of the numbers

$$\sigma_{\mathbb{Z}[i]}(1/2) = 2^{5/4}\pi^{1/2}e^{\pi/8}\Gamma(1/4)^{-2}$$

and (P. Bundschuh)

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1}{2} + \frac{\pi}{2} \cdot \frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}}.$$

D. Duverney, K. and K. Nishioka and I. Shiokawa as well as D. Bertrand derived from Nesterenko's Theorem 3.46 a number of interesting corollaries, including the following ones

Corollary 3.50. *Rogers-Ramanujan continued fraction:*

$$RR(\alpha) = 1 + \cfrac{\alpha}{1 + \cfrac{\alpha^2}{1 + \cfrac{\alpha^3}{1 + \ddots}}}$$

is transcendental for any algebraic α *with* $0 < |\alpha| < 1$.

Corollary 3.51. *Let* $(F_n)_{n\ge 0}$ *be the Fibonacci sequence:* $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$. *Then the number*

$$\sum_{n=1}^{\infty} \frac{1}{F_n^2}$$

is transcendental.

Jacobi Theta Series are defined by

$$\theta_2(q) = 2q^{1/4} \sum_{n\ge 0} q^{n(n+1)} = 2q^{1/4} \prod_{n=1}^{\infty} (1-q^{4n})(1+q^{2n}),$$

$$\theta_3(q) = \sum_{n\in\mathbb{Z}} q^{n^2} = \prod_{n=1}^{\infty} (1-q^{2n})(1+q^{2n-1})^2,$$

$$\theta_4(q) = \theta_3(-q) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} = \prod_{n=1}^{\infty} (1-q^{2n})(1-q^{2n-1})^2.$$

Corollary 3.52. . *Let* i, j *and* $k \in \{2,3,4\}$ *with* $i \ne j$. *Let* $q \in \mathbb{C}$ *satisfy* $0 < |q| < 1$. *Then each of the two fields*

$$\mathbb{Q}\big(q, \theta_i(q), \theta_j(q), D\theta_k(q)\big) \quad \textit{and} \quad \mathbb{Q}\big(q, \theta_k(q), D\theta_k(q), D^2\theta_k(q)\big)$$

has transcendence degree ≥ 3 *over* $\mathbb{Q}$.

As an example, *for an algebraic number* $q \in \mathbb{C}$ *with* $0 < |q| < 1$, *the three numbers*

$$\sum_{n\ge 0} q^{n^2}, \quad \sum_{n\ge 1} n^2 q^{n^2}, \quad \sum_{n\ge 1} n^4 q^{n^2}$$

are algebraically independent. In particular *the number*

$$\theta_3(q) = \sum_{n\in\mathbb{Z}} q^{n^2}$$

is transcendental. The number $\theta_3(q)$ was explicitly considered by Liouville as far back as 1851.

The proof of Yu.V. Nesterenko is effective and yields quantitative refinements (measures of algebraic independence).

References

[1] K. Barré-Sirieix, G. Diaz, F. Gramain & G. Philibert – "Une preuve de la conjecture de Mahler-Manin", *Invent. Math.* **124** (1996), no. 1-3, p. 1–9.

[2] A. O. Gel′fond – *Transcendental and algebraic numbers*, Translated from the first Russian edition by Leo F. Boron, Dover Publications Inc., New York, 1960.

[3] M. Kontsevich & D. Zagier – "Periods", in *Mathematics unlimited—2001 and beyond*, Springer, Berlin, 2001, p. 771–808.

[4] Th. Schneider –*Einführung in die transzendenten Zahlen*, Springer-Verlag, Berlin, 1957.

[5] M. Waldschmidt, *Open Diophantine Problems*, Moscow Mathematical Journal **4** N°1, 2004, 245–305.

[6] M. Waldschmidt, *Elliptic Functions and Transcendence*, Developments in Mathematics, Springer Verlag, to appear. http://www.math.jussieu.fr/∼miw/articles/pdf/SurveyTrdceEllipt2005.pdf