The University of Arizona
The Southwest Center for Arithmetic Geometry
AWS 2008: *Special Functions and Transcendence*

Updated: March 4, 2008

2008 Arizona Winter School, March 15-19, 2008
http://swc.math.arizona.edu/aws/08/index.html

Michel Waldschmidt

`Rough outlines of the project`

a) (Expanding a remark by S. Lang – [1]). Define $K_0 = \overline{\mathbb{Q}}$. Inductively, for $n \geq 1$, define K_n as the algebraic closure of the field generated over K_{n-1} by the numbers e^x, where x ranges over K_{n-1}. Let Ω_+ be the union of K_n, $n \geq 0$. Show that the numbers

$$\pi,\ \log\pi,\ \log\log\pi,\ \log\log\log\pi,\ldots$$

are algebraically independent over Ω_+.

References

[1] S. Lang – *Introduction to transcendental numbers*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1966. *Collected papers. Vol. I*, Springer-Verlag, New York, 2000, 1952–1970.

b) Try to get a (conjectural) generalisation involving the field Ω_- defined as follows. Define $E_0 = \overline{\mathbb{Q}}$. Inductively, for $n \geq 1$, define L_n as the algebraic closure of the field generated over L_{n-1} by the numbers y, where y ranges over the set of complex numbers such that $e^y \in L_{n-1}$. Let Ω_- be the union of L_n, $n \geq 0$.'

c)

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De :  Emmanuel.Kowalski@math.u-bordeaux.fr
Objet :  The field generated by p^it, p prime, t fixed...
Date :  22 octobre 2003 14:05:03 HAEC
À : NMBRTHRY@LISTSERV.NODAK.EDU
Répondre à :  Emmanuel.Kowalski@math.u-bordeaux.fr
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Hello,

I wonder if anyone has any ideas about the following question: let t be a real number, and define K_t to be the subfield of $\mathbb{C}$ generated over $\mathbb{Q}$ by all p^{it} where p is prime. Is it true that the intersection of K_t with the field of algebraic numbers is always a number field?

This is true if one takes only finitely many primes, as a subfield of a finitely generated field is also finitely generated. It is also easy to show that it is true for all t except at most countably many.

Note that this is not really a classical transcendence / algebraic independence problem because even the most degenerate case, $t = 0$, works with $K_0 = \mathbb{Q}$; no question is asked about the transcendence degree of K_t, or the independence of the various p^{it}.

Background of the problem: this field K_t is the field generated (over $\mathbb{Q}$) by eigenvalues of the Hecke operators for the Eisenstein series $E(z, 1/2 + it)$ of $\mathrm{SL}(2, \mathbb{Z})$; if it is true that the analogue field for cusp forms (which are newforms) only contains a finite algebraic extension of $\mathbb{Q}$, it would for instance follow that Katz's L-function with Kloosterman sums **is not that** of a Maass (new)form – as widely expected, but of course one can't really hope to prove it this way!

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De : poonen@math.berkeley.edu
Objet : Rép : The field generated by p^{it}, p prime, t fixed...
Date : 23 octobre 2003 04:42:42 HAEC
À : NMBRTHRY@LISTSERV.NODAK.EDU
Répondre à : poonen@math.berkeley.edu

On Wed, Oct 22, 2003 at 08:05:03AM -0400, Emmanuel Kowalski wrote:

I wonder if anyone has any ideas about the following question: let t be a real number, and define K_t to be the subfield of $\mathbb{C}$ generated over $\mathbb{Q}$ by all p^{it} where p is prime. Is it true that the intersection of K_t with the field of algebraic numbers is always a number field?

It would follow from Schanuel's Conjecture.

Let p_n be the n-th prime. If t is a nonzero real number, the $2n$ numbers

$$\log 2, \ldots, \log p_n, (\log 2)it, \ldots, (\log p_n)it$$

are$\mathbb{Q}$-linearly independent, so Schanuel's Conjecture implies that they together with their exponentials generate a field of transcendence degree at least $2n$. But these $2n$ numbers together with the exponentials of the first n generate a field of transcendence degree at most $n + 1$, so the remaining exponentials

$$2^{it}, \ldots, p_n^{it}$$

generate a field $K_t^{(n)}$ of transcendence degree at least $n - 1$. This holds for all n, so there is at most one $m \geq 1$ such that p_{m+1}^{it} is algebraic over $K_t^{(m)}$. Hence your field $K_t := \mathbb{Q}(2^{it}, 3^{it}, \ldots)$ is a purely transcendental extension of a finitely generated field L. Thus the field

$$K_t \cap \overline{\mathbb{Q}} = L \cap \overline{\mathbb{Q}}$$

is a number field.

Bjorn Poonen