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An introduction to irrationality and transcendence methods.

Michel Waldschmidt

Abstract and Notation [2]

Abstract

Diophantine approximation is a chapter in number theory which has witnessed outstanding progress together with a number of deep applications during the recent years. The proofs have long been considered as technically difficult. However, we understand better now the underlying ideas, hence it becomes possible to introduce the basic methods and the fundamental tools in a more clear way.

The first lecture deals with irrationality, as an introduction to transcendence results. After a short historical survey of this topic, we reproduce the easy proof of irrationality of e by Fourier (1815) and explain how Liouville extended it up to a proof that e^2 is not a quadratic number. Such irrationality proofs rest on a criterion for irrationality, which can be extended to a criterion of linear independence.

The second lecture is an historical introduction to transcendence.

The transcendence proofs for constants of analysis are essentially all based on the seminal work by Ch. Hermite: his proof of the transcendence of the number e in 1873 is the prototype of the methods which have been subsequently developed. The founding paper by Hermite was influenced by earlier authors (Lambert, Euler, Fourier, Liouville). We explain how his arguments have been expanded in several directions: Padé approximants, interpolation series, auxiliary functions. We complete this lecture by a historical survey of transcendental number theory.

The third lecture is devoted to elliptic functions and transcendence: the first results are due to Siegel, more recent achievements are due to Chudnovsky (1978) and Nesterenko (1996). Even if one is interested only in numbers related to the classical exponential function, like π and e^{π}, one finds that elliptic functions are required to prove transcendence results and get a better understanding of the situation.

[2] http://www.math.jussieu.fr/∼miw/articles/pdf/AWSLecture2.pdf

Transcendence proofs most often involve an auxiliary function. Such functions can take several forms. Historically, the first ones were Padé approximations, in Hermite's proof of the transcendence of e (1873). Next came functions whose existence is proved by means of the Dirichlet's box principle, with the work of Thue (early 1900) and Siegel (in the 1920's). Another tool was provided by interpolation formulae, mainly Newton interpolation (involving Hermite's formulae again) in the study by G. Polya (1914) and A.O. Gel'fond (1929) of integer valued entire functions. Along these lines, recent developments are due to T. Rivoal (to appear), who renewed the forgotten rational interpolation formulae of R. Lagrange (1935). In 1991 M. Laurent introduced interpolation determinants, and two years later J.B. Bost used Arakhelov theory to prove slope inequalities, which dispens of the choice of bases.

The last lecture is devoted to conjectures and open problems. Kontsevich and Zagier introduced the notion of periods and their suggestion would imply a number of results which are yet unknown. Examples deal with multiple zeta values. Schanuel's conjecture is a far reaching statement which would solve most transcendence problems related to the values of the exponential and elliptic function. The main special case, which is yet open, is the conjecture on algebraic independence of logarithms of algebraic numbers. We survey recent work on this topic, mainly tue to D. Roy. Further open problems deal with elliptic functions, modular functions, Fibonacci numbers and various series. We also quote some open problems on the expansion of irrational algebraic numbers.

Notation

We denote by $\mathbb{Z}$ the ring of rational integers, by $\mathbb{Q}$ the field of rational numbers, by $\mathbb{R}$ the field of real numbers and by $\mathbb{C}$ the field of complex numbers. Given a real number, we want to know whether it is rational or not, that means whether he belongs to $\mathbb{Q}$ or not. The set of irrational numbers $\mathbb{R}\setminus\mathbb{Q}$ has no nice algebraic properties: it is not stable by addition nor by multiplication.

Irrationality is the first step, the second one is transcendence. Given a complex number, one wants to know whether it is algebraic of not. The set of algebraic numbers, which is the set of roots of all non-zero polynomials with rational coefficients, is nothing else than the algebraic closure of $\mathbb{Q}$ into $\mathbb{C}$. We denote it by $\overline{\mathbb{Q}}$. The set of transcendental numbers is defined as $\mathbb{C}\setminus\overline{\mathbb{Q}}$. Since $\overline{\mathbb{Q}}$ is a field, the set of transcendental numbers is not stable by addition nor by multiplication.

We denote by $\mathcal{L}$ the $\mathbb{Q}$-vector space of logarithms of algebraic numbers:

$$\mathcal{L} = \left\{\log\alpha\,;\ \alpha\in\overline{\mathbb{Q}}^\times\right\} = \left\{\ell\in\mathbb{C}\,;\ e^\ell\in\overline{\mathbb{Q}}^\times\right\} = \exp^{-1}(\overline{\mathbb{Q}}^\times).$$