November 9 – 19, 2021. Limbe (Cameroun)

Number Theory
I : Linear Recurrent Sequences
African Institute for Mathematical Sciences (AIMS)

Michel Waldschmidt, Sorbonne Université

Assignment 1

Let $d \geq 1$ be an integer and $c_1, \ldots, c_d$ complex numbers with $c_d \neq 0$. Define

$$P(T) = T^d - c_1 T^{d-1} - \cdots - c_{d-1} T - c_d \in \mathbb{C}[T].$$

Denote by $\gamma_1, \ldots, \gamma_\ell$ the distinct roots of P in $\mathbb{C}$, and, for $1 \leq j \leq \ell$, by t_j the multiplicity of the root γ_j, so that

$$P(T) = \prod_{j=1}^{\ell} (T - \gamma_j)^{t_j}$$

with $t_j \geq 1, \;\; t_1 + \cdots + t_\ell = d\,.$

Denote by E the vector space over $\mathbb{C}$ of the sequences which satisfy the linear recurrence relation of order d given by

$$u_{n+d} = c_1 u_{n+d-1} + \cdots + c_d u_n \quad \text{for } n \geq 0. \tag{1}$$

(1). Prove that E is a complex vector subspace of $\mathbb{C}^{\mathbb{N}}$ of dimension d.
(2). The goal of this exercise is to prove that a basis of E is given by the d sequences

$$\left(n^i \gamma_j^n\right)_{n \in \mathbb{N}} \quad (1 \leq j \leq \ell, \;\; 0 \leq i \leq t_j - 1). \tag{2}$$

(a) Prove this result in the special case $t_1 = \cdots = t_\ell = 1$.
(b) Prove this result in the special case $\ell = 1$.

`Hint.`
Check, for $n \geq 0$ and $0 \leq i \leq d-1$,

$$\sum_{k=0}^{d} (-1)^k \binom{d}{k} (d + n - k)^i = 0.$$

(c) From (a) and (b), deduce this result in the special case $d = 2$.
Consider now the general case.
(d) Prove that the d sequences (2) satisfy (1).
(e) Prove that the d sequences (2) give a basis of E.

`Hint.` There are several ways of proving (b), (d) and (e). You may use some of the following suggestions.

(i) For $i \geq 0$, check that the map

$$\left(T\tfrac{\mathrm{d}}{\mathrm{d}T}\right)^i : \begin{array}{ccc} \mathbb{C}[T] & \longrightarrow & \mathbb{C}[T] \\ \sum_{h\geq 0} a_h T^h & \longmapsto & \sum_{h\geq 0} a_h h^i T^h \end{array}$$

is a linear map. We agree that $h^i = 1$ for $i = h = 0$. Check, for $n \geq 0$, $1 \leq j \leq \ell$ and $0 \leq i \leq t_j - 1$,

$$\left(T\frac{\mathrm{d}}{\mathrm{d}T}\right)^i (T^n P)(\gamma_j) = 0.$$

Deduce

$$(n+d)^i \gamma_j^{n+d} = \sum_{k=1}^{d} (n+d-k)^i c_k \gamma_j^{n+d-k} \qquad (n \geq 0),$$

with the convention that for $k = n + d$, the term $(n + d - k)^i$ takes the value 1 for $i = 0$ and the value 0 for $i \geq 1$. Deduce that for $1 \leq j \leq \ell$ and $0 \leq i \leq t_j - 1$, the sequence $\left(n^i \gamma_j^a\right)_{n\geq 0}$ satisfies (1).

(ii) Write the linear recurrence relation in a matrix form

$$U_{n+1} = CU_n$$

with

$$U_n = \begin{pmatrix} u_n \\ u_{n+1} \\ \vdots \\ u_{n+d-1} \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ c_d & c_{d-1} & c_{d-2} & \cdots & c_1 \end{pmatrix}.$$

Write the matrix C in its Jordan normal form.

(iii) Introduce the formal power series

$$U(T) = \sum_{n \geq 0} u_n T^n.$$

Check that $U(T)$ is a rational fraction, with denominator

$$1 - \sum_{i=1}^{d} c_i T^i = T^d P(1/T) = \prod_{j=1}^{\ell} (1 - \gamma_j T)^{t_j},$$

while the numerator is of degree $< d$. Use a partial fraction decomposition. Develop $(1 - \gamma_j T)^{-i-1}$ for $0 \leq i \leq t_j - 1$ as a power series expansion. Deduce that the d sequences (2) generate E.

(iv) Consider the matrix A made of ℓ vertical blocks $A_1, A_2, \ldots, A_\ell$ where for $1 \leq j \leq \ell$, A_j is the $t_j \times d$ matrix

$$A_j = \begin{pmatrix} 1 & \gamma_j & \gamma_j^2 & \cdots & \gamma_j^{t_j-1} & \gamma_j^{t_j} & \cdots & \gamma_j^{d-1} \\ 0 & 1 & \binom{2}{1}\gamma_j & \ldots & \binom{t_j-1}{1}\gamma_j^{t_j-2} & \binom{t_j}{1}\gamma_j^{t_j-1} & \ldots & \binom{d-1}{1}\gamma_j^{d-2} \\ 0 & 0 & 1 & \ldots & \binom{t_j-1}{2}\gamma_j^{t_j-3} & \binom{t_j}{2}\gamma_j^{t_j-2} & \ldots & \binom{d-1}{2}\gamma_j^{d-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & \binom{t_j}{t_j-1}\gamma_j & \cdots & \binom{d-1}{t_j-1}\gamma_j^{d-t_j} \end{pmatrix}.$$

Note that $\binom{r}{k} = 0$ for $r < k$.

Show that the d columns of A are linearly independent over $\mathbb{C}$.

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Assignment 1 — Solution

(1). We have seen in the course that E is a vector subspace of $\mathbb{C}^{\mathbb{N}}$ of dimension d. Let us repeat the proofs. The sum of two elements in E is in E, the product of an element in E with a constant is in E, hence E is a vector subspace of $\mathbb{C}^{\mathbb{N}}$. We now check that a basis of E is given by the d so–called *impulse* sequences $e^{(0)}, \dots, e^{(d-1)}$ defined by the initial conditions

$$e_n^{(j)} = \delta_{jn} \quad (0 \le j, n \le d-1),$$

δ_{jn} being the Kronecker symbol

$$\delta_{jn} = \begin{cases} 1 & \text{if } \ j = n, \\ 0 & \text{if } \ j \neq n. \end{cases}$$

We first check that $\{e^{(0)}, \dots, e^{(d-1)}\}$ is a generating set for E. Let $(u_n)_{n\ge 0}$ be in E. By the definition of $e^{(0)}, \dots, e^{(d-1)}$, for $0 \le n \le d-1$ we have

$$u_n = u_0 e_n^{(0)} + u_1 e_n^{(1)} + \cdots + u_{d-1} e_n^{(d-1)}.$$

Since both u and $u_0 e^{(0)} + u_1 e^{(1)} + \cdots + u_{d-1} e^{(d-1)}$ are in E, it follows that there relations are true for all $n \ge 0$. In other words we have

$$u = u_0 e^{(0)} + u_1 e^{(1)} + \cdots + u_{d-1} e^{(d-1)}.$$

Hence $\{e^{(0)}, \dots, e^{(d-1)}\}$ is a generating set for E.

Since the matrix $\left(e_i^{(n)}\right)_{0 \le i, n \le d-1}$ is the identity matrix, the d sequences $e^{(0)}, \dots, e^{(d-1)}$ are linearly independent.

(2) The answers have also been given during the course. Here they are.
(a) When $t_1 = \cdots = t_\ell = 1$ we have $\ell = d$ and $\gamma_1, \ldots, \gamma_d$ are d distinct roots of P. In this case the d sequences $(\gamma_j^n)_{n\geq 0}$ satisfy (1) :

$$\gamma_j^{n+d} = c_1\gamma_j^{n+d-1} + \cdots + c_d\gamma_j^n \quad \text{for } n \geq 0 \text{ and } 1 \leq j \leq d$$

and are linearly independent since the determinant

$$\begin{pmatrix} 1 & \gamma_1 & \gamma_1^2 & \cdots & \gamma_1^{d-1} \\ 1 & \gamma_2 & \gamma_2^2 & \cdots & \gamma_2^{d-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \gamma_d & \gamma_d^2 & \cdots & \gamma_d^{d-1} \end{pmatrix}$$

is not zero (Vandermonde).
(b) When $\ell = 1$ we have $t_1 = d$, $P(T) = (T - \gamma_1)^d$. We want to prove that the d sequences

$$(n^i\gamma_1^n)_{n\geq 0} \quad 0 \leq i \leq d-1 \tag{3}$$

give a basis for E. We first check that they belong to E. Since

$$(T-\gamma_1)^d = T^d - dT^{d-1}\gamma_1 + \binom{d}{2}T^{d-2}\gamma_1^2 - \cdots + (-1)^k\binom{d}{k}T^{d-k}\gamma_1^k + \cdots + (-1)^d\gamma_1^d,$$

the linear recurrence relations of which this polynomial is the characteristic polynomial is

$$u_{n+d} = du_{n+d-1}\gamma_1 - \binom{d}{2}u_{n+d-2}\gamma_1^2 + \cdots + (-1)^{k-1}\binom{d}{k}u_{n+d-k}\gamma_1^k + \cdots + (-1)^{d+1}u_n\gamma_1^n.$$

Hence the relation that we need to prove is

$$(n+d)^i = d(n+d-1)^i - \binom{d}{2}(n+d-2)^i + \cdots + (-1)^{k-1}\binom{d}{k}(n+d-k)^i + \cdots + (-1)^{d+1}n^i \tag{4}$$

for $0 \leq i \leq d-1$ and $n \geq 0$. Here is a proof.

For $n \geq 0$ the polynomial $T^n(T-1)^d$ has a zero at $T = 1$ of multiplicity d, hence for $1 \leq i \leq d-1$ and $n \geq 0$ the polynomial

$$\left(T\frac{\mathrm{d}}{\mathrm{d}T}\right)^i \left(T^n(T-1)^d\right)$$

vanishes at $T = 1$. From

$$T^n(T-1)^d = T^{n+d} - dT^{n+d-1} + \binom{d}{2} T^{n+d-2} - \cdots + (-1)^k \binom{d}{k} T^{n+d-k} + \cdots + (-1)^d T^n$$

and

$$\left(T\frac{\mathrm{d}}{\mathrm{d}T}\right)^i (T^h) = h^i T^h$$

we deduce

$$\left(T\frac{\mathrm{d}}{\mathrm{d}T}\right)^i \left(T^n(T-1)^d\right) = (n+d)^i T^{n+d} - d(n+d-1)^i T^{n+d-1}$$
$$+ \binom{d}{2}(n+d-2)^i T^{n+d-2} - \cdots + (-1)^k \binom{d}{k}(n+d-k)^i T^{n+d-k} + \cdots + (-1)^d n^i T^n.$$

Substituting $T = 1$ gives (4).

That these d sequences (3) give a basis for E amounts to saying that the matrix

$$\begin{pmatrix} 1 & \gamma_1 & \gamma_1^2 & \gamma_1^3 & \cdots & \gamma_1^n & \cdots & \gamma_1^{d-1} \\ 0 & \gamma_1 & 2\gamma_1^2 & 3\gamma_1^3 & \cdots & n\gamma_1^n & \cdots & (d-1)\gamma_1^{d-1} \\ 0 & \gamma_1 & 4\gamma_1^2 & 9\gamma_1^3 & \cdots & n^2\gamma_1^n & \cdots & (d-1)^2\gamma_1^{d-1} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\ 0 & \gamma_1 & 2^{d-1}\gamma_1^2 & 3^{d-1}\gamma_1^3 & \cdots & n^{d-1}\gamma_1^n & \cdots & (d-1)^{d-1}\gamma_1^{d-1} \end{pmatrix}$$

is regular. Since γ_1 is not zero, this is equivalent to saying that the determinant of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & \cdots & 1 \\ 0 & 1 & 2 & 3 & \cdots & n & \cdots & (d-1) \\ 0 & 1 & 4 & 9 & \cdots & n^2 & \cdots & (d-1)^2 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\ 0 & 1 & 2^{d-1} & 3^{d-1} & \cdots & n^{d-1} & \cdots & (d-1)^{d-1} \end{pmatrix}$$

is nonzero. Since the polynomial

$$\frac{X(X-1)\cdots(X-n+1)}{n!}$$

has degree n, by linear combinations of the rows we see that this determinant is the product by $1!2!\cdots(d-1)!$ with the determinant of the upper triangular matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & \cdots & 1 \\ 0 & 1 & \binom{2}{1} & \binom{3}{1} & \cdots & \binom{n}{1} & \cdots & \binom{d-1}{1} \\ 0 & 0 & 1 & \binom{3}{2} & \cdots & \binom{n}{2} & \cdots & \binom{d-1}{2} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \binom{n}{k} & \cdots & \binom{d-1}{k} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix}$$

hence it is not zero.
(c) When $d = 2$, the polynomial P has degree 2, and either
• P has two distinct roots γ_1 and γ_2, which means $\ell = d = 2$, $t_1 = t_2 = 1$. In this case we are in the situation of (a), the two sequences $(\gamma_1^n)_{n\geq 0}$ and $(\gamma_2^n)_{n\geq 0}$ give a basis of E,
or
• P has a double root γ, which means $\ell = 1$, $t_1 = 2$, and in this case we are in the situation of (b). The sequence $(n\gamma_1^n)_{n\geq 0}$ satisfies (1) and the two sequences $(\gamma_1^n)_{n\geq 0}$ and $(n\gamma_1^n)_{n\geq 0}$ give a basis of E.

In fact Quizz 1 gives a direct answer to (c).

(e) We use the method suggested by (i) to prove that the d sequences (2) are in E.

Since γ_j is a root of multiplicity t_j of $T^nP(T)$, γ_j is a root of the polynomials

$$\left(\frac{\mathrm{d}}{\mathrm{d}T}\right)^i (T^nP(T))$$

for $0 \leq i \leq t_j - 1$. For $0 \leq i \leq t_j - 1$, the polynomial $\left(T\frac{\mathrm{d}}{\mathrm{d}T}\right)^i (T^nP(T))$ is a linear combinations of the polynomials $\left(\frac{\mathrm{d}}{\mathrm{d}T}\right)^k (T^nP(T))$ with $0 \leq k \leq i$. Hence for $1 \leq j \leq \ell$ and $0 \leq i \leq t_j - 1$ the number γ_j is a root of the polynomials

$$\left(T\frac{\mathrm{d}}{\mathrm{d}T}\right)^i (T^nP(T)).$$

From

$$T^nP(T) = T^{n+d} - c_1T^{n+d-1} - \cdots - c_dT^n$$

we deduce

$$\left(T\frac{\mathrm{d}}{\mathrm{d}T}\right)^i (T^nP(T)) = (n+d)^iT^{n+d} - c_1(n+d-1)^iT^{n+d-1} - \cdots - c_dn^iT^n.$$

Since γ_j is a root of this polynomial, we get, for $n \geq 0$, $1 \leq j \leq \ell$ and $0 \leq i \leq t_j$,

$$(n+d)^i\gamma_j^{n+d} = c_1(n+d-1)^i\gamma_j^{n+d-1} - \cdots - c_dn^i\gamma_j^n.$$

This means that the sequences $(n^i\gamma_j^n)_{n\geq 0}$ for $1 \leq j \leq \ell,\ 0 \leq i \leq t_j$ satisfy (1).

(e) We use the method suggested by (ii) to prove that the d sequences (2) generate E.

Let $(u_n)_{n\geq 0}$ be an element in E. Write the linear recurrence relation in a matrix form

$$U_{n+1} = CU_n$$

with

$$U_n = \begin{pmatrix} u_n \\ u_{n+1} \\ \vdots \\ u_{n+d-1} \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ c_d & c_{d-1} & c_{d-2} & \cdots & c_1 \end{pmatrix}.$$

Write the matrix C in its Jordan normal form :

$$P^{-1}CP = J \quad \text{where} \quad J = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_\ell \end{pmatrix}$$

and

$$J_j = \begin{pmatrix} \gamma_j & 1 & 0 & \cdots & 0 & 0 \\ 0 & \gamma_j & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & \gamma_j & 1 \\ 0 & 0 & 0 & \cdots & 0 & \gamma_j \end{pmatrix} \qquad (1 \leq j \leq \ell)$$

For $n \geq 0$ we have

$$J^n = \begin{pmatrix} J_1^n & 0 & \cdots & 0 \\ 0 & J_2^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_\ell^n \end{pmatrix}$$

and, for $1 \leq j \leq \ell$,

$$J_j^n = \begin{pmatrix} \gamma_j^n & \binom{n}{1}\gamma_j^{n-1} & \binom{n}{2}\gamma_j^{n-2} & \cdots & \binom{n}{t_j-2}\gamma_j^{n-t_j+2} & \binom{n}{t_j-1}\gamma_j^{n-t_j+1} \\ 0 & \gamma_j^n & \binom{n}{1}\gamma_j^{n-1} & \cdots & \binom{n}{t_j-3}\gamma_j^{n-t_j+3} & \binom{n}{t_j-2}\gamma_j^{n-t_j+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{n}{1}\gamma_j^{n-1} & \binom{n}{2}\gamma_j^{n-2} \\ 0 & 0 & 0 & \cdots & \gamma_j^n & \binom{n}{1}\gamma_j^{n-1} \\ 0 & 0 & 0 & \cdots & 0 & \gamma_j^n \end{pmatrix}.$$

From the matrix relation $U_n = C^n U_0$ we deduce

$$U_n = P^{-1} J^n P U_0.$$

Hence there exist complex numbers c_{ij} such that, for all $n \geq 0$,

$$u_n = \sum_{j=1}^{\ell} \sum_{i=0}^{t_j-1} c_{ij} n^i \gamma_j^n.$$

This proves that any element of E is a linear combination of the sequences (2), hence that these d sequences are a system of generators of E. Since E has dimension d, it shows that these sequences (2) are a basis of E.

(e) We use the method suggested by (iii) to prove again that the d sequences (2) generate E.

The left hand side of the product $\left(1 - \sum_{i=1}^d c_i T^i\right) U(T)$ is a telescoping series

$$\left(1 - \sum_{i=1}^{d} c_i T^i\right) U(T) = \sum_{j=0}^{d-1} \left(u_j - \sum_{i=1}^{j} c_i u_{j-i}\right) T^j.$$

Hence $U(T)$ is a rational fraction, with denominator

$$1 - \sum_{i=1}^{d} c_i T^i = T^d P(1/T) = \prod_{j=1}^{\ell} (1 - \gamma_j T)^{t_j},$$

while the numerator is of degree $< d$. Using a partial fraction decomposition, we write this rational fraction as

$$U(T) = \sum_{j=1}^{\ell} \sum_{i=0}^{t_j - 1} \frac{q_{ij}}{(1-\gamma_j T)^{i+1}}.$$

For $1 \leq j \leq \ell$, we develop $(1-\gamma_j T)^{-i-1}$ as a power series expansion :

$$\frac{1}{(1-\gamma_j T)^{i+1}} = \frac{1}{i!\gamma_j^i}\left(\frac{\mathrm{d}}{\mathrm{d}T}\right)^i \frac{1}{1-\gamma_j T} = \sum_{n\geq 0} \frac{(n+1)(n+2)\cdots(n+i)}{i!} \gamma_j^n T^n.$$

It follows that u_n is a linear combination of the elements γ_j^n with coefficients being polynomials of degree $< t_j$ evaluated at n.

(e) We use the method suggested by (iii) to prove again that the d sequences (2) are linearly independent.

This amounts to showing that the determinant of the matrix A is different from 0. Let us define s_j to be

$$s_j = t_1 + \cdots + t_{j-1} \quad \text{for} \quad 1 \leq j \leq \ell \ \text{ with } \ s_1 = 0.$$

For $1 \leq j \leq \ell$, $0 \leq i \leq t_j - 1$, $0 \leq k \leq d-1$, the $(s_j + i, k)$ entry of the matrix A is

$$\frac{1}{i!}\left(\frac{\mathrm{d}}{\mathrm{d}T}\right)^i T^k \Bigg|_{T=\gamma_j} = \binom{k}{i} \gamma_j^{k-i}.$$

Let $C_0, \ldots, C_{d-1}$ denote the d columns of A and let $b_0, \ldots, b_{d-1}$ be complex numbers such that

$$b_0 C_0 + \cdots + b_{d-1} C_{d-1} = 0.$$

The left side of this equality is an element of $\mathbb{C}^d$, the d components of which are all 0, and these d relations mean that the polynomial

$$b_0 + b_1 T + \cdots + b_{d-1} T^{d-1}$$

vanishes at the point γ_j with multiplicity at least t_j for $1 \leq j \leq \ell$. Since $t_1 + \cdots + t_\ell = d$, we deduce that $b_0 = \cdots = b_{d-1} = 0$.

Références

[1] Everest, Graham ; van der Poorten, Alf ; Shparlinski, Igor ; Ward, Thomas. *Recurrence sequences,* Mathematical Surveys and Monographs (AMS, 2003), volume **104** `(1290 references)`. Zbl MR

[2] Andrica, Dorin ; Bagdasar, Ovidiu. Recurrent sequences. Key results, applications, and problems. Problem Books in Mathematics, Springer 2020. Zbl MR

[3] Levesque, Claude and Waldschmidt, Michel. Linear recurrence sequences and twisted binary forms. Proceedings of the International Conference on Pure and Applied Mathematics ICPAM–GOROKA 2014, South Pacific Journal of Pure and Applied Mathematics, vol. 2, No 3 (2015), 65-83. arXiv:1802.05154 [math.NT].

Further references are available from my website

`http://www.imj-prg.fr/~michel.waldschmidt/`

See in particular

`http://www.imj-prg.fr/~michel.waldschmidt/AgendaArchives.html`

January 25 - 28, 2021. Limbe (Cameroun) - online

A course on *linear recurrent sequences* at the African Institute for Mathematical Sciences (AIMS) Cameroun.

1. *The square root of 2, the Golden ratio and the Fibonacci sequence.* Link to the video.
2. *Linear recurrence sequences.* Part I, examples. Link to the video.
3. *Linear recurrence sequences.* Part II, the theory.. Link to the video.
4. *On the Brahmagupta-Fermat-Pell Equation* Link to the video.

Quizz 1. Quizz 2. Tutorial 1, January 26, 2021. *Tutorial 2*, January 28, 2021. *Assignment. Tutorial 3.*