Bangalore India/AMS Conference

December 19, 2003

Multiple Zeta Values

Michel Waldschmidt

http://www.math.jussieu.fr/∼miw/

Euler Numbers

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} \qquad \text{for } s \geq 2.$$

These are special values of the Riemann Zeta Function: $s \in \mathbf{C}$.

Euler: $\pi^{-2k}\zeta(2k) \in \mathbf{Q}$ *for* $k \geq 1$. (Bernoulli numbers).

Diophantine Question: *What are the algebraic relations among the numbers*

$$\zeta(2), \quad \zeta(3), \quad \zeta(5), \quad \zeta(7)\ldots ?$$

Conjecture. *There is no algebraic relation at all: these numbers*

$$\zeta(2), \quad \zeta(3), \quad \zeta(5), \quad \zeta(7)\ldots$$

are algebraically independent.

Conjecture. *There is no algebraic relation at all: these numbers*

$$\zeta(2), \quad \zeta(3), \quad \zeta(5), \quad \zeta(7)\ldots$$

are algebraically independent.

Known:

- Hermite-Lindemann: π *is transcendental, hence* $\zeta(2k)$ *also for* $k \geq 1$.

Conjecture. *There is no algebraic relation at all: these numbers*

$$\zeta(2), \quad \zeta(3), \quad \zeta(5), \quad \zeta(7)\ldots$$

are algebraically independent.

Known:

- Hermite-Lindemann: π *is transcendental, hence* $\zeta(2k)$ *also for* $k \geq 1$.

- Apéry (1978): $\zeta(3)$ *is irrational.*

Conjecture. *There is no algebraic relation at all: these numbers*

$$\zeta(2), \quad \zeta(3), \quad \zeta(5), \quad \zeta(7) \ldots$$

are algebraically independent.

Known:

• Hermite-Lindemann: π *is transcendental, hence* $\zeta(2k)$ *also for* $k \geq 1$.

• Apéry (1978): $\zeta(3)$ *is irrational.*

• Rivoal (2000) + Ball, Zudilin. . . *Infinitely many* $\zeta(2k+1)$ *are irrational + lower bound for the dimension of the* $\mathbf{Q}$*-space they span.*

Let $\epsilon > 0$. For a be a sufficiently large odd integer the dimension of the $\mathbf{Q}$-space spanned by $1,\ \zeta(3),\ \zeta(5),\ \cdots,\zeta(a)$ is at least

$$\frac{1-\epsilon}{1+\log 2}\log a.$$

W. Zudilin.

• *One at least of the four numbers*

$$\zeta(5), \quad \zeta(7), \quad \zeta(9), \quad \zeta(11)$$

is irrational.

• *There is an odd integer* j *in the range* $[5, 69]$ *such that the three numbers* 1, $\zeta(3)$, $\zeta(j)$ *are linearly independent over* $\mathbf{Q}$.

Linearization of the problem *(Euler).* The product of two zeta values is not quite a zeta value, but something similar.

Linearization of the problem *(Euler).* The product of two zeta values is not quite a zeta value, but something similar. From

$$\sum_{n_1\ge 1} n_1^{-s_1} \sum_{n_2\ge 1} n_2^{-s_2} = \sum_{n_1>n_2\ge 1} n_1^{-s_1} n_2^{-s_2} + \sum_{n_2>n_1\ge 1} n_2^{-s_2} n_1^{-s_1} + \sum_{n\ge 1} n^{-s_1-s_2}$$

one deduces, for $s_1 \ge 2$ and $s_2 \ge 2$,

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2)$$

with

$$\zeta(s_1, s_2) = \sum_{n_1>n_2\ge 1} n_1^{-s_1} n_2^{-s_2}.$$

For k, $s_1, \ldots, s_k$ positive integers with $s_1 \geq 2$, define $\underline{s} = (s_1, \ldots, s_k)$ and

$$\zeta(\underline{s}) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}.$$

For k, $s_1, \ldots, s_k$ positive integers with $s_1 \geq 2$, define $\underline{s} = (s_1, \ldots, s_k)$ and

$$\zeta(\underline{s}) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}.$$

For $k = 1$ one recovers Euler's numbers $\zeta(s)$.

For k, $s_1, \dots, s_k$ positive integers with $s_1 \geq 2$, define $\underline{s} = (s_1, \dots, s_k)$ and

$$\zeta(\underline{s}) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}.$$

For $k = 1$ one recovers Euler's numbers $\zeta(s)$.

The product of two Multiple Zeta Values is a linear combination, with integer coefficients, of Multiple Zeta Values.

These numbers satisfy a quantity of linear relations with rational coefficients.

A complete description of these relations would in principle settle the problem of the algebraic independence of

$$\pi, \quad \zeta(3), \quad \zeta(5), \ldots, \quad \zeta(2k+1).$$

Goal: *Describe all linear relations among Multiple Zeta Values.*

Main tool: *Multiple zeta values are special values of multiple polylogarithms*.

Examples of linear relations.

Euler:

$$\zeta(2,1) = \zeta(3).$$

Sum Theorem

Ohno: *Fix* $k \geq 1$, $p \geq 2$ *and denote by* $\mathcal{S}_{k,p}$ *the set of* $(s_1, \ldots, s_k)$ *in* $\mathbf{Z}^k$ *with* $s_1 \geq 2$, $s_j \geq 1$ *for* $j = 2, \ldots, k$ *and* $s_1 + \cdots + s_k = p$. *Then*

$$\sum_{\underline{s} \in \mathcal{S}_{k,p}} \zeta(\underline{s}) = \zeta(p).$$

Examples:

$$k=2,\qquad p=3,\qquad \zeta(2,1)=\zeta(3).$$
$$k=2,\qquad p=4,\qquad \zeta(3,1)+\zeta(2,2)=\zeta(4)$$
$$k=p-1,\ p\ge 3,\qquad \zeta(2,\{1\}_{p-2})=\zeta(p)$$

where $\{a\}_n=(a,a,\dots,a)$ with n occurences of a.

Example: $k=n+1$, $p=3n+4$

$$\zeta(4,\{3\}_n)=\zeta(\{3\}_{n+1},1)+\zeta(2,\{3\}_n,2).$$

Zagier-Broadhurst formula *For any* $n \geq 1$,

$$\zeta\big(\{3,1\}_n\big) = 4^{-n}\zeta\big(\{4\}_n\big).$$

Zagier-Broadhurst formula *For any* $n \geq 1$,

$$\zeta\big(\{3,1\}_n\big) = 4^{-n}\zeta\big(\{4\}_n\big).$$

Hoffman's derivation Theorem Fix $(s_1, \ldots, s_k)$ in $\mathbf{Z}^k$ with $s_1 \geq 2$, and $s_j \geq 1$ for $j = 2, \ldots, k$. Then

$$\sum_{h=1}^{k} \zeta(s_1, \ldots, s_{h-1}, s_h + 1, s_{h+1}, \ldots, s_p) =$$

$$\sum_{\substack{1 \leq h \leq k \\ s_h \geq 2}} \sum_{j=0}^{s_h - 2} \zeta(s_1, \ldots, s_{h-1}, s_h - j, j + 1, , s_{h+1}, \ldots, s_p).$$

Fact: *the Multiple Zeta Values satisfy two sets of quadratic relations: one arises from product of series, the other from an expression of them in terms of integrals.*

We describe these integrals for multiple polylogarithms, then we specialize to MZV.

Usual logarithm

For $z \in \mathbf{C}$, $|z| \leq 1$ and $z \neq 1$,

$$\mathrm{Li}_1(z) = \sum_{n \geq 1} \frac{z^n}{n} = -\log(1-z) = \int_0^z \frac{dt}{1-t}$$

Classical polylogarithms

(Definition as series)

For $s \in \mathbf{Z}$ with $s \geq 1$ and for $z \in \mathbf{C}$ with $|z| \leq 1$ satisfying $(s, z) \neq (1, 1)$, define

$$\mathrm{Li}_s(z) = \sum_{n \geq 1} \frac{z^n}{n^s}.$$

For $s \geq 2$, the value of $\mathrm{Li}_s(z)$ at $z = 1$ produces Euler zeta values (Riemann zeta function)

$$\zeta(s) = \mathrm{Li}_s(1).$$

Definition as solutions of differential equations

These functions Li_s are also defined inductively by the differential equations

$$\frac{d}{dz}\mathrm{Li}_1(z) = \frac{1}{1-z}$$

and

$$\frac{d}{dz}\mathrm{Li}_s(z) = \frac{1}{z}\mathrm{Li}_{s-1}(z) \qquad \text{for } s \geq 2,$$

with the initial conditions $\mathrm{Li}_s(0) = 0$.

Integral representation

$$\mathrm{Li}_1(z) = \sum_{n\geq 1} \frac{z^n}{n} = -\log(1-z) = \int_0^z \frac{dt}{1-t}$$

Integral representation

$$\mathrm{Li}_1(z) = \sum_{n\geq 1} \frac{z^n}{n} = -\log(1-z) = \int_0^z \frac{dt}{1-t}$$

and by induction, for $s \geq 2$,

$$\mathrm{Li}_s(z) = \int_0^z \mathrm{Li}_{s-1}(t)\frac{dt}{t}$$

$$= \int_0^z \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{t_2} \cdots \int_0^{t_{s-2}} \frac{dt_{s-1}}{t_{s-1}} \int_0^{t_{s-1}} \frac{dt_s}{1-t_s}.$$

For $s \geq 1$ and $z > 0$,

$$\mathrm{Li}_s(z) = \int_{z>t_1>\cdots>t_s>0} \frac{dt_1}{t_1} \cdot \frac{dt_2}{t_2} \cdots \frac{dt_{s-1}}{t_{s-1}} \cdot \frac{dt_s}{1-t_s}.$$

For $s \geq 1$ and $z > 0$,

$$\mathrm{Li}_s(z) = \int_{z>t_1>\cdots>t_s>0} \frac{dt_1}{t_1} \cdot \frac{dt_2}{t_2} \cdots \frac{dt_{s-1}}{t_{s-1}} \cdot \frac{dt_s}{1-t_s}.$$

Examples.

$$\zeta(2) = \int_{1>t_1>t_2>0} \frac{dt_1}{t_1} \cdot \frac{dt_2}{1-t_2}.$$

$$\zeta(3) = \int_{1>t_1>t_2>t_3>0} \frac{dt_1}{t_1} \cdot \frac{dt_2}{t_2} \cdot \frac{dt_3}{1-t_3}.$$

Chen Iterated Integrals

For a holomorphic 1-form φ,

$$\int_0^z \varphi$$

is the primitive of φ which vanishes at $z = 0$.
For 1-forms $\varphi_1, \ldots, \varphi_k$, define inductively

$$\int_0^z \varphi_1 \cdots \varphi_k := \int_0^z \varphi_1(t) \int_0^t \varphi_2 \cdots \varphi_k.$$

Chen Iterated Integrals

$$\int_0^z \varphi_1 \cdots \varphi_k := \int_0^z \varphi_1(t) \int_0^t \varphi_2 \cdots \varphi_k.$$

Define

$$\omega_0 = \frac{dt}{t}, \qquad \omega_1 = \frac{dt}{1-t}.$$

Then

$$\mathrm{Li}_s(z) = \int_0^z \omega_0^{s-1}\omega_1 \quad \text{for} \quad s \geq 1 \quad \text{and} \quad |z| < 1$$

while

$$\zeta(s) = \int_0^1 \omega_0^{s-1}\omega_1 \quad \text{for} \quad s \geq 2.$$

Product of polylogarithms
Example

For $0 < z < 1$:

$$\mathrm{Li}_1(z)\mathrm{Li}_2(z) = \int_{z>t>0} \frac{dt}{1-t} \int_{z>u>v>0} \frac{du}{u} \cdot \frac{dv}{1-v}$$

Product of polylogarithms
Example

For $0 < z < 1$:

$$\mathrm{Li}_1(z)\mathrm{Li}_2(z) = \int_{z>t>0} \frac{dt}{1-t} \int_{z>u>v>0} \frac{du}{u} \cdot \frac{dv}{1-v}$$

$$\begin{aligned}\{(t,u,v)\,;\;\; z>t>0,\; z>u>v>0\} \simeq& \\ &\{(t,u,v)\,;\; z>t>u>v>0\} \\ &\times\{(t,u,v)\,;\; z>u>t>v>0\} \\ &\times\{(t,u,v)\,;\; z>u>v>t>0\}\end{aligned}$$

Product of polylogarithms
Example

For $0 < z < 1$:

$$
\begin{aligned}
\mathrm{Li}_1(z)\mathrm{Li}_2(z) = & \int_{z>t>0} \frac{dt}{1-t} \int_{z>u>v>0} \frac{du}{u} \cdot \frac{dv}{1-v} \\
= & \int_{z>t>u>v>0} \frac{dt}{1-t} \cdot \frac{du}{u} \cdot \frac{dv}{1-v} \\
+ & \int_{z>u>t>v>0} \frac{du}{u} \cdot \frac{dt}{1-t} \cdot \frac{dv}{1-v} \\
+ & \int_{z>u>v>t>0} \frac{du}{u} \cdot \frac{dv}{1-v} \cdot \frac{dt}{1-t}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Li}_1(z)\mathrm{Li}_2(z) &= \int_0^z \omega_1 \int_0^z \omega_0\omega_1 \\
&= \int_0^z \omega_1\omega_0\omega_1 + 2\int_0^z \omega_0\omega_1^2 \\
&= \int_0^z (\omega_1\omega_0\omega_1 + 2\omega_0\omega_1^2).
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Li}_1(z)\mathrm{Li}_2(z) &= \int_0^z \omega_1 \int_0^z \omega_0\omega_1 \\
&= \int_0^z \omega_1\omega_0\omega_1 + 2\int_0^z \omega_0\omega_1^2 \\
&= \int_0^z (\omega_1\omega_0\omega_1 + 2\omega_0\omega_1^2).
\end{aligned}
$$

$$
\omega_1 ш(\omega_0\omega_1) = \omega_1\omega_0\omega_1 + 2\omega_0\omega_1^2.
$$

$$\begin{aligned}\mathrm{Li}_1(z)\mathrm{Li}_2(z) &= \int_0^z \omega_1 \int_0^z \omega_0\omega_1 \\ &= \int_0^z \omega_1\omega_0\omega_1 + 2\int_0^z \omega_0\omega_1^2 \\ &= \int_0^z (\omega_1\omega_0\omega_1 + 2\omega_0\omega_1^2).\end{aligned}$$

$$\omega_1 ш (\omega_0\omega_1) = \omega_1\omega_0\omega_1 + 2\omega_0\omega_1^2.$$

$$\varphi_1 ш (\varphi_2\varphi_3) = \varphi_1\varphi_2\varphi_3 + \varphi_2\varphi_1\varphi_3 + \varphi_2\varphi_3\varphi_1.$$

Shuffle product of differential forms

$$\varphi_1 \cdots \varphi_n \text{ ш } \psi_1 \cdots \psi_k \ = \qquad \varphi_1(\varphi_2 \cdots \varphi_n \text{ш} \psi_1 \cdots \psi_k)$$

$$+ \ \psi_1(\varphi_1 \cdots \varphi_n \text{ш} \psi_2 \cdots \psi_k).$$

$$\varphi_1 \text{ш} \psi_1 = \varphi_1\psi_1 + \psi_1\varphi_1.$$

For $k = 0$ (e the empty product)

$$\varphi_1 \cdots \varphi_n \text{ш} e = \varphi_1 \cdots \varphi_n.$$

Product of iterated integrals:

Let $\varphi_1, \ldots, \varphi_n$, $\psi_1, \ldots, \psi_k$ *be differential forms with* $n \geq 0$ *and* $k \geq 0$. *Then*

$$\int_0^z \varphi_1 \cdots \varphi_n \int_0^z \psi_1 \cdots \psi_k = \int_0^z \varphi_1 \cdots \varphi_n ш \psi_1 \cdots \psi_k.$$

Proof. Assume $z > 0$. Decompose the Cartesian product

$$\{\underline{t} \in \mathbf{R}^n \; ; \; z \geq t_1 \geq \cdots \geq t_n \geq 0\} \times \{\underline{u} \in \mathbf{R}^k \; ; \; z \geq u_1 \geq \cdots \geq u_k \geq 0\}$$

into a disjoint union of simplices (up to sets of zero measure)

$$\{\underline{v} \in \mathbf{R}^{n+k} \; ; \; z \geq v_1 \geq \cdots \geq v_{n+k} \geq 0\}.$$

The product of two polylogarithms:

$$\mathrm{Li}_s(z)\mathrm{Li}_{s'}(z) = \int_0^z \omega_s \int_0^z \omega_{s'} = \int_0^z \omega_s ш \omega_{s'}$$

where $\omega_s = \omega_0^{s-1}\omega_1$ involves more general polylogarithms like

$$\int_0^z \omega_0^{s_1-1}\omega_1\omega_0^{s_2-1}\omega_1.$$

Hence we need to introduce

$$\omega_{\underline{s}} = \omega_{s_1}\cdots\omega_{s_k} = \omega_0^{s_1-1}\omega_1\cdots\omega_0^{s_k-1}\omega_1$$

for $\underline{s} = (s_1, \ldots, s_k)$.

Multiple Polylogarithms in One Variable

Definition as series

For k, $s_1, \ldots, s_k$ positive integers and $z \in \mathbf{C}$, $|z| < 1$, define $\underline{s} = (s_1, \ldots, s_k)$ and

$$\mathrm{Li}_{\underline{s}}(z) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{z^{n_1}}{n_1^{s_1} \cdots n_k^{s_k}}.$$

For $k = 1$ one recovers the usual $\mathrm{Li}_s(z)$ while for $z = 1$, $s_1 \geq 2$ one recovers

$$\zeta(\underline{s}) = \mathrm{Li}_{\underline{s}}(1).$$

Definition as solutions of differential equations

$$\frac{d}{dz}\mathrm{Li}_{(s_1,\ldots,s_k)}(z) = \frac{1}{z}\mathrm{Li}_{(s_1-1,s_2,\ldots,s_k)}(z) \qquad (s_1 \geq 2)$$

Definition as solutions of differential equations

$$\frac{d}{dz}\mathrm{Li}_{(s_1,\dots,s_k)}(z) = \frac{1}{z}\mathrm{Li}_{(s_1-1,s_2,\dots,s_k)}(z) \qquad (s_1 \geq 2)$$

$$\frac{d}{dz}\mathrm{Li}_{(1,s_2,\dots,s_k)}(z) = \frac{1}{1-z}\mathrm{Li}_{(s_2,\dots,s_k)}(z) \qquad (s_1 = 1).$$

Definition as solutions of differential equations

$$\frac{d}{dz}\mathrm{Li}_{(s_1,\dots,s_k)}(z) = \frac{1}{z}\mathrm{Li}_{(s_1-1,s_2,\dots,s_k)}(z) \qquad (s_1 \geq 2)$$

$$\frac{d}{dz}\mathrm{Li}_{(1,s_2,\dots,s_k)}(z) = \frac{1}{1-z}\mathrm{Li}_{(s_2,\dots,s_k)}(z) \qquad (s_1 = 1).$$

Initial condition: $\mathrm{Li}_{\underline{s}}(0) = 0$.

Definition as solutions of differential equations

$$\frac{d}{dz}\mathrm{Li}_{(s_1,\dots,s_k)}(z) = \frac{1}{z}\mathrm{Li}_{(s_1-1,s_2,\dots,s_k)}(z) \qquad (s_1 \geq 2)$$

$$\frac{d}{dz}\mathrm{Li}_{(1,s_2,\dots,s_k)}(z) = \frac{1}{1-z}\mathrm{Li}_{(s_2,\dots,s_k)}(z) \qquad (s_1 = 1).$$

Initial condition: $\mathrm{Li}_{\underline{s}}(0) = 0$.

Recall

$$\omega_{\underline{s}} = \omega_0^{s_1-1}\omega_1 \cdots \omega_0^{s_k-1}\omega_1.$$

Hence

$$\mathrm{Li}_{\underline{s}}(z) = \int_0^z \omega_{\underline{s}}.$$

Example: $\underline{s} = (2,1)$, $\omega_{\underline{s}} = \omega_0\omega_1^2$

$$\zeta(2,1) = \int_0^1 \omega_0\omega_1^2 = \int_{1>t_1>t_2>t_3>0} \frac{dt_1}{t_1} \cdot \frac{dt_2}{1-t_2} \cdot \frac{dt_3}{1-t_3}.$$

Example: $\underline{s} = (2,1)$, $\omega_{\underline{s}} = \omega_0\omega_1^2$

$$\zeta(2,1) = \int_0^1 \omega_0\omega_1^2 = \int_{1>t_1>t_2>t_3>0} \frac{dt_1}{t_1}\cdot\frac{dt_2}{1-t_2}\cdot\frac{dt_3}{1-t_3}.$$

Recall:

$$\zeta(3) = \int_0^1 \omega_0^2\omega_1 = \int_{1>t_1>t_2>t_3>0} \frac{dt_1}{t_1}\cdot\frac{dt_2}{t_2}\cdot\frac{dt_3}{1-t_3}.$$

Example: $\underline{s} = (2,1)$, $\omega_{\underline{s}} = \omega_0\omega_1^2$

$$\zeta(2,1) = \int_0^1 \omega_0\omega_1^2 = \int_{1>t_1>t_2>t_3>0} \frac{dt_1}{t_1} \cdot \frac{dt_2}{1-t_2} \cdot \frac{dt_3}{1-t_3}.$$

Recall:

$$\zeta(3) = \int_0^1 \omega_0^2\omega_1 = \int_{1>t_1>t_2>t_3>0} \frac{dt_1}{t_1} \cdot \frac{dt_2}{t_2} \cdot \frac{dt_3}{1-t_3}.$$

Remark. From $(t_1,t_2,t_3) \mapsto (1-t_3, 1-t_2, 1-t_1)$ one deduces

(Euler)

$$\zeta(2,1) = \zeta(3).$$

Proposition. *The product of two multiple polylogarithms is a linear combination of multiple polylogarithms:*

$$\mathrm{Li}_{\underline{s}}(z)\mathrm{Li}_{\underline{s}'}(z) = \int_0^z \omega_{\underline{s}} ш \omega_{\underline{s}'}.$$

Proposition. *The product of two multiple polylogarithms is a linear combination of multiple polylogarithms:*

$$\mathrm{Li}_{\underline{s}}(z)\mathrm{Li}_{\underline{s}'}(z) = \int_0^z \omega_{\underline{s}} ш \omega_{\underline{s}'}.$$

Next goal: Associate a multiple polylogarithm to a linear combination of $\omega_{\underline{s}}$, so that the product of two multiple polylogarithms is a multiple polylogarithm.

Tool: Free algebra on $\{\omega_0, \omega_1\}$.

The free monoid X^*

Let $X = \{x_0, x_1\}$ denote the *alphabet* with two letters x_0, x_1 and X^* the free monoid on X. The elements of X^* are *words*. A word can be written

$$x_{\epsilon_1} \cdots x_{\epsilon_k}$$

with $k \geq 0$ and where each ϵ_j is 0 or 1.

This law is called *concatenation.* It is not commutative:

$$x_0 x_1 \neq x_1 x_0.$$

Its unit is the *empty word* $e \in X^*$: the word for which $k = 0$.

The words which end with x_1 are the elements of X^*x_1.

The words which end with x_1 are the elements of $X^* x_1$.

Let $w \in X^* x_1$. Write $w = x_{\epsilon_1} \cdots x_{\epsilon_p}$ where each ϵ_i is 0 or 1 and $\epsilon_p = 1$.
If k is the number of x_1, we define positive integers $s_1, \ldots, s_k$ by

$$w = x_0^{s_1 - 1} x_1 \cdots x_0^{s_k - 1} x_1.$$

The words which end with x_1 are the elements of $X^* x_1$.

Let $w \in X^* x_1$. Write $w = x_{\epsilon_1} \cdots x_{\epsilon_p}$ where each ϵ_i is 0 or 1 and $\epsilon_p = 1$.
If k is the number of x_1, we define positive integers $s_1, \ldots, s_k$ by

$$w = x_0^{s_1 - 1} x_1 \cdots x_0^{s_k - 1} x_1.$$

For $s \geq 1$ define $y_s = x_0^{s-1} x_1$.

The words which end with x_1 are the elements of X^*x_1.

Let $w \in X^*x_1$. Write $w = x_{\epsilon_1} \cdots x_{\epsilon_p}$ where each ϵ_i is 0 or 1 and $\epsilon_p = 1$.
If k is the number of x_1, we define positive integers $s_1, \ldots, s_k$ by

$$w = x_0^{s_1-1} x_1 \cdots x_0^{s_k-1} x_1.$$

For $s \geq 1$ define $y_s = x_0^{s-1} x_1$.

Hence

$$w = y_{s_1} \cdots y_{s_k}.$$

This means that w *is a word on the alphabet*

$$Y = \{y_1, y_2, \ldots, y_s, \ldots\}.$$

This means that w is a word on the alphabet

$$Y = \{y_1, y_2, \ldots, y_s, \ldots\}.$$

For $\underline{s} = (s_1, \ldots, s_k)$ with $s_i \geq 1$, set

$$y_{\underline{s}} = y_{s_1} \cdots y_{s_k} = x_0^{s_1-1} x_1 \cdots x_0^{s_k-1} x_1.$$

The free monoid Y^* on Y is

$$Y^* = \{y_{\underline{s}}\ ;\ \underline{s} = (s_1, \ldots, s_k),\ k \geq 0,\ s_j \geq 1\ (1 \leq j \leq k)\}.$$

This is also the set $\{e\} \cup X^* x_1$ of words which do not end with x_0.

Proposition. *Set* $Y = \{y_1, y_2, y_3 \ldots\}$ *Then the free monoid* Y^* *on* Y *is a submonoid of* X^**.*

Any message can be coded with only two letters.

The Algebra $\mathfrak{H} = \mathbf{Q}\langle x_0, x_1 \rangle$

The free $\mathbf{Q}$-vector space with basis X^* is the free algebra on X, denoted by $\mathfrak{H} = \mathbf{Q}\langle X \rangle$. Its elements are non commutative polynomials in the two variables x_0, x_1.

Its unit is the $empty$ word e.

The Subalgebra $\mathfrak{H}^1 = \mathbf{Q}e + \mathfrak{H}x_1$.

The words which end with x_1 are the elements of X^*x_1.

The words which do not end with x_0 are the elements of $\{e\} \cup X^*x_1$. The $\mathbf{Q}$-vector subspace they span in $\mathfrak{H}$ is a subalgebra $\mathfrak{H}^1$ of $\mathfrak{H}$:

$$\mathfrak{H}^1 = \mathbf{Q}e + \mathfrak{H}x_1.$$

$\mathfrak{H}^1$ is a free algebra.

Recall $y_s = x_0^{s-1}x_1$ for $s \geq 1$ and

$$Y = \{y_1, y_2, y_3 \ldots\}.$$

Since a basis of the $\mathbf{Q}$-vector space $\mathfrak{H}^1$ is $\{e\} \cup X^*x_1 = Y^*$, we deduce:

Proposition. *The algebra $\mathfrak{H}^1$ is the free algebra on Y:*

$$\mathfrak{H}^1 = \mathbf{Q}\langle Y \rangle.$$

Polylogarithms associated to words in X^*x_1

For $\underline{s} = (s_1, \ldots, s_k)$ with $k \geq 1$ and $s_j \geq 1$, set

$$\widehat{\mathrm{Li}}_{y_{\underline{s}}}(z) = \mathrm{Li}_{\underline{s}}(z).$$

This defines $\widehat{\mathrm{Li}}_w(z)$ for $w \in X^*x_1$.

When $w = x_{\epsilon_1} \cdots x_{\epsilon_p}$ where each ϵ_i is 0 or 1 and $\epsilon_p = 1$,

$$\widehat{\mathrm{Li}}_w(z) = \int_0^z \omega_{\epsilon_1} \cdots \omega_{\epsilon_p}.$$

Polylogarithms associated to elements in $\mathfrak{H}^1$

By linearity, extend the definition of $\widehat{\mathrm{Li}}_w(z)$ to $w \in \mathfrak{H}^1$ with $\widehat{\mathrm{Li}}_e(z) = 1$ for the empty word e.

Let

$$P = \sum_{w \in X^*} \langle P, w \rangle w \in \mathfrak{H}^1.$$

The coefficients $\langle P, w \rangle$ are rational numbers. Further, the support $\{w \in X^* \; ; \; \langle P, w \rangle \neq 0\}$ is finite and contained in $\{e\} \cup X^* x_1 = Y^*$.

Then

$$\widehat{\mathrm{Li}}_P(z) = \sum_{w \in X^*} \langle P, w \rangle \widehat{\mathrm{Li}}_w(z).$$

Proposition. *For any w and w' in $\mathfrak{H}^1$, we have*

$$w \sqcup\!\sqcup w' \in \mathfrak{H}^1$$

and

$$\widehat{\mathrm{Li}}_w(z)\widehat{\mathrm{Li}}_{w'}(z) = \widehat{\mathrm{Li}}_{w \sqcup\!\sqcup w'}(z).$$

The shuffle $\sqcup\!\sqcup$ endows $\mathfrak{H}$ with a structure of commutative algebra $\mathfrak{H}_{\sqcup\!\sqcup}$ and $\mathfrak{H}^1$ is a subalgebra $\mathfrak{H}^1_{\sqcup\!\sqcup}$

Multizeta values associated to words

Recall: For $\underline{s} = (s_1, \ldots, s_k)$ with $s_1 \geq 2$,

$$\zeta(\underline{s}) = \mathrm{Li}_{\underline{s}}(1).$$

The condition $s_1 \geq 2$ means that $y_{\underline{s}}$ starts with x_0.

The set of words in X^* which start with x_0 and end with x_1 is $x_0 X^* x_1$.

The set of words in X^* which do not start with x_1 and do not end with x_0 is $\{e\} \cup x_0 X^* x_1$.

The Subalgebra $\mathfrak{H}^0 = \mathbf{Q}e + x_0\mathfrak{H}x_1$.

For $w \in x_0X^*x_1$, define

$$\widehat{\zeta}(w) = \widehat{\mathrm{Li}}_w(1).$$

Define also $\widehat{\zeta}(e) = 1$ and extend by $\mathbf{Q}$-linearity the definition of $\widehat{\zeta}$ to the $\mathbf{Q}$-vector space spanned by $\{e\} \cup x_0X^*x_1$ in $\mathfrak{H}^1$, which is the sub-algebra

$$\mathfrak{H}^0 = \mathbf{Q}e + x_0\mathfrak{H}x_1$$

of $\mathfrak{H}$.

Shuffle relations among MZV

For w and w' in $\mathfrak{H}^0$, the shuffle product $w ш w'$ belongs to $\mathfrak{H}^0$. Furthermore,

$$\widehat{\zeta}(w)\widehat{\zeta}(w') = \widehat{\zeta}(w ш w')$$

for any w and w' in $\mathfrak{H}^0$.

Proposition. *The map* $\widehat{\zeta} : \mathfrak{H}^0 \to \mathbf{R}$ *is a morphism of algebras of* $\mathfrak{H}^0_{ш}$ *into* $\mathbf{R}$.

The Harmonic Algebra

Recall:

For $s \geq 2$ and $s' \geq 2$:

$$\sum_{n\geq 1} n^{-s} \sum_{m\geq 1} m^{-s'} = \sum_{n>m\geq 1} n^{-s} m^{-s'} + \sum_{m>n\geq 1} m^{-s'} n^{-s} + \sum_{n\geq 1} n^{-s-s'},$$

$$\zeta(s)\zeta(s') = \zeta(s, s') + \zeta(s', s) + \zeta(s + s')$$

For instance

$$\zeta(2)^2 = 2\zeta(2, 2) + \zeta(4).$$

The map $\star : X^* \times X^* \to \mathfrak{H}$ is defined by induction, starting with

$$x_0^n \star w = w \star x_0^n = w x_0^n$$

for any $w \in X^*$ and any $n \geq 0$ (for $n = 0$ it means $e \star w = w \star e = w$ for all $w \in X^*$), and then

$$y_s u \star y_t v = y_s(u \star y_t v) + y_t(y_s u \star v) + y_{s+t}(u \star v)$$

for u and v in X^*, s and t positive integers.

Hoffman's harmonic algebra is denoted by $\mathfrak{H}_\star$.

Example.

$$y_2^{\star 3} = y_2 \star y_2 \star y_2 = 6y_2^3 + 3y_2y_4 + 3y_4y_2 + y_6.$$

Quadratic relations arising from the product of series

The map $\widehat{\zeta}:\mathfrak{H}^0\to\mathbf{R}$ is a morphism of algebras of $\mathfrak{H}^0_\star$ into $\mathbf{R}$:

$$\widehat{\zeta}(u\star v)=\widehat{\zeta}(u)\widehat{\zeta}(v).$$

for u and v in $\mathfrak{H}^0$.

Quadratic relations arising from the product of series

The map $\widehat{\zeta} : \mathfrak{H}^0 \rightarrow \mathbf{R}$ is a morphism of algebras of $\mathfrak{H}^0_\star$ into $\mathbf{R}$:

$$\widehat{\zeta}(u \star v) = \widehat{\zeta}(u)\widehat{\zeta}(v).$$

for u and v in $\mathfrak{H}^0$.

Consequence of the two sets of quadratic relations:

$$\widehat{\zeta}(u \sqcup\!\sqcup v - u \star v) = 0$$

for u and v in $\mathfrak{H}^0$.

Hoffman Third Standard Relations

For any $w \in \mathfrak{H}^0$, we have $x_1 ш w - x_1 \star w \in \mathfrak{H}^0$ and

$$\widehat{\zeta}(x_1 ш w - x_1 \star w) = 0.$$

Hoffman Third Standard Relations

For any $w \in \mathfrak{H}^0$, we have $x_1 \text{ш} w - x_1 \star w \in \mathfrak{H}^0$ and

$$\widehat{\zeta}(x_1 \text{ш} w - x_1 \star w) = 0.$$

Example. For $w = x_0 x_1$,

$$x_1 \text{ш} x_0 x_1 = x_1 x_0 x_1 + 2 x_0 x_1^2 = y_1 y_2 + 2 y_2 y_1,$$

$$x_1 \star x_0 x_1 = y_1 \star y_2 = y_1 y_2 + y_2 y_1 + y_3,$$

hence

$$y_2 y_1 - y_3 \in \ker \widehat{\zeta}$$

and (Euler)

$$\zeta(2,1) = \zeta(3).$$

Diophantine Conjecture *(simple form)*

Conjecture (Petitot, Hoang Ngoc Minh. . .). *The kernel of* $\widehat{\zeta}$ *is spanned by the standard relations*

$$\widehat{\zeta}(u \sqcup\!\sqcup v - u \star v) = 0 \quad \textit{and} \quad \widehat{\zeta}(x_1 \sqcup\!\sqcup w - x_1 \star w) = 0$$

for u, v *and* w *in* $x_0 X^* x_1$.

Minh, H.N, Jacob, G., Oussous, N. E., Petitot, M. –
Aspects combinatoires des polylogarithmes et des sommes d'Euler-Zagier.
J. Électr. Sém. Lothar. Combin. **43** (2000), Art. B43e, 29 pp.

Regularized Double Shuffle Relations

The map $\widehat{\zeta} : \mathfrak{H}^0 \rightarrow \mathbf{R}$ is a morphism of algebras for $ш$ and for $\star$:

$$\widehat{\zeta}(u ш v) = \widehat{\zeta}(u)\widehat{\zeta}(v) \quad \text{and} \qquad \widehat{\zeta}(u \star v) = \widehat{\zeta}(u)\widehat{\zeta}(v).$$

Question: Is-it possible to extend $\widehat{\zeta}$ to $\mathfrak{H}^1$ into a morphism of algebras both for $ш$ and $\star$?

Regularized Double Shuffle Relations

The map $\widehat{\zeta} : \mathfrak{H}^0 \to \mathbf{R}$ is a morphism of algebras for $ш$ and for $\star$:

$$\widehat{\zeta}(u ш v) = \widehat{\zeta}(u)\widehat{\zeta}(v) \quad \text{and} \qquad \widehat{\zeta}(u \star v) = \widehat{\zeta}(u)\widehat{\zeta}(v).$$

Question: Is-it possible to extend $\widehat{\zeta}$ to $\mathfrak{H}^1$ into a morphism of algebras both for $ш$ and $\star$?

Answer: NO!

$$x_1 ш x_1 = 2x_1^2, \qquad x_1 \star x_1 = y_1 \star y_1 = 2x_1^2 + y_2$$

$$\widehat{\zeta}(y_2) = \zeta(2) \neq 0.$$

Radford's Theorem:

$$\mathfrak{H}_{ш} = \mathfrak{H}^1_{ш}[x_0]_{ш} = \mathfrak{H}^0_{ш}[x_0, x_1]_{ш} \quad \textit{and} \quad \mathfrak{H}^1_{ш} = \mathfrak{H}^0_{ш}[x_1]_{ш}.$$

Radford's Theorem:

$$\mathfrak{H}_{\sqcup\!\sqcup} = \mathfrak{H}^1_{\sqcup\!\sqcup}[x_0]_{\sqcup\!\sqcup} = \mathfrak{H}^0_{\sqcup\!\sqcup}[x_0, x_1]_{\sqcup\!\sqcup} \quad \textit{and} \quad \mathfrak{H}^1_{\sqcup\!\sqcup} = \mathfrak{H}^0_{\sqcup\!\sqcup}[x_1]_{\sqcup\!\sqcup}.$$

Hoffman's Theorem:

$$\mathfrak{H}_\star = \mathfrak{H}^1_\star[x_0]_\star = \mathfrak{H}^0_\star[x_0, x_1]_\star \quad \textit{and} \quad \mathfrak{H}^1_\star = \mathfrak{H}^0_\star[x_1]_\star.$$

Radford's Theorem:

$$\mathfrak{H}_{ш} = \mathfrak{H}^1_{ш}[x_0]_{ш} = \mathfrak{H}^0_{ш}[x_0, x_1]_{ш} \quad \textit{and} \qquad \mathfrak{H}^1_{ш} = \mathfrak{H}^0_{ш}[x_1]_{ш}.$$

Hoffman's Theorem:

$$\mathfrak{H}_\star = \mathfrak{H}^1_\star[x_0]_\star = \mathfrak{H}^0_\star[x_0, x_1]_\star \quad \textit{and} \quad \mathfrak{H}^1_\star = \mathfrak{H}^0_\star[x_1]_\star.$$

From $\mathfrak{H}^1_{ш} = \mathfrak{H}^0_{ш}[x_1]_{ш}$ and $\mathfrak{H}^1_\star = \mathfrak{H}^0_\star[x_1]_\star$ we deduce that there are two uniquely determined algebra morphisms

$$\widehat{Z}_{ш} : \mathfrak{H}^1_{ш} \longrightarrow \mathbf{R}[T] \quad \text{and} \qquad \widehat{Z}_\star : \mathfrak{H}^1_\star \longrightarrow \mathbf{R}[T]$$

which extend $\widehat{\zeta}$ and map x_1 to T.

Theorem (Boutet de Monvel, Zagier). *There is a* $\mathbf{R}$*-linear isomorphism* $\varrho : \mathbf{R}[T] \rightarrow \mathbf{R}[X]$ *which makes commutative the following diagram:*

$$
\begin{array}{ccc}
 & & \mathbf{R}[X] \\
 & \widehat{Z}_{ш} \nearrow & \\
\mathfrak{H}^1 & & \uparrow \varrho \\
 & \widehat{Z}_{\star} \searrow & \\
 & & \mathbf{R}[T]
\end{array}
$$

An explicit formula for ϱ is given by means of the generating series

$$\sum_{\ell \geq 0} \varrho(T^\ell) \frac{t^\ell}{\ell!} = \exp\left(Xt + \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n} t^n \right).$$

Compare with the formula giving the expansion of the logarithm of Euler Gamma function:

$$\Gamma(1+t) = \exp\left(-\gamma t + \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n} t^n \right).$$

One may see ϱ as the differential operator of infinite order

$$\exp\left(\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n}\left(\frac{\partial}{\partial T}\right)^n\right)$$

(just consider the image of e^{tT}).

Denote by $\mathrm{reg}_{ш}$ the $\mathbf{Q}$-linear map $\mathfrak{H} \to \mathfrak{H}^0$ which maps $w \in \mathfrak{H}$ onto its constant term when w is written as a polynomial in x_0, x_1 in the shuffle algebra $\mathfrak{H}^0[x_0, x_1]_{ш}$. Then $\mathrm{reg}_{ш}$ is a morphism of algebras $\mathfrak{H}_{ш} \to \mathfrak{H}^0_{ш}$.

Theorem. *(Regularized Double Shuffle Relations – Ihara+Kaneko). For $w \in \mathfrak{H}^1$ and $w_0 \in \mathfrak{H}^0$,*

$$\mathrm{reg}_{ш}(w ш w_0 - w \star w_0) \in \ker \widehat{\zeta}.$$

Example. Take $w = x_1$. Since $x_1 ш w_0 - x_1 \star w_0 \in \mathfrak{H}^0$ for any $w_0 \in \mathfrak{H}^0$, one recovers the third standard relations of Hoffman.

Diophantine Conjectures

Conjecture (Zagier, Cartier, Ihara-Kaneko,. . .). *All existing algebraic relations between the real numbers* $\zeta(\underline{s})$ *are in the ideal generated by the ones described above.*

Petitot and Hoang Ngoc Minh: up to weight $s_1 + \cdots s_k \leq 16$, the three standard relations for u, v and w in $x_0X^*x_1$

$$\widehat{\zeta}(u)\widehat{\zeta}(v) = \widehat{\zeta}(u ш v), \quad \widehat{\zeta}(u)\widehat{\zeta}(v) = \widehat{\zeta}(u \star v),$$

$$\widehat{\zeta}(x_1 ш w - x_1 \star w) = 0$$

suffice.

Goncharov's Conjecture

Let $\mathfrak{Z}$ denote the $\mathbf{Q}$-vector space spanned in $\mathbf{C}$ by the numbers

$$(2i\pi)^{-|s|}\zeta(\underline{s})$$

$\underline{s} = (s_1, \ldots, s_k) \in \mathbf{N}^k$ with $k \geq 1$, $s_1 \geq 2$, $s_i \geq 1$ $(2 \leq i \leq k)$.

Hence $\mathfrak{Z}$ is a $\mathbf{Q}$-subalgebra of $\mathbf{C}$ bifiltered by the weight and by the depth.

For a graded Lie algebra $C_\bullet$ denote by $\mathfrak{U}C_\bullet$ its universal envelopping algebra and by

$$\mathfrak{U}C_\bullet^\vee = \bigoplus_{n\geq 0} (\mathfrak{U}C)_n^\vee$$

its graded dual, which is a commutative Hopf algebra.

Conjecture (Goncharov). *There exists a free graded Lie algebra $C_\bullet$ and an isomorphism of algebras*

$$\mathfrak{Z} \simeq \mathfrak{U}C_\bullet^\vee$$

filtered by the weight on the left and by the degree on the right.

References:

Goncharov A.B. – Multiple polylogarithms, cyclotomy and modular complexes. *Math. Research Letter* **5** (1998), 497–516.

References on multiple zeta values and Euler sums

compiled by Michael Hoffman

`http://www.usna.edu/Users/math/meh/biblio.html`

Duality

Let τ denote the anti-homomorphism of $\mathfrak{H}$ which exchanges x_0 and x_1. Notice that $\mathfrak{H}^0$ and $\mathfrak{H}^1$ are stable under τ. Then, for $w \in \mathfrak{H}^0$,

$$\widehat{\zeta}(\tau w) = \widehat{\zeta}(w).$$

Proof. We have

$$\tau(x_{\epsilon_1} \cdots x_{\epsilon_p}) = x_{1-\epsilon_p} \cdots x_{1-\epsilon_1}$$

and

$$\widehat{\zeta}(x_{\epsilon_1} \cdots x_{\epsilon_p}) = \int_0^1 \omega_{\epsilon_1} \cdots \omega_{\epsilon_p}.$$

In the integral, change the variables

$$t_i \longmapsto 1 - t_{p-i}, \qquad (1 \le i \le p).$$

Hoffman's derivation Theorem

Theorem *(Hoffman).* Let D be the derivation on $\mathfrak{H}$ with $Dx_0 = 0$ and $Dx_1 = x_0x_1$. Then for $w \in \mathfrak{H}^0$

$$\widehat{\zeta}(Dw) = \widehat{\zeta}(D\tau w).$$

Hoffman's derivation Theorem

Theorem *(Hoffman).* Let D be the derivation on $\mathfrak{H}$ with $Dx_0 = 0$ and $Dx_1 = x_0x_1$. Then for $w \in \mathfrak{H}^0$

$$\widehat{\zeta}(Dw) = \widehat{\zeta}(D\tau w).$$

Equivalent statement: Fix $(s_1, \dots, s_k)$ in $\mathbf{Z}^k$ with $s_1 \geq 2$, and $s_j \geq 1$ for $j = 2, \dots, k$. Then

$$\sum_{h=1}^{k} \zeta(s_1, \dots, s_{h-1}, s_h + 1, s_{h+1}, \dots, s_p) =$$

$$\sum_{\substack{1 \leq h \leq k \\ s_h \geq 2}} \sum_{j=0}^{s_h - 2} \zeta(s_1, \dots, s_{h-1}, s_h - j, j + 1, , s_{h+1}, \dots, s_p).$$

A generalization of Hoffman's DerivationTheorem

Theorem *(Y. Ohno, K. Ihara and M. Kaneko)* Fix $n \geq 1$. Define the antisymmetric derivation δ_n on $\mathfrak{H}$ by

$$\delta_n x_0 = -\delta_n x_1 = x_0(x_0 + x_1)^{n-1} x_1.$$

Then for any $w \in \mathfrak{H}^0$,

$$\widehat{\zeta}(\delta_n w) = 0.$$

Remark: $\delta_1 = \tau D \tau - D$:

$$\delta_1(w) = x_1 \text{ш} w - x_1 \star w.$$

Theorem *(Y. Ohno).* Let $\underline{s} = (s_1, \ldots, s_k)$ be a tuple of positive integers with $s_1 \geq 2$. Define $\underline{s}' = (s'_1, \ldots, s'_{k'})$ by the relation $y_{\underline{s}'} = \tau y_{\underline{s}}$. Further let $\ell \geq 0$ be a given integer. Then

$$\sum_{\substack{e_1+\cdots+e_k=\ell \\ e_i \geq 0}} \zeta(s_1+e_1, \ldots, s_k+e_k) = \sum_{\substack{e'_1+\cdots+e'_{k'}=\ell \\ e_j \geq 0}} \zeta(s'_1+e'_1, \ldots, s'_{k'}+e_{k'}).$$

Cyclic derivations

Define a derivation $C : \mathfrak{H} \to \mathfrak{H}$ as $\tilde{\mu} \circ \widetilde{C}$ where $\tilde{\mu} : \mathfrak{H} \otimes \mathfrak{H} \to \mathfrak{H}$ is $\mu(a \otimes b) = ba$ and $\widetilde{C} : \mathfrak{H} \to \mathfrak{H} \otimes \mathfrak{H}$ maps x_0 to 0 and x_1 to $x_1 \otimes x_0$.

Theorem *(Ohno, conjectured by Hoffman).* For any $w \in \mathfrak{H}^1 \setminus \{x_1, x_1^2, \ldots\}$,

$$\widehat{\zeta}(Cw) = \widehat{\zeta}(\tau C \tau w).$$

Example:

$$\zeta(4, \{3\}_n) = \zeta(\{3\}_{n+1}, 1) + \zeta(2, \{3\}_n, 2).$$

Zagier-Broadhurst formula

Theorem *(Broadhurst - Conjecture of Zagier). For any* $n \geq 1$,

$$\zeta\big(\{3,1\}_n\big) = 4^{-n}\zeta\big(\{4\}_n\big).$$

Remark.

$$\zeta\big(\{2\}_n\big) = \frac{\pi^{2n}}{(2n+1)!}$$

and

$$\frac{1}{2n+1}\zeta\big(\{2\}_{2n}\big) = \frac{1}{2^{2n}}\zeta\big(\{4\}_n\big).$$

hence

$$\zeta\big(\{3,1\}_n\big) = 2\cdot\frac{\pi^{4n}}{(4n+2)!}\cdot$$

Theorem *(Broadhurst - Conjecture of Zagier).* *For any* $n \geq 1$,

$$y_4^n - (4y_3y_1)^n \in \ker \widehat{\zeta}.$$

Theorem *(Broadhurst - Conjecture of Zagier). For any* $n \geq 1$,

$$y_4^n - (4y_3y_1)^n \in \ker \widehat{\zeta}.$$

$$\widehat{\zeta}(y_4^n) = \zeta\bigl(\{4\}_n\bigr)$$

$$\widehat{\zeta}\bigl((y_3y_1)^n\bigr) = \zeta\bigl(\{3,1\}_n\bigr).$$

Syntaxic identities

Definition: For $w \in X^* \setminus \{0\}$,

$$w^* = e + w + w^2 + \cdots$$

Hence $(e - w)w^* = w^*(e - w) = e$.

Lemma 1.

$$y_2^\star ш (-y_2)^\star = (-4y_3y_1)^\star.$$

Lemma 2.

$$y_2^* \star (-y_2)^* = (-y_4)^*.$$

Proof of $y_4^n - (4y_3y_1)^n \in \ker \widehat{\zeta}$.

From
$$y_2^* \star (-y_2)^* = (-y_4)^* \quad \text{and} \quad y_2^* ш (-y_2)^* = (-4y_3y_1)^*$$
one deduces, for any $n \geq 1$,

$$\sum_{i+j=2n} (-1)^j y_2^{2i} \star y_2^{2j} = (-y_4)^n$$

and
$$\sum_{i+j=2n} (-1)^j y_2^{2i} ш y_2^{2j} = (-4y_3y_1)^n,$$

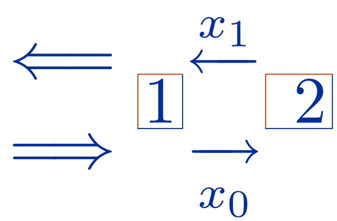
hence

$$y_4^n - (4y_3y_1)^n = \sum_{i+j=2n} (-1)^{n-j} (y_2^{2i} \star y_2^{2j} - y_2^{2i} ш y_2^{2j}) \in \ker \widehat{\zeta}.$$

Lemma 1.

$$(x_0x_1)^\star ш (-x_0x_1)^\star = (-4x_0^2x_1^2)^\star.$$

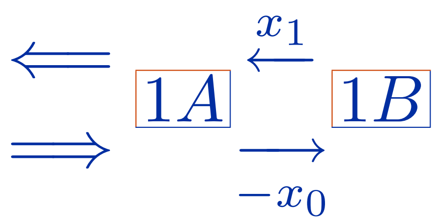
Proof. The series associated to the automaton



is

$$S_1 = e + x_0x_1 + (x_0x_1)^2 + \cdots + (x_0x_1)^n + \cdots = (x_0x_1)^\star,$$

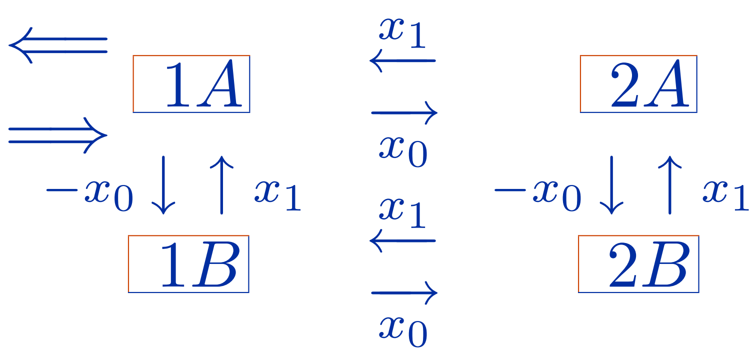
The series associated to



is

$$S_A = e - x_0x_1 + (x_0x_1)^2 + \cdots + (-x_0x_1)^n + \cdots = (-x_0x_1)^\star.$$

The following automaton is the cartesian product of the automata associated with S_1 and S_A:



One computes the associated series $S_{1A} = S_1 ш S_A$ by solving a system of linear (noncommutative) equations as follows. Define also S_{1B}, S_{2A} and S_{2B} as the series of labels of the paths starting at the corresponding state and ending at a terminal state. Then

$$\begin{aligned} S_{1A} &= e - x_0 S_{1B} + x_0 S_{2A}, \\ S_{1B} &= x_1 S_{1A} + x_0 S_{2B}, \\ S_{2A} &= x_1 S_{1A} - x_0 S_{2B}, \\ S_{2B} &= x_1 S_{1B} + x_1 S_{2A}. \end{aligned}$$

One deduces

$$S_{1A} = e - x_0(S_{1B} - S_{2A}), \quad S_{1B} - S_{2A} = -2x_0 S_{2B},$$

$$S_{2B} = x_1(S_{1B} + S_{2A}), \quad S_{1B} + S_{2A} = 2x_1 S_{1A}$$

and therefore

$$S_{1A} = e + 4x_0^2 x_1^2 S_{1A},$$

Lemma 2.

$$y_2^* \star (-y_2)^* = (-y_4)^*.$$

Proof. Denote by $\underline{t} = (t_1, t_2, \ldots)$ a sequence of commutative variables. Consider the quasisymmetric series

$$\phi(y_{\underline{s}}) = \sum_{n_1 > \cdots > n_k \geq 1} t_{n_1}^{s_1} \cdots t_{n_k}^{s_k}$$

and extend by linearity. Then

$$\phi(u \star v) = \phi(u)\phi(v).$$

On the other hand

$$\phi(y_2^*) = \sum_{k=0}^{\infty} \sum_{n_1 > \cdots > n_k \geq 1} t_{n_1}^2 \cdots t_{n_k}^2,$$

$$\phi\Big((-y_2)^*\Big) = \sum_{k=0}^{\infty} (-1)^k \sum_{n_1 > \cdots > n_k \geq 1} t_{n_1}^2 \cdots t_{n_k}^2$$

$$\phi\Big((-y_4)^*\Big) = (-1)^k \sum_{n_1 > \cdots > n_k \geq 1} t_{n_1}^4 \cdots t_{n_k}^4.$$

Hence from the identity

$$\prod_{n=1}^{\infty} (1 + t_n t) = \sum_{k=0}^{\infty} t^k \sum_{n_1 > \cdots > n_k \geq 1} t_{n_1} \cdots t_{n_k}$$

one deduces

$$\phi(y_2^*) = \prod_{n=1}^{\infty}(1+t_n^2), \quad \phi\Big((-y_2)^*\Big) = \prod_{n=1}^{\infty}(1-t_n^2)$$

$$\phi\Big((-y_4)^*\Big) = \prod_{n=1}^{\infty}(1-t_n^4),$$

which implies the Lemma.

Knizhnik-Zamolodchikov Differential Equation
Polylogarithms associated to elements in $\mathfrak{H}$

We want (need) to extend the definition of $\widehat{\mathrm{Li}}_w$ to $w \in \mathfrak{H}$ so that for any w and w' in $\mathfrak{H}$, we have

$$\widehat{\mathrm{Li}}_w(z)\widehat{\mathrm{Li}}_{w'}(z) = \widehat{\mathrm{Li}}_{w \sqcup\!\sqcup w'}(z).$$

It suffices to define $\widehat{\mathrm{Li}}_{x_0}(z)$. The definition is

$$\widehat{\mathrm{Li}}_{x_0}(z) = \int_1^z \omega_0 = \int_1^z \frac{dt}{t} = \log z \quad \text{for} \quad |z-1| < 1.$$

By induction, for $n\geq 1$

$$\widehat{\mathrm{Li}}_{x_0^n}(z)=\int_1^z \omega_0^n=\frac{1}{n!}(\log z)^n$$

while for $w\in X^*\setminus\{e,x_0,x_0^2,\ldots\}$ and $i\in\{0,1\}$,

$$\widehat{\mathrm{Li}}_{x_iw}(z)=\int_0^z \omega_i(t)\widehat{\mathrm{Li}}_w(t).$$

Knizhnik-Zamolodchikov Differential Equation

Proposition. *The generating series*

$$\widehat{\mathrm{Li}}(z) = \sum_{w \in X^*} \widehat{\mathrm{Li}}_w(z) w.$$

is the solution of the differential equation

$$\frac{d}{dz} F(z) = \left(\frac{x_0}{z} + \frac{x_1}{1-z} \right) F(z)$$

satisfying the initial condition

$$\lim_{z \to 0} e^{-x_0 \log z} \widehat{\mathrm{Li}}(z) = 1.$$

The Knizhnik-Zamolodchikov differential equation means

$$d\ \widehat{\mathrm{Li}}_{x_i w}(z) = \omega_i(z)\widehat{\mathrm{Li}}_w(z)$$

for $i \in \{0, 1\}$ and $w \in X^*$.

Remark.

$$e^{x_0 \log z} = \sum_{n\geq 0} \frac{1}{n!}(\log z)^n x_0^n = \sum_{n\geq 0} \widehat{\mathrm{Li}}_{x_0^n}(z) x_0^n.$$

A result of Hoang Ngoc Minh, M. Petitot, van der Hoeven

The map $w \mapsto \widehat{\mathrm{Li}}_w(z)$ defines an **injective** *homomorphism of algebras from $\mathfrak{H}^1_{\text{ш}}$ into the algebra of analytic functions in the unit disc.*

More precisely the functions $\widehat{\mathrm{Li}}_w(z)$ for $w \in X^*$ are linearly independent over the field of meromorphic functions on $\mathbf{C} \setminus \{0, 1\}$.

The proof rests on the study of the monodromy of the Knizhnik-Zamolodchikov differential equation.