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New variations on the six exponentials theorem

Michel Waldschmidt

miw@math.jussieu.fr http://www.math.jussieu.fr/~miw/

S. Ramanujan.– Highly composite numbers, Proc. London Math. Soc. **2** 14 (1915), 347–409.

For $n \geq 1$ let $d(n)$ denote the number of divisors of n. A *superior highly composite number* is a positive integer n for which there exists $\epsilon > 0$ such that the function $d(m)/m^\epsilon$ reaches its maximum at n. The first superior highly composite numbers are

$$2,\quad 6,\quad 12,\quad 60,\quad 120,\quad 360,\quad 2\,520,\quad 5\,040,\quad 55\,440,\quad 720\,720, \ldots$$

and their successive quotients are prime numbers

$$3,\quad 2,\quad 5,\quad 2,\quad 3,\quad 7,\quad 2,\quad 11,\quad 13,\quad 2,\quad 3\ldots$$

L. Alaoglu, P. Erdős.– On highly composite and similar numbers, Trans. Amer. Math. Soc., **56** (1944), 448–469.

Colossally abundant numbers : replace

$$d(n) = \sum_{d|n} 1$$

by

$$\sigma(n) = \sum_{d|n} d$$

Quotients of consecutive colossally abundant numbers are prime.

¿ If x is a real number such that p_1^x and p_2^x are rational integers for two distinct primes p_1 and p_2, then $x \in \mathbf{Z}$?

L. Alaoglu, P. Erdős.– On highly composite and similar numbers, Trans. Amer. Math. Soc., **56** (1944), 448–469.

S. Lang.– Nombres transcendants, Sém. Bourbaki 18ème année (1965/66), N° 305.

S. Lang.– Algebraic values of meromorphic functions, II, Topology, **5** (1966), 363–370.

S. Lang.– *Introduction to Transcendental Numbers*; Addison-Wesley 1966.

L. Alaoglu, P. Erdős.– On highly composite and similar numbers, Trans. Amer. Math. Soc., **56** (1944), 448–469.

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S. Lang.– *Introduction to Transcendental Numbers*; Addison-Wesley 1966.

K. Ramachandra.– Contributions to the theory of transcendental numbers (I); Acta Arith., **14** (1968), 65–72; (II), id., 73–88.

K. Ramachandra.– *Lectures on transcendental numbers*; The Ramanujan Institute, Univ. of Madras, 1969, 72 p.

Six Exponentials Theorem *(special case).* If x is a real number such that p_1^x, p_2^x and p_3^x are rational integers for three distinct primes p_1, p_2 and p_3, then $x \in \mathbf{Z}$.

Six Exponentials Theorem *(special case).* If x is a real number such that p_1^x, p_2^x and p_3^x are rational integers for three distinct primes p_1, p_2 and p_3, then $x \in \mathbf{Z}$.

Six Exponentials Theorem *(again a special case).* If x is a complex number such that α_1^x, α_2^x and α_3^x are algebraic for three multiplicatively independent numbers α_1, α_2 and α_3, then $x \in \mathbf{Q}$.

Six Exponentials Theorem. *If x_1, x_2 are two complex numbers which are $\mathbf{Q}$-linearly independent, if y_1, y_2, y_3 are three complex numbers which are $\mathbf{Q}$-linearly independent, then one at least of the six numbers*

$$e^{x_i y_j} \quad (i = 1, 2,\ j = 1, 2, 3)$$

is transcendental.

Six Exponentials Theorem. *If* x_1, x_2 *are two complex numbers which are* $\mathbf{Q}$*-linearly independent, if* y_1, y_2, y_3 *are three complex numbers which are* $\mathbf{Q}$*-linearly independent, then one at least of the six numbers*

$$e^{x_i y_j} \quad (i = 1, 2,\ j = 1, 2, 3)$$

is transcendental.

Set

$$x_i y_j = \lambda_{ij} \quad (i = 1, 2;\ j = 1, 2, 3).$$

A 2×3 matrix has rank one iff it is of the form

$$\begin{pmatrix} x_1 y_1 & x_1 y_2 & x_1 y_3 \\ x_2 y_1 & x_2 y_2 & x_2 y_3 \end{pmatrix}.$$

Denote by $\overline{\mathbf{Q}}$ the set of algebraic numbers and define

$$\mathcal{L} = \{\log \alpha \,;\; \alpha \in \overline{\mathbf{Q}}^{\times}\}. = \{\lambda \in \mathbf{C} \,;\; e^{\lambda} \in \overline{\mathbf{Q}}^{\times}\}.$$

Six Exponentials Theorem *(logarithmic form). For $i = 1, 2$ and $j = 1, 2, 3$, let $\lambda_{ij} \in \mathcal{L}$. Assume $\lambda_{11}, \lambda_{12}, \lambda_{13}$ are linearly independent over $\mathbf{Q}$ and also $\lambda_{11}, \lambda_{21}$ are linearly independent over $\mathbf{Q}$. Then the matrix*

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{12} & \lambda_{22} & \lambda_{23} \end{pmatrix}$$

has rank 2.

Four Exponentials Conjecture

History.

A. Selberg (50's).

Th. Schneider(1957) - first problem.

S Lang (60's).

K. Ramachandra (1968).

Leopoldt's Conjecture on the p-adic rank of the units of an algebraic number field (non vanishing of the p-adic regulator).

Four Exponentials Conjecture *(exponential form). Let x_1, x_2 be two $\mathbf{Q}$-linearly independent complex numbers and y_1, y_2 also two $\mathbf{Q}$-linearly independent complex numbers. Then one at least of the four numbers*

$$e^{x_1y_1},\ e^{x_1y_2},\ e^{x_2y_1},\ e^{x_2y_2}$$

is transcendental.

Four Exponentials Conjecture *(logarithmic form).* *For* $i = 1, 2$ *and* $j = 1, 2$, *let* α_{ij} *be a non zero algebraic number and* λ_{ij} *a complex number satisfying* $e^{\lambda_{ij}} = \alpha_{ij}$. *Assume* $\lambda_{11}, \lambda_{12}$ *are linearly independent over* $\mathbf{Q}$ *and also* $\lambda_{11}, \lambda_{21}$ *are linearly independent over* $\mathbf{Q}$. *Then*

$$\lambda_{11}\lambda_{22} \neq \lambda_{12}\lambda_{21}.$$

Four Exponentials Conjecture *(logarithmic form).* *For $i=1,2$ and $j=1,2$, let α_{ij} be a non zero algebraic number and λ_{ij} a complex number satisfying $e^{\lambda_{ij}}=\alpha_{ij}$. Assume $\lambda_{11},\lambda_{12}$ are linearly independent over $\mathbf{Q}$ and also $\lambda_{11},\lambda_{21}$ are linearly independent over $\mathbf{Q}$. Then*

$$\lambda_{11}\lambda_{22}\neq\lambda_{12}\lambda_{21}.$$

Notice:

$$\lambda_{11}\lambda_{22}-\lambda_{12}\lambda_{21}=\det\begin{vmatrix}\lambda_{11} & \lambda_{12}\\ \lambda_{12} & \lambda_{22}\end{vmatrix}.$$

Algebraic independence of logarithms of algebraic numbers

Conjecture *Let* $\alpha_1, \ldots, \alpha_n$ *be non zero algebraic numbers. For* $1 \leq j \leq n$ *let* $\lambda_j \in \mathbf{C}$ *satisfy* $e^{\lambda_j} = \alpha_j$. *Assume* $\lambda_1, \ldots, \lambda_n$ *are linearly independent over* $\mathbf{Q}$. *Then* $\lambda_1, \ldots, \lambda_n$ *are algebraically independent.*

Algebraic independence of logarithms of algebraic numbers

Conjecture *Let* $\alpha_1, \ldots, \alpha_n$ *be non zero algebraic numbers. For* $1 \leq j \leq n$ *let* $\lambda_j \in \mathbf{C}$ *satisfy* $e^{\lambda_j} = \alpha_j$. *Assume* $\lambda_1, \ldots, \lambda_n$ *are linearly independent over* $\mathbf{Q}$. *Then* $\lambda_1, \ldots, \lambda_n$ *are algebraically independent.*

Write $\lambda_j = \log \alpha_j$.

If $\log \alpha_1, \ldots, \log \alpha_n$ *are* $\mathbf{Q}$*-linearly independent then they are algebraically independent.*

Algebraic independence of logarithms of algebraic numbers

Conjecture *Let $\alpha_1, \ldots, \alpha_n$ be non zero algebraic numbers. For $1 \le j \le n$ let $\lambda_j \in \mathbf{C}$ satisfy $e^{\lambda_j} = \alpha_j$. Assume $\lambda_1, \ldots, \lambda_n$ are linearly independent over $\mathbf{Q}$. Then $\lambda_1, \ldots, \lambda_n$ are algebraically independent.*

Open problem:

$$\text{transc.deg}_{\mathbf{Q}}\mathbf{Q}(\mathcal{L}) \ge 2?$$

Quadratic relations between logarithms of algebraic numbers

Homogeneous quadratic relations *(Four Exponentials Conjecture):*

$$\lambda_1\lambda_2 = \lambda_3\lambda_4 \ ?$$

Transcendence of $\alpha^{(\log \beta)/\log \gamma}$:

$$(\log \alpha)(\log \beta) = (\log \gamma)(\log \delta)?$$

Quadratic relations between logarithms of algebraic numbers

Non homogeneous quadratic relations:

$$(\log \alpha)(\log \beta) = \log \gamma$$

.

Open problem: Transcendence of $2^{\log 2}$:

$$(\log 2)^2 = \log \gamma?$$

Quadratic relations between logarithms of algebraic numbers

Non homogeneous relations

Three Exponentials Conjecture *(logarithmic form).* *Let* λ_1, λ_2, λ_3 *be three elements in* $\mathcal{L}$ *and* γ *a non zero algebraic number. Assume* $\lambda_1\lambda_2 = \gamma\lambda_3$. *Then* $\lambda_1\lambda_2 = \gamma\lambda_3 = 0$.

Quadratic relations between logarithms of algebraic numbers

Non homogeneous relations

Three Exponentials Conjecture *(logarithmic form). Let λ_1, λ_2, λ_3 be three elements in $\mathcal{L}$ and γ a non zero algebraic number. Assume $\lambda_1\lambda_2 = \gamma\lambda_3$. Then $\lambda_1\lambda_2 = \gamma\lambda_3 = 0$.*

Special case: $\lambda_1 = \lambda_2 = \log\alpha$, $\gamma = 1$: transcendence of $\alpha^{\log\alpha}$?

Example: transcendence of e^{π^2}?

Quadratic relations between logarithms of algebraic numbers

Non homogeneous relations

Three Exponentials Conjecture *(logarithmic form).* *Let* λ_1, λ_2, λ_3 *be three elements in* $\mathcal{L}$ *and* γ *a non zero algebraic number. Assume* $\lambda_1\lambda_2 = \gamma\lambda_3$. *Then* $\lambda_1\lambda_2 = \gamma\lambda_3 = 0$.

Three Exponentials Conjecture *(exponential form).* *Let* x_1, x_2, y *be non zero complex numbers and* γ *a non zero algebraic number. Then one at least of the three numbers*

$$e^{x_1 y}, \ \ e^{x_2 y}, \ \ e^{\gamma x_1/x_2}$$

is transcendental.

Five Exponentials Theorem *(exponential form).* *If* x_1, x_2 *are* $\mathbf{Q}$*-linearly independent,* y_1, y_2 *are* $\mathbf{Q}$*-linearly independent and* γ *is a non zero algebraic number, then one at least of the five numbers*

$$e^{x_1y_1},\ e^{x_1y_2},\ e^{x_2y_1},\ e^{x_2y_2},\ e^{\gamma x_2/x_1}$$

is transcendental.

Five Exponentials Theorem *(exponential form).* *If* x_1, x_2 *are* $\mathbf{Q}$*-linearly independent,* y_1, y_2 *are* $\mathbf{Q}$*-linearly independent and* γ *is a non zero algebraic number, then one at least of the five numbers*

$$e^{x_1y_1},\ e^{x_1y_2},\ e^{x_2y_1},\ e^{x_2y_2},\ e^{\gamma x_2/x_1}$$

is transcendental.

Five Exponentials Theorem *(logarithmic form).* *For* $i = 1, 2$ *and* $j = 1, 2$*, let* $\lambda_{ij} \in \mathcal{L}$*. Assume* $\lambda_{11}, \lambda_{12}$ *are linearly independent over* $\mathbf{Q}$*. Further let* $\gamma \in \overline{\mathbf{Q}}^{\times}$ *and* $\lambda \in \mathcal{L}$*. Then the matrix*

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} & \gamma \\ \lambda_{21} & \lambda_{22} & \lambda \end{pmatrix}$$

has rank 2*.*

Sharp Six Exponentials Theorem *(logarithmic form).* *For* $i = 1, 2$ *and* $j = 1, 2, 3$, *let* $\lambda_{ij} \in \mathcal{L}$ *and* $\beta_{ij} \in \overline{\mathbf{Q}}$. *Assume* $\lambda_{11}, \lambda_{12}, \lambda_{13}$ *are linearly independent over* $\mathbf{Q}$ *and also* $\lambda_{11}, \lambda_{21}$ *are linearly independent over* $\mathbf{Q}$. *Then the matrix*

$$\begin{pmatrix} \lambda_{11} + \beta_{11} & \lambda_{12} + \beta_{12} & \lambda_{13} + \beta_{13} \\ \lambda_{21} + \beta_{21} & \lambda_{22} + \beta_{22} & \lambda_{23} + \beta_{23} \end{pmatrix}$$

has rank 2.

Sharp Six Exponentials Theorem *(exponential form). If x_1, x_2 are two complex numbers which are $\mathbf{Q}$-linearly independent, if y_1, y_2, y_3 are three complex numbers which are $\mathbf{Q}$-linearly independent and if β_{ij} are six algebraic numbers such that*

$$e^{x_i y_j - \beta_{ij}} \in \overline{\mathbf{Q}} \quad \text{for} \quad i = 1, 2,\ j = 1, 2, 3,$$

then $x_i y_j = \beta_{ij}$ for $i = 1, 2$ and $j = 1, 2, 3$.

Sharp Six Exponentials Theorem *(exponential form). If* x_1, x_2 *are two complex numbers which are* $\mathbf{Q}$*-linearly independent, if* y_1, y_2, y_3 *are three complex numbers which are* $\mathbf{Q}$*-linearly independent and if* β_{ij} *are six algebraic numbers such that*

$$e^{x_i y_j - \beta_{ij}} \in \overline{\mathbf{Q}} \quad \textit{for} \quad i = 1, 2,\ j = 1, 2, 3,$$

then $x_i y_j = \beta_{ij}$ *for* $i = 1, 2$ *and* $j = 1, 2, 3$.

The sharp six exponentials Theorem implies the five exponentials Theorem: set $y_3 = \gamma/x_1$ and use Baker's Theorem for checking that y_1, y_2, y_3 are linearly independent over $\mathbf{Q}$.

A consequence of the sharp six exponentials Theorem: *One at least of the two numbers*

$$e^{\lambda^2} = \alpha^{\log\alpha}, \; e^{\lambda^3} = \alpha^{(\log\alpha)^2}$$

is transcendental.

$$\mathrm{rank}\begin{pmatrix} 1 & \lambda & \lambda^2 \\ \lambda & \lambda^2 & \lambda^3 \end{pmatrix} = 1.$$

First proof in 1970 (also by W.D. Brownawell) as a consequence of a result of algebraic independence.

Sharp Four Exponentials Conjecture *(exponential form). If x_1, x_2 are two complex numbers which are $\mathbf{Q}$-linearly independent, if y_1, y_2, are two complex numbers which are $\mathbf{Q}$-linearly independent and if $\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}$ are four algebraic numbers such that the four numbers*

$$e^{x_1 y_1 - \beta_{11}}, \ e^{x_1 y_2 - \beta_{12}}, \ e^{x_2 y_1 - \beta_{21}}, \ e^{x_2 y_2 - \beta_{22}}$$

are algebraic, then $x_i y_j = \beta_{ij}$ for $i = 1, 2$ and $j = 1, 2$.

Sharp Four Exponentials Conjecture *(logarithmic form). For $i = 1, 2$ and $j = 1, 2$, let $\lambda_{ij} \in \mathcal{L}$ and $\beta_{ij} \in \overline{\mathbf{Q}}$. Assume $\lambda_{11}, \lambda_{12}$ are linearly independent over $\mathbf{Q}$ and also $\lambda_{11}, \lambda_{21}$ are linearly independent over $\mathbf{Q}$. Then*

$$\det \begin{vmatrix} \lambda_{11} + \beta_{11} & \lambda_{12} + \beta_{12} \\ \lambda_{21} + \beta_{21} & \lambda_{22} + \beta_{22} \end{vmatrix} \neq 0.$$

Sharp Three Exponentials Conjecture *(exponential form).* *If* x_1, x_2, y *are non zero complex numbers and* $\alpha, \beta_1, \beta_2, \gamma$ *are algebraic numbers such that the three numbers*

$$e^{x_1 y - \beta_1}, \quad e^{x_2 y - \beta_2}, \quad e^{(\gamma x_1 / x_2) - \alpha},$$

are algebraic, then either $x_2 y = \beta_2$ *or* $\gamma x_1 = \alpha x_2$.

Sharp Three Exponentials Conjecture *(exponential form).* *If* x_1, x_2, y *are non zero complex numbers and* $\alpha, \beta_1, \beta_2, \gamma$ *are algebraic numbers such that the three numbers*

$$e^{x_1 y - \beta_1}, \quad e^{x_2 y - \beta_2}, \quad e^{(\gamma x_1/x_2) - \alpha},$$

are algebraic, then either $x_2 y = \beta_2$ *or* $\gamma x_1 = \alpha x_2$.

Sharp Three Exponentials Conjecture *(logarithmic form).* *Let* $\lambda_1, \lambda_2, \lambda_3$ *be three elements of* $\mathcal{L}$ *with* $\lambda_1 \lambda_3 \neq 0$ *and* $\beta_1, \beta_2, \beta_3, \gamma$ *four algebraic numbers. Then*

$$\det \begin{vmatrix} \lambda_1 + \beta_1 & \gamma \\ \lambda_2 + \beta_2 & \lambda_3 + \beta_3 \end{vmatrix} \neq 0.$$

Sharp Five Exponentials Conjecture. *If* x_1, x_2 *are* $\mathbf{Q}$-*linearly independent, if* y_1, y_2 *are* $\mathbf{Q}$-*linearly independent and if* $\alpha, \beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}, \gamma$ *are six algebraic numbers with* $\gamma \neq 0$ *such that*

$$e^{x_1y_1-\beta_{11}}, \; e^{x_1y_2-\beta_{12}}, \; e^{x_2y_1-\beta_{21}}, \; e^{x_2y_2-\beta_{22}}, \; e^{(\gamma x_2/x_1)-\alpha}$$

are algebraic, then $x_iy_j = \beta_{ij}$ *for* $i = 1, 2,\; j = 1, 2$ *and also* $\gamma x_2 = \alpha x_1$.

Sharp Five Exponentials Conjecture. *If x_1, x_2 are $\mathbf{Q}$-linearly independent, if y_1, y_2 are $\mathbf{Q}$-linearly independent and if $\alpha, \beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}, \gamma$ are six algebraic numbers with $\gamma \neq 0$ such that*

$$e^{x_1y_1-\beta_{11}},\; e^{x_1y_2-\beta_{12}},\; e^{x_2y_1-\beta_{21}},\; e^{x_2y_2-\beta_{22}},\; e^{(\gamma x_2/x_1)-\alpha}$$

are algebraic, then $x_iy_j = \beta_{ij}$ for $i = 1, 2,\ j = 1, 2$ and also $\gamma x_2 = \alpha x_1$.

Difficult case: when $y_1, y_2, \gamma/x_1$ are $\mathbf{Q}$-linearly dependent.

Example: $x_1 = y_1 = \gamma = 1$.

Sharp Five Exponentials Conjecture. *If x_1, x_2 are $\mathbf{Q}$-linearly independent, if y_1, y_2 are $\mathbf{Q}$-linearly independent and if $\alpha, \beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}, \gamma$ are six algebraic numbers with $\gamma \neq 0$ such that*

$$e^{x_1y_1-\beta_{11}},\ e^{x_1y_2-\beta_{12}},\ e^{x_2y_1-\beta_{21}},\ e^{x_2y_2-\beta_{22}},\ e^{(\gamma x_2/x_1)-\alpha}$$

are algebraic, then $x_iy_j = \beta_{ij}$ for $i = 1, 2$, $j = 1, 2$ and also $\gamma x_2 = \alpha x_1$.

Consequence: *Transcendence of the number e^{π^2}.*

Proof. Set $x_1 = y_1 = 1$, $x_2 = y_2 = i\pi$, $\gamma = 1$, $\alpha = 0$, $\beta_{11} = 1$, $\beta_{ij} = 0$ for $(i, j) \neq (1, 1)$.

Denote by $\widetilde{\mathcal{L}}$ the $\overline{\mathbf{Q}}$-vector space spanned by 1 and $\mathcal{L}$ (linear combinations of logarithms of algebraic numbers with algebraic coefficients):

$$\widetilde{\mathcal{L}} = \left\{ \beta_0 + \sum_{h=1}^{\ell} \beta_h \log \alpha_h \; ; \; \ell \geq 0, \; \alpha\text{'s in } \overline{\mathbf{Q}}^{\times}, \; \beta\text{'s in } \overline{\mathbf{Q}} \right\}$$

Strong Six Exponentials Theorem (*D. Roy*). *If x_1, x_2 are $\overline{\mathbf{Q}}$-linearly independent and if y_1, y_2, y_3 are $\overline{\mathbf{Q}}$-linearly independent, then one at least of the six numbers*

$$x_i y_j \quad (i = 1, 2, \; j = 1, 2, 3)$$

does not belong to $\widetilde{\mathcal{L}}$.

Strong Four Exponentials Conjecture. *If* x_1, x_2 *are* $\overline{\mathbf{Q}}$*-linearly independent and if* y_1, y_2, *are* $\overline{\mathbf{Q}}$*-linearly independent, then one at least of the four numbers*

$$x_1y_1,\ x_1y_2,\ x_2y_1,\ x_2y_2$$

does not belong to $\widetilde{\mathcal{L}}$.

Strong Three Exponentials Conjecture. *If* x_1, x_2, y *are non zero complex numbers with* $x_1/x_2 \notin \overline{\mathbf{Q}}$ *and* $x_1/x_2 \notin \overline{\mathbf{Q}}$, *then one at least of the three numbers*

$$x_1y, \quad x_2y, \quad x_2/x_1$$

is not in $\widetilde{\mathcal{L}}$.

Strong Five Exponentials Conjecture. *Let* x_1, x_2 *be* $\overline{\mathbf{Q}}$*-linearly independent and* y_1, y_2 *be* $\overline{\mathbf{Q}}$*-linearly independent. Then one at least of the five numbers*

$$x_1y_1,\ x_1y_2,\ x_2y_1,\ x_2y_2,\ x_1/x_2$$

does not belong to $\widetilde{\mathcal{L}}$.

12 statements

Three exponentials

sharp Four exponentials

strong Five exponentials

Six exponentials

12 statements

Three exponentials

sharp	Four exponentials	Conjecture
strong	Five exponentials	Theorem

Six exponentials

12 statements

	Three exponentials	
sharp	Four exponentials	Conjecture
strong	Five exponentials	Theorem
	Six exponentials	

Three exponentials: three conjectures

Four exponentials: three conjectures

Six exponentials: three theorems

Five exponentials: two conjectures (for sharp and strong)
one theorem

$$
\begin{array}{ccccccc}
 & & \text{Alg. indep. C} & & & & \\
 & & \Downarrow & & & & \\
\text{Strong 3 exp C} & \Leftarrow & \text{Strong 4 exp C} & \Rightarrow & \text{Strong 5 exp C} & \Rightarrow & \text{Strong 6 exp T} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\text{Sharp 3 exp C} & \Leftarrow & \text{Sharp 4 exp C} & \Rightarrow & \text{Sharp 5 exp C} & \Rightarrow & \text{Sharp 6 exp T} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\text{3 exp C} & & \text{4 exp C} & \Rightarrow & \text{5 exp T} & \Rightarrow & \text{6 exp T}
\end{array}
$$

Remark:
The sharp 6 exponentials Theorem implies the 5 exponentials Theorem.

Consequences of the 4 exponentials Conjecture

$$\lambda_{11} - \frac{\lambda_{12}\lambda_{21}}{\lambda_{22}} \neq 0, \qquad \frac{\lambda_{11}\lambda_{22}}{\lambda_{12}\lambda_{21}} \neq 0,$$

$$\frac{\lambda_{11}}{\lambda_{12}} - \frac{\lambda_{21}}{\lambda_{22}} \neq 0, \qquad \lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21} \neq 0.$$

Consequence of the sharp 4 exponentials Conjecture

Let λ_{ij} $(i = 1, 2,\ j = 1, 2)$ be four non zero logarithms of algebraic numbers.

Consequence of the sharp 4 exponentials Conjecture

Let λ_{ij} $(i = 1, 2,\ j = 1, 2)$ be four non zero logarithms of algebraic numbers.

Assume

$$\lambda_{11} - \frac{\lambda_{12}\lambda_{21}}{\lambda_{22}} \in \overline{\mathbf{Q}}.$$

Then

$$\lambda_{11}\lambda_{22} = \lambda_{12}\lambda_{21}.$$

Proof. Assume
$$\lambda_{11} - \frac{\lambda_{12}\lambda_{21}}{\lambda_{22}} = \beta \in \overline{\mathbf{Q}}.$$

Use the sharp four exponentials conjecture with

$$(\lambda_{11} - \beta)\lambda_{22} = \lambda_{12}\lambda_{21}.$$

Consequence of the strong 4 exponentials Conjecture

Let λ_{ij} $(i = 1, 2$, $j = 1, 2)$ be four non zero logarithms of algebraic numbers.

Assume

$$\frac{\lambda_{11}\lambda_{22}}{\lambda_{12}\lambda_{21}} \in \overline{\mathbf{Q}}.$$

Then

$$\frac{\lambda_{11}\lambda_{22}}{\lambda_{12}\lambda_{21}} \in \mathbf{Q}.$$

Proof: Assume

$$\frac{\lambda_{11}\lambda_{22}}{\lambda_{12}\lambda_{21}} = \beta \in \overline{\mathbf{Q}}.$$

Use the strong four exponentials conjecture with

$$\lambda_{11}\lambda_{22} = \beta\lambda_{12}\lambda_{21}.$$

Consequence of the strong 4 exponentials Conjecture

Let λ_{ij} $(i=1,2,\ j=1,2)$ be four non zero logarithms of algebraic numbers.

Assume

$$\frac{\lambda_{11}}{\lambda_{12}} - \frac{\lambda_{21}}{\lambda_{22}} \in \overline{\mathbf{Q}}.$$

Then

• either $\lambda_{11}/\lambda_{12} \in \mathbf{Q}$ and $\lambda_{21}/\lambda_{22} \in \mathbf{Q}$

• or $\lambda_{12}/\lambda_{22} \in \mathbf{Q}$ and

$$\frac{\lambda_{11}}{\lambda_{12}} - \frac{\lambda_{21}}{\lambda_{22}} \in \mathbf{Q}.$$

Remark:

$$\frac{\lambda_{11}}{\lambda_{12}} - \frac{b\lambda_{11} - a\lambda_{12}}{b\lambda_{12}} = \frac{a}{b}.$$

Proof: Assume

$$\frac{\lambda_{11}}{\lambda_{12}} - \frac{\lambda_{21}}{\lambda_{22}} = \beta \in \overline{\mathbf{Q}}.$$

Use the strong four exponentials conjecture with

$$\lambda_{12}(\beta\lambda_{22} + \lambda_{21}) = \lambda_{11}\lambda_{22}.$$

Question: Let λ_{ij} $(i=1,2,\ j=1,2)$ be four non zero logarithms of algebraic numbers. Assume

$$\lambda_{11}\lambda_{22}-\lambda_{12}\lambda_{21}\in\overline{\mathbf{Q}}.$$

Deduce

$$\lambda_{11}\lambda_{22}=\lambda_{12}\lambda_{21}.$$

Question: Let λ_{ij} ($i = 1, 2$, $j = 1, 2$) be four non zero logarithms of algebraic numbers. Assume

$$\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21} \in \overline{\mathbf{Q}}.$$

Deduce

$$\lambda_{11}\lambda_{22} = \lambda_{12}\lambda_{21}.$$

Answer: *This is a consequence of the Conjecture on algebraic independence of logarithms of algebraic numbers.*

Consequences of the strong 6 exponentials Theorem

Let λ_{ij} $(i = 1, 2,\ \ j = 1, 2, 3)$ be six non zero logarithms of algebraic numbers. Assume

• $\lambda_{11}, \lambda_{21}$ are linearly independent over $\mathbf{Q}$

and

• $\lambda_{11}, \lambda_{12}, \lambda_{13}$ are linearly independent over $\mathbf{Q}$.

- *One at least of the two numbers*

$$\lambda_{12} - \frac{\lambda_{11}\lambda_{22}}{\lambda_{21}}, \quad \lambda_{13} - \frac{\lambda_{11}\lambda_{23}}{\lambda_{21}}$$

is transcendental.

- *One at least of the two numbers*

$$\frac{\lambda_{12}\lambda_{21}}{\lambda_{11}\lambda_{22}}, \quad \frac{\lambda_{13}\lambda_{21}}{\lambda_{11}\lambda_{23}}$$

is transcendental.

- *One at least of the two numbers*

$$\frac{\lambda_{12}}{\lambda_{11}} - \frac{\lambda_{22}}{\lambda_{21}}, \quad \frac{\lambda_{13}}{\lambda_{11}} - \frac{\lambda_{23}}{\lambda_{21}}$$

is transcendental.

- *Also one at least of the two numbers*

$$\frac{\lambda_{21}}{\lambda_{11}} - \frac{\lambda_{22}}{\lambda_{12}}, \quad \frac{\lambda_{21}}{\lambda_{11}} - \frac{\lambda_{23}}{\lambda_{13}}$$

is transcendental.

Replacing λ_{21} by 1.

- *One at least of the two numbers*

$$\lambda_{12}-\lambda_{11}\lambda_{22},\quad \lambda_{13}-\lambda_{11}\lambda_{23}$$

is transcendental.

- *The same holds for*

$$\frac{\lambda_{12}}{\lambda_{11}}-\lambda_{22},\quad \frac{\lambda_{13}}{\lambda_{11}}-\lambda_{23}.$$

is transcendental.

• *Finally one at least of the two numbers*

$$\frac{\lambda_{11}\lambda_{22}}{\lambda_{12}}, \quad \frac{\lambda_{11}\lambda_{23}}{\lambda_{13}}$$

is transcendental, and also one at least of the two numbers

$$\frac{1}{\lambda_{11}} - \frac{\lambda_{22}}{\lambda_{12}}, \quad \frac{1}{\lambda_{11}} - \frac{\lambda_{23}}{\lambda_{13}}.$$

is transcendental.

Missing:

One at least of the two numbers

$$\lambda_{12}\lambda_{21} - \lambda_{22}\lambda_{11}, \quad \lambda_{13}\lambda_{21} - \lambda_{23}\lambda_{11}$$

is transcendental ?

Theorem. *Let* λ_{ij} *(*$i=1,2,\ j=1,2,3,4,5$*) be ten non zero logarithms of algebraic numbers. Assume*

- $\lambda_{11}, \lambda_{21}$ *are linearly independent over* $\mathbf{Q}$

and

- $\lambda_{11}, \ldots, \lambda_{15}$ *are linearly independent over* $\mathbf{Q}$.

Then one at least of the four numbers

$$\lambda_{1j}\lambda_{21} - \lambda_{2j}\lambda_{11}, \quad (j = 2,3,4,5)$$

is transcendental.

Further related transcendence results

Theorem *(W.D. Brownawell, M. W., 1970) For $i = 1, 2$ and $j = 1, 2$, let α_{ij} be a non zero algebraic number and λ_{ij} a complex number satisfying $e^{\lambda_{ij}} = \alpha_{ij}$. Assume $\lambda_{11}, \lambda_{12}$ are linearly independent over $\mathbf{Q}$ and also $\lambda_{11}, \lambda_{21}$ are linearly independent over $\mathbf{Q}$. Then one at least of the following two statements holds*

- $$\lambda_{11}\lambda_{22} \neq \lambda_{12}\lambda_{21}$$

- *the field $\mathbf{Q}(\lambda_{11}, \lambda_{12}, \lambda_{21}\lambda_{22})$ has transcendence degree ≥ 2.*

Consequences

- One at least of the two numbers e^e, e^{e^2} is transcendental.

- For $\lambda \in \mathcal{L} \setminus \{0\}$, one at least of the two numbers e^{λ^2}, e^{λ^3} is transcendental.

- One at least of the two following statements is true:
  - the two numbers e and π are algebraically independent
  - the number e^{π^2} is transcendental.

- For $\lambda \in \mathcal{L} \setminus \{0\}$, one at least of the two following statements is true:
  - the two numbers e and λ are algebraically independent
  - the number e^{λ^2} is transcendental.

Generalization

Theorem *(D. Roy–M. W., 1995) Let* $Q \in \mathbf{Q}[X_1, \ldots, X_n]$ *be a homogeneous quadratic polynomial and* $\lambda_1, \ldots, \lambda_n$ *be elements in* $\mathcal{L}$ *such that*

$$Q(\lambda_1, \ldots, \lambda_n) = 0.$$

Assume the field $\mathbf{Q}(\lambda_1, \ldots, \lambda_n)$ *has transcendence degree* 1 *over* $\mathbf{Q}$*. Then the point* $(\lambda_1, \ldots, \lambda_n)$ *belongs to a linear subspace of* $\mathbf{C}^n$ *contained in the hypersurface* $Q = 0$*.*

Next step: investigate the transcendence of numbers

$$Q(\lambda_1, \ldots, \lambda_n).$$

Happy Birthday Professor Ramachandra!