January 23, 2019 Arba Minch University

Continued fractions: an introduction

Michel Waldschmidt

Sorbonne Université

Institut de Mathématiques de Jussieu

http://www.imj-prg.fr/~michel.waldschmidt/

Continued fractions

The Euclidean algorithm for computing the Greatest Common Divisor (gcd) of two positive integers is arguably the oldest mathematical algorithms : it goes back to antiquity and was known to Euclid. A closely related algorithm yields the continued fraction expansion of a real number, which is a very efficient process for finding the best rational approximations of a real number. Continued fractions is a versatile tool for solving problems related with movements involving two different periods. This is how it occurs in number theory, in complex analysis, in dynamical systems, as well as questions related with music, calendars, gears... We will quote some of these applications.

What is the connection between the following questions ?

How to construct a calendar ?

How to design a planetarium ?

How to build a musical scale ?

How to find two positive integers x, y such that

$$x^2 - 61y^2 = 1?$$

How to prove the irrationality of constants from analysis ?

Answer : continued fractions.

Number of days in a year

What is a year ? Astronomical year (Sidereal, tropical, anomalistic...).

A year is the orbital period of the Earth moving in its orbit around the Sun. For an observer on the Earth, this corresponds to the period it takes to the Sun to complete one course throughout the zodiac along the ecliptic.

A year is approximately 365.2422 days.

A good approximation is

$$365 + \frac{1}{4} = 365.25$$

with a leap year every 4 years. This is a little bit too much.

A better approximation is

$$365 + \frac{8}{33} = 365.2424\ldots$$

The Gregorian calendar

The Gregorian calendar is based on a cycle of 400 years : there is one leap year every year which is a multiple of 4 but not of 100 unless it is a multiple of 400.

It is named after Pope Gregory XIII, who introduced it in 1582.

The Gregorian calendar

In 400 years, in the Gregorian calendar, one omits 3 leap years, hence there are $365 \cdot 400 + 100 - 3 = 146\,097$ days.

Since $400 = 4(33 \cdot 3 + 1)$, in 400 years, the number of days counted with a year of $365 + \frac{8}{33}$ days is

$$\left(365 + \frac{8}{33}\right) \cdot 400 = 365 \cdot 400 + 3 \cdot 32 + \frac{32}{33} = 146\,096.9696\ldots$$

Further correction needed

In $10\,000$ years, the number of days in reality is

$$365.2422 \cdot 10\,000 = 3\,652\,422$$

while the number of days in the Gregorian calendar is

$$146\,097 \cdot 25 = 3\,652\,425.$$

Hence one should omit three more leap years every $10\,000$ years.

Approximating 365.2422

Write

$$365.2422 = 365 + \frac{1}{4.1288\ldots}$$

The first approximation is $365 + \frac{1}{4}$.

Next write

$$4.1288\cdots = 4 + \frac{1}{7.7628\ldots}.$$

The second approximation is

$$365 + \cfrac{1}{4 + \cfrac{1}{7}} = 365 + \frac{7}{29}.$$

Replacing 365.2422 with $365+\dfrac{8}{33}$

Next write

$$7.7628\cdots = 7+\frac{1}{1.3109\dots}$$

The third approximation is

$$365+\cfrac{1}{4+\cfrac{1}{7+\cfrac{1}{1}}} = 365+\cfrac{1}{4+\cfrac{1}{8}} = 365+\frac{8}{33}.$$

Using $1.3109\cdots = 1+\dfrac{1}{3.2162\dots}$, one could continue by writing

$$365.2422 = 365+\cfrac{1}{4+\cfrac{1}{7+\cfrac{1}{1+\cfrac{1}{3.2162}}}}$$

Continued fraction notation

Write

$$365.2422 = 365+\cfrac{1}{4+\cfrac{1}{7+\cfrac{1}{1+\cfrac{1}{3+\ddots}}}} = [365,4,7,1,3,\dots].$$

Third approximation :

$$[365,4,7,1] = [365,4,8] = 365+\cfrac{1}{4+\cfrac{1}{8}} = 365+\frac{8}{33}.$$

References on calendars

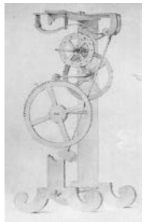
`Exercise` ; During 4000 years, the number of Fridays 13 is 6880, the number of Thursdays 13 is 6840 (and there are 6850 Mondays or Tuesdays 13).

V. Frederick Rickey, *Mathematics of the Gregorian Calendar*, The Mathematical Intelligencer **7** n°1 (1985) 53–56.

Jacques Dutka, *On the Gregorian revision of the Julian Calendar*, The Mathematical Intelligencer **10** n°1 (1988) 56–64.

Description and design of a planetarium

Automati planetarii of Christiaan Huygens (1629 –1695) astronomer, physicist, probabilist and horologist.
Huygens designed more accurate clocks than the ones available at the time. His invention of the pendulum clock was a breakthrough in timekeeping, and he made a prototype by the end of 1656.



Earth and Saturn

The ratio between the angle covered by the Earth and the angle covered by Saturn is

$$\frac{77\,708\,431}{2\,640\,858} = 29.425\,448\ldots$$

Not to scale !

Continued fraction of $77\,708\,431/2\,640\,858$

The ratio between the angle covered by the Earth and the angle covered by Saturn is

$$\frac{77\,708\,431}{2\,640\,858} = 29.425\,448\cdots = 29 + \cfrac{1}{2 + \cfrac{1}{2+\ldots}}\cdot$$

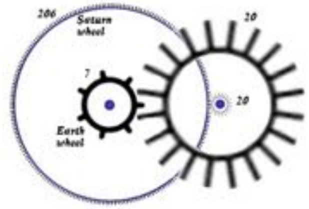
The continued fraction of this ratio is

$$[29, 2, 2, 1, 5, 1, 4, \ldots]$$

and

$$[29, 2, 2, 1] = 29 + \frac{3}{7} = \frac{206}{7}\cdot$$

$$\frac{206}{7} = 29.428\,5\ldots$$



http://plus.maths.org/content/chaos-numberland-secret-life-continued-fractions

The algorithm of continued fractions

Let $x \in \mathbf{R}$. Euclidean division of x by 1 :

$$x = \lfloor x \rfloor + \{x\} \quad \text{with } \lfloor x \rfloor \in \mathbf{Z} \text{ and } 0 \le \{x\} < 1.$$

If x is not an integer, then $\{x\} \ne 0$. Set $x_1 = \dfrac{1}{\{x\}}$, so that

$$x = \lfloor x \rfloor + \frac{1}{x_1} \quad \text{with } \lfloor x \rfloor \in \mathbf{Z} \text{ and } x_1 > 1.$$

If x_1 is not an integer, set $x_2 = \dfrac{1}{\{x_1\}}$:

$$x = \lfloor x \rfloor + \cfrac{1}{\lfloor x_1 \rfloor + \cfrac{1}{x_2}} \quad \text{with } x_2 > 1.$$

Continued fraction expansion

Set $a_0 = \lfloor x \rfloor$ and $a_i = \lfloor x_i \rfloor$ for $i \ge 1$.
Then :

$$x = \lfloor x \rfloor + \cfrac{1}{\lfloor x_1 \rfloor + \cfrac{1}{\lfloor x_2 \rfloor + \cfrac{1}{\ddots}}} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

the algorithm stops after finitely many steps if and only if x is rational.
We use the notation

$$x = [a_0,\ a_1,\ a_2,\ a_3, \ldots]$$

Remark : if $a_k \ge 2$, then
$[a_0,\ a_1,\ a_2,\ a_3, \ldots, a_k] = [a_0,\ a_1,\ a_2,\ a_3, \ldots, a_k - 1, 1]$.

Continued fractions and rational approximation

For
$$x = [a_0,\ a_1,\ a_2, \dots, a_k, \dots]$$
the rational numbers in the sequence
$$\frac{p_k}{q_k} = [a_0,\ a_1,\ a_2, \dots, a_k] \qquad (k = 1, 2, \dots)$$
give rational approximations for x which are the best ones when comparing the quality of the approximation and the size of the denominator.

$a_0, a_1, a_2, \dots$ are the *partial quotients*,

$\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2} \dots$ are the *convergents*.

Connection with the Euclidean algorithm

If x is rational, $x = \frac{p}{q}$, this process is nothing else than Euclidean algorithm of dividing p by q :

$$p = a_0 q + r_0, \quad 0 \le r_0 < q.$$

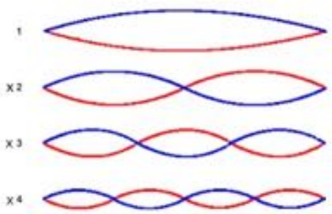
If $r_0 \ne 0$,
$$x_1 = \frac{q}{r_0} > 1.$$

Euclide :
(∼ -306, ∼ -283)

$$q = a_1 r_0 + r_1, \quad x_2 = \frac{r_0}{r_1}.$$

Harmonics

The successive harmonics of a note of frequency n are the vibrations with frequencies $2n$, $3n$, $4n$, $5n$, . . .



Octaves

The successive octaves of a note of frequency n are vibrations with frequencies $2n$, $4n$, $8n$, $16n$,. . .The ear recognizes notes at the octave one from the other.

Using octaves, one replaces each note by a note with frequency in a given interval, say $[n, 2n)$. The classical choice in Hertz is $[264, 528)$. For simplicity we take rather $[1, 2)$.

Hence a note with frequency f is replaced by a note with frequency r with $1 \le r < 2$, where
$$f = 2^a r, \qquad a = [\log_2 f] \in \mathbf{Z}, \quad r = 2^{\{\log_2 f\}} \in [1, 2).$$

This is a multiplicative version of the Euclidean division.

The fourth and the fifth

A note with frequency 3 (which is a harmonic of 1) is at the octave of a note with frequency $\frac{3}{2}\cdot$

The musical interval $\left[1, \frac{3}{2}\right]$ is called *fifth*, the ratio of the endpoints of the interval is $\frac{3}{2}\cdot$

The musical interval $\left[\frac{3}{2}, 2\right]$ is *the fourth*, with ratio $\frac{4}{3}\cdot$

The successive fifths

The successive fifths are the notes in the interval $[1, 2]$, which are at the octave of notes with frequency

$$1,\ 3,\ 9,\ 27,\ 81, \dots$$

namely :

$$1,\ \frac{3}{2},\ \frac{9}{8},\ \frac{27}{16},\ \frac{81}{64}, \dots$$

We shall never come back to the initial value 1, since the Diophantine equation $3^a = 2^b$ has no solution in positive integers a, b.

Exponential Diophantine equations

We cannot solve exactly the equation $2^a = 3^b$ in positive rational integers a and b, but we can look for powers of 2 which are close to powers of 3.

There are just three solutions (m, n) to the equation $3^m - 2^n = \pm 1$ in positive integers m and n, namely

$$3 - 2 = 1, \quad 4 - 3 = 1, \quad 9 - 8 = 1.$$

Exponential Diophantine equations

Levi ben Gershon (1288–1344), a medieval Jewish philosopher and astronomer, answering a question of the French composer Philippe de Vitry, proved that $3^m - 2^n \neq 1$ when $m \geq 3$.

Levi ben Gershon

Philippe de Vitry

$3^m - 2^n \neq 1$ when $m \geq 3$.

Assume $3^m - 2^n = 1$ with $m \geq 3$. Then $n \geq 2$, which implies that $3^m \equiv 1 \bmod 4$, whence m is even.

Writing $m = 2k$, we obtain

$$(3^k - 1)(3^k + 1) = 2^n,$$

which implies that both $3^k - 1$ and $3^k + 1$ are powers of 2.

But the only powers of 3 which differ by 2 are 1 and 3.

Hence $k = 1$, contradicting the assumption $m \geq 3$.

Diophantine equations

This question leads to the study of so-called *exponential Diophantine equations*, which include the Catalan's equation $x^p - y^q = 1$ where x,y, p and q are unknowns in $\mathbf{Z}$ all ≥ 2 (this was solved recently ; the only solution is $3^2 - 2^3 = 1$, as suggested in 1844 by E. Catalan, the same year when Liouville produced the first examples of transcendental numbers).

Eugène Charles Catalan
(1814 – 1894)

Approximating 3^a by 2^b

Instead of looking at Diophantine equations, one can consider rather the question of approximating 3^a by 2^b from another point of view. The fact that the equation $3^a = 2^b$ has no solution in positive integers a, b means that the logarithm in basis 2 of 3 :

$$\log_2 3 = \frac{\log 3}{\log 2} = 1.58496250072\ldots,$$

which is the solution x of the equation $2^x = 3$, is irrational.

Powers of 2 which are close to powers of 3 correspond to rational approximations $\frac{a}{b}$ to $\log_2 3$:

$$\log_2 3 \simeq \frac{a}{b}, \qquad 2^a \simeq 3^b.$$

Approximating $\log_2 3$ by rational numbers

Hence it is natural to consider the continued fraction expansion

$$\log_2 3 = 1.58496250072\cdots = [1,\ 1,\ 1,\ 2,\ 2,\ 3,\ 1,\ 5,\ \ldots]$$

and to truncate this expansion :

$$[1] = 1,\ [1,\ 1] = 2,\ [1,\ 1,\ 1] = \frac{3}{2},\ [1,\ 1,\ 1,\ 2] = \frac{8}{5} = 1.6.$$

The approximation of $\log_2 3 = 1.58\ldots$ by $\frac{8}{5} = 1.6$ means that

$$2^8 = 256 \quad \text{is not too far from} \quad 3^5 = 243.$$

The number $\left(\frac{3}{2}\right)^5 = 7.593\ldots$ is close to 2^3 ; this means that 5 *fifths produce almost to* 3 *octaves*.

Approximating $\log_2 3$ by rational numbers

The next approximation is

$$[1,\ 1,\ 1,\ 2,\ 2] = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2}}}} = \frac{19}{12} = 1.5833\dots$$

It is related to the fact that 2^{19} is close to 3^{12} :

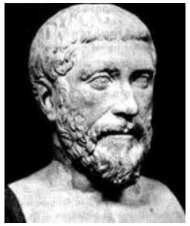
$$2^{19} = 524\,288 \simeq 3^{12} = 531\,441,$$

$$\left(\frac{3}{2}\right)^{12} = 129.74\dots \text{ is close to } 2^7 = 128.$$

In music it means that *twelve fifths is a bit more than seven octaves*.

Pythagoras

Pythagoras of Samos
(about 569 BC - about 475 BC)



The *comma of Pythagoras* is

$$\frac{3^{12}}{2^{19}} = 1.01364$$

It produces an error of about 1.36%, which most people cannot ear.

Further remarkable approximations

$5^3 = 125 \simeq 2^7 = 128$ $\qquad \left(\frac{5}{4}\right)^3 = 1.953\dots \simeq 2$

Three thirds (ratio $5/4$) produce almost one octave.

$\boxed{2^{10} = 1024 \simeq 10^3}$

• Computers (kilo octets)

• Acoustic : multiplying the intensity of a sound by 10 means adding 10 decibels (logarithmic scale).
Multiplying the intensity by k, amounts to add d decibels with $10^d = k^{10}$.
Since $2^{10} \simeq 10^3$, doubling the intensity, is close to adding 3 decibels.

Leonardo Pisano (Fibonacci)

Fibonacci sequence $(F_n)_{n\geq 0}$

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,$$

$$34,\ 55,\ 89,\ 144,\ 233,\dots$$

is defined by

$$F_0 = 0,\ F_1 = 1,$$

$$F_n = F_{n-1} + F_{n-2} \quad (n \geq 2).$$

`http://oeis.org/A000045`

Leonardo Pisano (Fibonacci)
(1170–1250)

Fibonacci sequence and Golden Ratio

The developments

$$[1], \quad [1,1], \quad [1,1,1], \quad [1,1,1,1], \quad [1,1,1,1,1], \quad [1,1,1,1,1,1], \dots$$

are the quotients

$$\begin{array}{cccccc} \frac{F_2}{F_1} & \frac{F_3}{F_2} & \frac{F_4}{F_3} & \frac{F_5}{F_4} & \frac{F_6}{F_5} & \frac{F_7}{F_6} \\ \| & \| & \| & \| & \| & \| \\ \frac{1}{1} & \frac{2}{1} & \frac{3}{2} & \frac{5}{3} & \frac{8}{5} & \frac{13}{8} \end{array} \quad \dots$$

of consecutive Fibonacci numbers.
The development $[1,1,1,1,1,\dots]$ is the continued fraction expansion of the *Golden Ratio*

$$\Phi = \frac{1+\sqrt{5}}{2} = \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = 1.6180339887499\dots$$

which satisfies

$$\Phi = 1 + \frac{1}{\Phi}.$$

Continued fraction of $\sqrt{2}$

The number

$$\sqrt{2} = 1.414\,213\,562\,373\,095\,048\,801\,688\,724\,209\,\dots$$

satisfies

$$\sqrt{2} = 1 + \frac{1}{\sqrt{2}+1}.$$

For instance

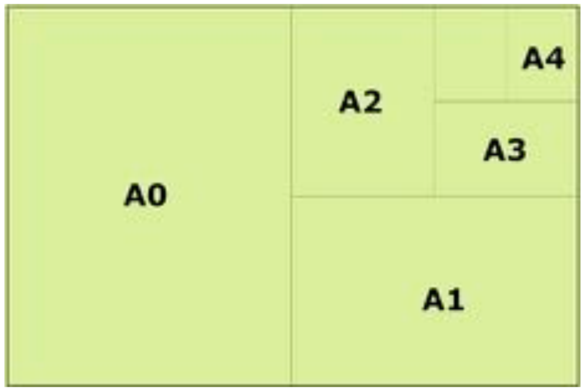
$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{\sqrt{2}+1}} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}$$

We write the continued fraction expansion of $\sqrt{2}$ using the shorter notation

$$\sqrt{2} = [1,\ 2,\ 2,\ 2,\ 2,\ 2,\ \dots] = [1,\ \overline{2}].$$

A4 format

The number $\sqrt{2}$ is twice its inverse : $\sqrt{2} = 2/\sqrt{2}$.
Folding a rectangular piece of paper with sides in proportion $\sqrt{2}$ yields a new rectangular piece of paper with sides in proportion $\sqrt{2}$ again.



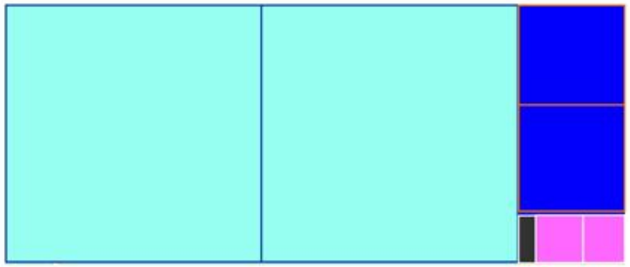
Irrationality of $\sqrt{2}$: geometric proof

• Start with a rectangle have side length 1 and $1+\sqrt{2}$.
• Decompose it into two squares with sides 1 and a smaller rectangle of sides $1+\sqrt{2}-2 = \sqrt{2}-1$ and 1.
• This second small rectangle has side lenghts in the proportion

$$\frac{1}{\sqrt{2}-1} = 1+\sqrt{2},$$

which is the same as for the large one.
• Hence the second small rectangle can be split into two squares and a third smaller rectangle, the sides of which are again in the same proportion.
• This process does not end.

Rectangles with proportion $1+\sqrt{2}$



Irrationality of $\sqrt{2}$: geometric proof

If we start with a rectangle having integer side lengths, then this process stops after finitely may steps (the side lengths of the successive rectangles produces a decreasing sequence of positive integers).

Also, for a rectangle with side lengths in a rational proportion, this process stops after finitely may steps (reduce to a common denominator and scale).

Hence $1+\sqrt{2}$ is an irrational number, and $\sqrt{2}$ also.

Geometric proof of irrationality

Set $t=\sqrt{2}+1=2.414\ 213\ 56\ \ldots$ The continued fraction expansion of t is

$$[2,2,\ldots]=[\overline{2}].$$

Indeed, from

$$\sqrt{2}-1=\frac{1}{\sqrt{2}+1},$$

we deduce

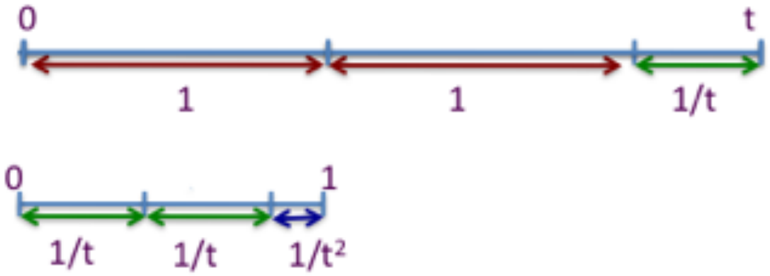
$$t=2+\frac{1}{t}\cdot$$

Decompose an interval of length t into three intervals, two of length 1 and one of length $1/t$.

$t=\sqrt{2}+1=2.414\ 213\ 56\ \cdots=2+1/t$

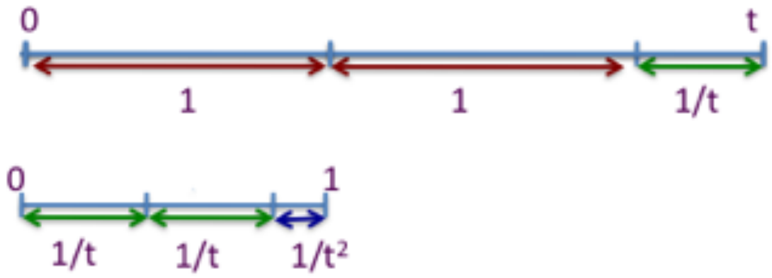
Decompose the interval of length 1

$$2+\frac{1}{t}=t, \qquad \frac{2}{t}+\frac{1}{t^2}=1.$$

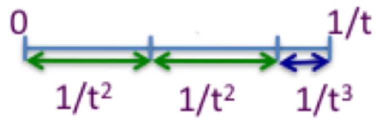


We go from the first picture to the second by a homothety $1/t$, because $t = 2 + 1/t$.

New homothety $1/t$



Intervalle $(0, 1/t)$ enlarged : $\quad \frac{2}{t^2}+\frac{1}{t^3}=\frac{1}{t}$:



Geometric proof of irrationality

An interval of length $t = \sqrt{2}+1$ is decomposed into two intervals of length 1 and one of length $1/t$.

After a homothety $1/t$, the interval of length 1 is decomposed into two intervals of length $1/t$ and one of length $1/t^2$.

The next step is to decompose an interval of length $1/t$ into two intervals of length $1/t^2$ and one of length $1/t^3$.

Next, an interval of length $1/t^2$ produces two intervals of length $1/t^3$ and one of length $1/t^4$.

At each step we get two large intervals and a small one. The process does not stop.

Decomposition of a rational interval

Start with a rational number $u = a/b$, say $a > b > 0$ with a and b integers.

Decompose an interval of length u into a whole number of intervals of length 1 plus an interval of length less than 1.

It is convenient to scale : it amounts to the same to decompose an interval of length a into a whole number of intervals of length b, plus an interval of length less than b, say c, which is an integer ≥ 0.
This is the Euclidean division, again.

The process stops after finitely many steps.

The Golden Ratio

The Golden Ratio

$$\Phi = \frac{1+\sqrt{5}}{2} = 1.6180339887499\ldots$$

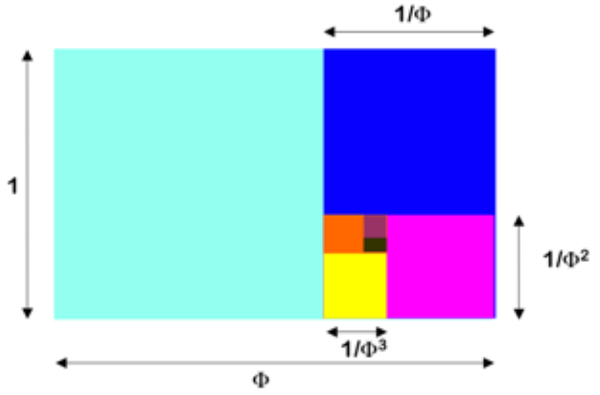
satisfies

$$\Phi = 1 + \frac{1}{\Phi}\cdot$$

If we start from a rectangle with the Golden ratio as proportion of length sides, at each step we get a square and a smaller rectangle with the same proportion for the length sides.

The Golden Ratio $(1+\sqrt{5})/2 = 1.6180339887499\ldots$

Golden Rectangle



Leonard Euler (1707 – 1783)

Leonhard Euler
De fractionibus continuis dissertatio,
Commentarii Acad. Sci. Petropolitanae,
9 (1737), 1744, p. 98–137 ;
Opera Omnia Ser. I vol. **14**,
Commentationes Analyticae, p. 187–215.

$$\mathrm{e} = \lim_{n\to\infty}(1+1/n)^n$$
$$= 2.718\,281\,828\,459\,045\,235\,360\,287\,471\,352\ldots$$

Continued fraction expansion for e

$$\mathrm{e} = 2 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{\ddots}}}}}}$$
$$= [2\,,\ 1,\ 2,\ 1,\ 1,\ 4,\ 1,\ 1,\ 6, \ldots]$$
$$= [2, \overline{1,\ 2m,\ 1}]_{m\ge 1}.$$

e is neither rational (J-H. Lambert, 1766) nor quadratic irrational (J-L. Lagrange, 1770).

Lagrange (1736 – 1813)

Joseph-Louis Lagrange was an Italian-born French mathematician who excelled in all fields of analysis and number theory and analytic and celestial mechanics.

Irrationality of π

Johann Heinrich Lambert (1728 – 1777)
Mémoire sur quelques propriétés remarquables des quantités transcendantes circulaires et logarithmiques,
Mémoires de l'Académie des Sciences de Berlin, **17** (1761), p. 265-322 ;
lu en 1767 ; Math. Werke, t. II.

$\tan(v)$ is irrational when $v \neq 0$ is rational.
Hence π is irrational, since $\tan(\pi/4) = 1$.

Lambert and King Frédérick II

— Que savez vous, Lambert ?
— Tout, Sire.
— Et de qui le tenez–vous ?
— De moi-même !

Continued fraction of π

The development of $\pi = 3.1415926535898\ldots$ starts with

$$\pi = [3,\ 7,\ 15,\ 1,\ 292,\ 1,\ 1,\ 1,\ 2,\ 1,\ 3,\ 1,\ 14,\ 2,\ 1,\ 1, \ldots]$$

Open problem : is the sequence of partial quotients bounded ?

Continued fraction expansion for $\mathrm{e}^{1/a}$

Starting point : $y = \tanh(x/a)$ satisfies the differential equation $ay' + y^2 = 1$.
This leads Euler to

$$\mathrm{e}^{1/a} = [1,\ a-1,\ 1,\ 1,\ 3a-1,\ 1,\ 1,\ 5a-1, \dots] = [1,\ \overline{(2m+1)a-1,\ 1}]_{m\geq 0}.$$

Continued fraction expansion of $\tan(x)$

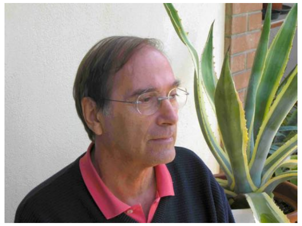
$$\tan(x) = \frac{1}{i}\tanh(ix), \qquad \tanh(x) = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{\mathrm{e}^x + \mathrm{e}^{-x}}.$$

$$\tan(x) = \cfrac{x}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{5 - \cfrac{x^2}{7 - \cfrac{x^2}{9 - \ddots}}}}}.$$

Padé approximants

Henri Eugène Padé,
1863 – 1953

Claude Brezinski
History of Continued Fractions and Padé Approximants.
Springer-Verlag, Berlin, 1991,
551 pages.



http://math.univ-lille1.fr/~brezinsk/

Diophantine approximation in the real life

Calendars : bissextile years

Spokes

Small divisors and dynamical systems (H. Poincaré)

Periods of Saturn orbits (Cassini divisions)

Chaotic systems.

Stability of the solar system. Expansion of the universe.

General theory of relativity. Cosmology. Black holes.

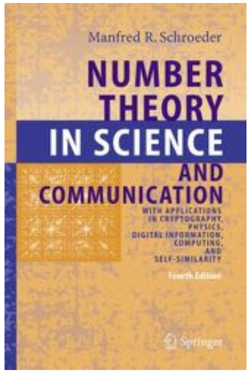
Resonance in astronomy

Quasi-cristals

Acoustic of concert halls

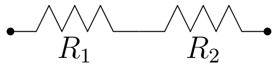
Number Theory in Science and communication

M.R. Schroeder.
Number theory in science and communication :
with applications in cryptography, physics, digital information, computing and self similarity
Springer series in information sciences **7** 1986.
4th ed. (2006) 367 p.



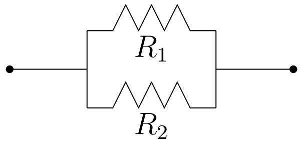
Electric networks

- The resistance of a network in series



is the sum $R_1 + R_2$.

- The resistance R of a network in parallel

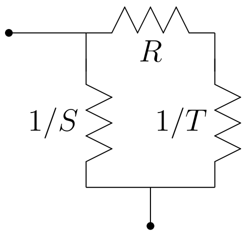


satisfies

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

Electric networks and continued fractions

The resistance U of the circuit

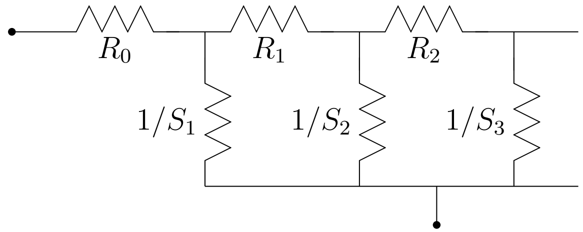


is given by

$$U = \cfrac{1}{S + \cfrac{1}{R + \cfrac{1}{T}}}$$

Decomposition of a square in squares

- For the network

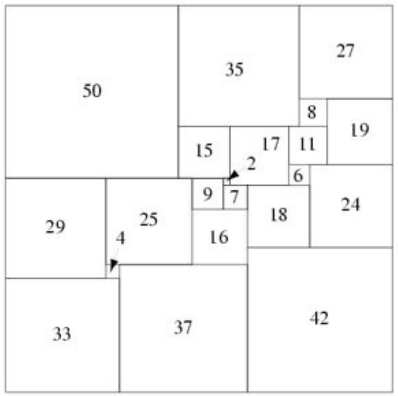


the resistance is given by a continued fraction expansion

$$[R_0, S_1, R_1, S_2, R_2, \dots]$$

- Electric networks and continued fractions have been used to find the first solution to the problem of decomposing an integer square into a disjoint union of integer squares, all of which are distinct.

Squaring the square



21-*square perfect square*

There is a unique simple perfect square of order 21 (the lowest possible order), discovered in 1978 by A. J. W. Duijvestijn (Bouwkamp and Duijvestijn 1992). It is composed of 21 squares with total side length 112, and is illustrated above.

Quadratic numbers

The continued fraction expansion of a real number is ultimately periodic if and only if the number is a quadratic number, that means root of a degree 2 polynomial with rational coefficients.

For a positive integer d which is not a square, the continued fraction expansion of the number $\sqrt{d}$ is

$$\sqrt{d} = [a_0,\ a_1,\ a_2,\ \dots,\ a_k,\ a_1,\ a_2,\ \dots,\ a_k,\ a_1,\ a_2,\ \dots],$$

which we write for simplicity

$$\sqrt{d} = [a_0,\ \overline{a_1,\ a_2,\ \dots,\ a_k}].$$

Hence $\sqrt{2} = [1,\ \overline{2}]$ and $\sqrt{3} = [1,\ \overline{1,\ 2}]$.

Connexion with the equation $x^2 - dy^2 = \pm 1$

Let d be a positive integer which is not a square. Consider the Diophantine equation

$$(1) \qquad x^2 - dy^2 = \pm 1$$

where the unknowns x, y take their values in $\mathbf{Z}$.

If (x, y) is a solution with $y \geq 1$, then $(x - \sqrt{d}y)(x + \sqrt{d}y) = 1$, hence $\frac{x}{y}$ is a rational approximation of $\sqrt{d}$ and this approximation is sharper when x is larger.

This is why a strategy for solving Pell's equation (1) is based on the continued fraction expansion of $\sqrt{d}$.

Problem of Brahmagupta (628)

Brahmasphutasiddhanta :

Solve in integers the equation

$$x^2 - 92y^2 = 1$$

Brahmagupta

If (x, y) is a solution, then $(x - \sqrt{92}y)(x + \sqrt{92}y) = 1$, hence $\frac{x}{y}$ is a good approximation of $\sqrt{92} = 9.591663046625\dots$.

Problem of Brahmagupta (628)

The continued fraction expansion of $\sqrt{92}$ is

$$\sqrt{92} = [9, \overline{1,\ 1,\ 2,\ 4,\ 2,\ 1,\ 1,\ 18}].$$

A solution of

$$x^2 - 92y^2 = 1$$

is obtained from

$$[9,\ 1,\ 1,\ 2,\ 4,\ 2,\ 1,\ 1] = \frac{1151}{120}.$$

Indeed $1151^2 - 92 \cdot 120^2 = 1\,324\,801 - 1\,324\,800 = 1.$

Bhaskara II (12th Century)

Lilavati

(*Bijaganita*, 1150)
$x^2 - 61y^2 = 1$ Solution :
$x = 1\,766\,319\,049$,
$y = 226\,153\,980$.

Cyclic method (Chakravala) of Brahmagupta.

$$\sqrt{61} = [7, \overline{1,\ 4,\ 3,\ 1,\ 2,\ 2,\ 1,\ 3,\ 4,\ 1,\ 14}]$$

$$[7,1,4,3,1,2,2,1,3,4,1,14,1,4,3,1,2,2,1,3,5] = \frac{1\,766\,319\,049}{226\,153\,980}$$

Narayana (14th Century)

Narayana cows
(Tom Johnson)

$x^2 - 103y^2 = 1$. Solution :

$x = 227\,528$, $y = 22\,419$.

$227\,528^2 - 103 \cdot 22\,419^2 = 51\,768\,990\,784 - 51\,768\,990\,783 = 1.$

$$\sqrt{103} = [10, \overline{6,\ 1,\ 2,\ 1,\ 1,\ 9,\ 1,\ 1,\ 2,\ 1,\ 6,\ 20}]$$

$$[10, 6,\ 1,\ 2,\ 1,\ 1,\ 9,\ 1,\ 1,\ 2,\ 1,\ 6] = \frac{227\,528}{22\,419}$$

Correspondence from Fermat to Brouncker

"*pour ne vous donner pas trop de peine*" (Fermat)
" *to make it not too difficult*"

$$X^2 - DY^2 = 1, \text{ with } D = 61 \text{ and } D = 109.$$

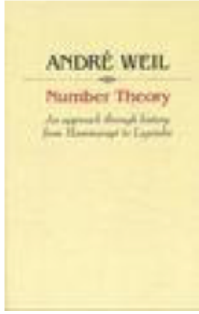
Solutions respectively :

$$(1\,766\,319\,049\ ,\ 226\,153\,980)$$
$$(158\,070\,671\,986\,249,\ 15\,140\,424\,455\,100)$$

$$158\,070\,671\,986\,249\ +\ 15\,140\,424\,455\,100\sqrt{109} = \left(\frac{261 + 25\sqrt{109}}{2}\right)^6.$$

A reference on the History of Numbers

André Weil
Number theory. :
An approach through history. From Hammurapi to Legendre.
Birkhäuser Boston, Inc., Boston, Mass., (1984) 375 pp.
MR 85c:01004

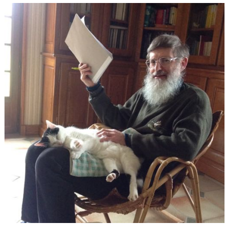


Farey dissection

John Farey
(1766 –1826) geologist

http://www-history.mcs.st-andrews.ac.uk/PictDisplay/Farey.html

Patrice Philippon.
A Farey Tail.
Notices of the AMS Volume **59**, (6), 2012, 746 – 757.



Riemannian varieties with negative curvature

The study of the so-called Pell-Fermat Diophantine equation yield the construction of Riemannian varieties with negative curvature : *arithmetic varieties.*

Nicolas Bergeron (Paris VI) :
"Sur la topologie de certains espaces provenant de constructions arithmétiques"

Ergodic theory

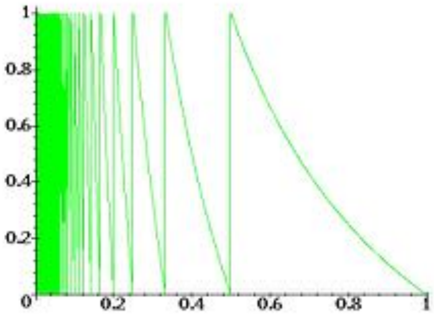
Torus of dimension 1 :

$$\mathbf{R}/\mathbf{Z} \simeq S^1$$

Gauss map

$$T : x \mapsto \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$$

Deterministic chaotic dynamical system.



This transformation is ergodic : any subset E of $[0, 1)$ such that $T^{-1}(E) \subset E$ has measure 0 or 1.

Birkhoff ergodic Theorem

Let T be an ergodic endomorphism of the probability space X and let $f : X \to \mathbf{R}$ be a real-valued measurable function. Then for almost every x in X, we have

$$\frac{1}{n}\sum_{j=1}^{n} f \circ T^j(x) \longrightarrow \int f \mathrm{d}m$$

as $n \to \infty$.

George David Birkhoff (1884–1944)

Connection with the Riemann zeta function

For s of real part > 1,

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}.$$

We have also

$$\zeta(s) = \frac{1}{s-1} - s\int_0^1 T(x)x^{s-1}\mathrm{d}x$$

with

$$T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$$

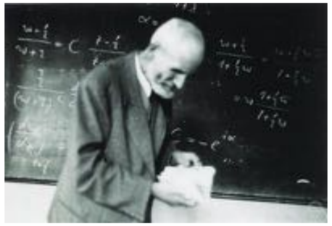
Generalization of the continued fraction expansion in higher dimension

Simultaneous rational approximation to real numbers is much more difficult than the rational approximation theory for a single number.

The continued fraction expansion algorithms has many specific features, so far there is no extension of this algorithm in higher dimension with all such properties.

Jacobi – Perron

Partial answers are known, like the Jacobi – Perron algorithm



Carl Gustav Jacob Jacobi (1804 –1851)

Oskar Perron (1880 – 1975) *Die Lehre von den Kettenbrüchen*, 1913.

Geometry of numbers

The geometry of numbers studies convex bodies and integer vectors in n-dimensional space. The geometry of numbers was initiated by Hermann Minkowski (1864 – 1909).

The LLL algorithm

Given a basis of $\mathbf{R}^n$, the LLL algorithm produces a basis of the lattice they generate, most often with smaller norm than the initial one.

Arjen Lenstra Hendrik Lenstra Laszlo Lovasz

Parametric geometry of numbers

Recent work by

Wolfgang M. Schmidt and Leo Summerer, Damien Roy

January 23, 2019 Arba Minch University

Continued fractions: an introduction

Michel Waldschmidt

Sorbonne Université

Institut de Mathématiques de Jussieu

http://www.imj-prg.fr/~michel.waldschmidt/