A course on interpolation

First Course : Introduction

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Abstract

The very first interpolation problem for analytic functions is to ask for an entire function f, the derivatives of which at the origin are given; the solution is given by the Taylor series. There are many other ways of interpolating analytic functions: in place of $f^{(n)}(0)$, we can consider $f(n)$, or $f^{(n)}(n)$ for instance. Lagrange interpolation polynomials involve the values of the function at several points; some derivatives may be included. The main goal of these lectures will be to consider further examples with Lidstone, Whittaker, Poritsky and Gontcharoff interpolation problems.

This first course includes the necessary background on entire functions, with Cauchy's inequalities, the order and type of an entire function, Stirling's and Jensen's formulae, Schwarz Lemma, Laplace transform, Weierstrass and Hadamard infinite products, Newton interpolation, Hermite identity, Abel interpolation.

Entire functions

An *entire function* is a function $\mathbb{C} \to \mathbb{C}$ which is analytic ($=$ holomorphic) in $\mathbb{C}$.

Examples are : polynomials, the exponential function

$$\mathrm{e}^z = \sum_{n \geq 0} \frac{z^n}{n!},$$

trigonometric functions $\sin z$, $\cos z$, $\sinh z$, $\cosh z \ldots$

An entire function which is not a polynomial is *transcendental*.

The interpolation problem

The graph $\{(z, f(z)) \mid z \in \mathbb{C}\} \subset \mathbb{C}^2$ of an entire function has the power of continuum.

However, such a function is uniquely determined by a countable set; for instance by the sequence of coefficients of its Taylor series at a given point z_0:

$$f(z) = \sum_{n \geq 0} f^{(n)}(z_0) \frac{(z - z_0)^n}{n!}.$$

Notation:

$$f^{(n)}(z) = \frac{\mathrm{d}^n}{\mathrm{d}z^n} f(z).$$

There are other sequences of numbers which determine uniquely an entire function, at least if we restrict to some classes of entire functions.

Interpolation data

Given complex numbers $\{\sigma_i\}_{i\in I}$, $\{a_i\}_{i\in I}$ and nonnegative integers $\{k_i\}_{i\in I}$, the *interpolation problem* is to decide whether there exists an analytic function f satisfying

$$f^{(k_i)}(\sigma_i) = a_i \text{ for all } i \in I.$$

We will consider this question for f analytic everywhere in $\mathbb{C}$ (i.e. f is an entire function) and $I = \mathbb{N}$.

The unicity is given by the answer to the same question with $a_i = 0$ for all $i \in I$ and requesting $f \neq 0$.

First example: Taylor series

Taylor series: data: $\sigma_n = 0$ and $k_n = n$ for all $n \geq 0$. The solution of the interpolation problem, if it exists, is unique and given with

$$f(z) = \sum_{n \geq 0} a_n \frac{z^n}{n!}, \qquad f^{(n)}(0) = a_n.$$

The polynomials $\frac{z^n}{n!}$ satisfy

$$\frac{\mathrm{d}^m}{\mathrm{d}z^m} \left(\frac{z^n}{n!} \right) (0) = \delta_{mn} = \begin{cases} 1 & \text{for } m = n, \\ 0 & \text{for } m \neq n, \end{cases}$$

for all $m, n \geq 0$.

Calculus of finite differences

Another classical interpolation problem is given by the data $k_n = 0$ and $\sigma_n = n$ for all $n \geq 0$. Given complex numbers a_n, does there exist an entire function f satisfying

$$f(n) = a_n \text{ for all } n \geq 0?$$

The answer depends on the growth of the sequence $(a_n)_{n\geq 0}$. The example of the function $\sin(\pi z)$ shows that the solution is not unique in general. However we recover unicity by adding a condition on the growth of the solution f.
For the existence, one uses interpolation formulae based on

$$f(z) = f(0) + z f_1(z), \quad f_1(z) = f_1(1) + (z-1) f_2(z),$$
$$f_n(z) = f_n(n) + (z-n) f_{n+1}(z), \quad \dots$$

Lagrange interpolation

Let $\alpha_1, \ldots, \alpha_s$ be pairwise distinct complex numbers. For $1 \leq \sigma_0 \leq s$, the polynomial

$$A_{\sigma_0} = \prod_{\substack{1 \leq \sigma \leq s \\ \sigma \neq \sigma_0}} \frac{z - \alpha_\sigma}{\alpha_{\sigma_0} - \alpha_\sigma}$$

Joseph-Louis Lagrange

(1736 – 1813)

satisfies

$$A_{\sigma_0}(\alpha_\sigma) = \begin{cases} 1 & \text{if } \sigma = \sigma_0, \\ 0 & \text{for } \sigma \neq \sigma_0,\ 1 \leq \sigma \leq s. \end{cases}$$

Edward Waring (1779), Leonhard Euler (1783), Lagrange (1795).

Lagrange–Hermite interpolation

Let $\alpha_1, \ldots, \alpha_n$ be pairwise distinct complex numbers and $t_1, \ldots, t_n$ be positive integers. There exist polynomials
$A_{i\nu}(z)$
($1 \leq i \leq n$, $0 \leq \nu < t_i$)
such that

Charles Hermite

(1822 – 1901)

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\tau} A_{i\nu}(\alpha_j) = \delta_{\tau,\nu}\delta_{ij}$$

for $1 \leq i, j \leq n$, $0 \leq \tau, \nu < t_i$.

Integral formulae by Hermite:
`Reference:` S. Lang, Complex analysis, Chap. XII § 4.

Abel interpolation

Given a sequence $(a_n)_{n\geq 0}$ of complex numbers, is there an entire function f such that $f^{(n)}(n) = a_n$ for all $n \geq 0$?

Solution: start with $a_n = \delta_{mn}$ for all $m \geq 0$: for each $n \geq 0$, ask for a polynomial P_n satisfying

$$P_n^{(m)}(m) = \delta_{mn} \text{ for } m \geq 0.$$

Niels Henrik Abel

(1802 – 1829)

Answer:

$$f(z) = \sum_{n\geq 0} f^{(n)}(n) P_n(z).$$

Further interpolation problems

We are going to consider the following interpolation problems:

- (Lidstone):

$$f^{(2n)}(0) = a_n, \quad f^{(2n)}(1) = b_n \text{ for } n \geq 0.$$

- (Whittaker):

$$f^{(2n+1)}(0) = a_n, \quad f^{(2n)}(1) = b_n \text{ for } n \geq 0.$$

- (Poritsky): For $m \geq 2$ and $\sigma_0, \ldots, \sigma_{m-1}$ in $\mathbb{C}$,

$$f^{(mn)}(\sigma_j) = a_{nj} \text{ for } n \geq 0 \quad \text{and} \quad j = 0, 1, \ldots, m-1.$$

- (Gontcharoff): For $(\sigma_n)_{n \geq 0}$ a sequence of complex numbers,

$$f^{(n)}(\sigma_n) = a_n \text{ for } n \geq 0.$$

Applications of interpolation

- Numerical analysis: numerical integration, differential equations, approximation of functions, boundary value problem, scientific calculus.

- Actuariat, statistics, electrical engineering, physics, computer science.

- Cryptography, Shamir's Secret Sharing.

- Number theory; integer valued entire functions of finite exponential type, transcendence theory.

Entire functions

Analytic functions, Taylor series

$$f(z) = \sum_{n \geq 0} a_n z^n,$$

$$a_n = \frac{1}{n!} f^{(n)}(0) = \frac{1}{2i\pi} \int_{|z|=r} f(z) \frac{\mathrm{d}z}{z^{n+1}}.$$

The zeroes of a nonzero entire function are isolated.
Multiplicity of a zero.
Number of zeroes in a compact counting multiplicities.
An entire function f has no zero if and only if $f(z) = \mathrm{e}^{g(z)}$:

$$g(z) = \log f(0) + \int_0^z \frac{f'(\zeta)}{f(\zeta)} \mathrm{d}\zeta.$$

Order and type of entire functions

For $\varrho \in \mathbb{Z}$, $\varrho \geq 0$, the function e^{z^ϱ} is an entire function of order ϱ.
For $\tau \in \mathbb{C}$, $\tau \neq 0$, the function $\mathrm{e}^{\tau z}$ is an entire function of order 1 and exponential type $|\tau|$.

For $t \in \mathbb{C}$, $t \neq 0$, the function

$$\sin(tz/\pi) = \frac{\mathrm{e}^{itz} - \mathrm{e}^{-itz}}{2i}$$

has order 1 and exponential type $|t|$.

Order and type of entire functions

Maximum modulus principle:

$$|f|_r := \sup_{|z|=r} |f(z)| = \sup_{|z|\le r} |f(z)|.$$

The order of an entire function f is

$$\varrho(f) := \limsup_{r\to\infty} \frac{\log\log|f|_r}{\log r},$$

while the exponential type of an entire function is

$$\tau(f) := \limsup_{r\to\infty} \frac{\log|f|_r}{r}.$$

Order and type of entire functions

If the exponential type is finite, then f has order ≤ 1.
If f has order < 1, then the exponential type is 0.

Examples:
A polynomial has order 0, hence exponential type 0.
The function e^{z^2} has order 2, hence infinite exponential type.
The function $\mathrm{e}^{\mathrm{e}^z}$ has infinite order, hence infinite exponential type.

Entire functions of finite exponential type

Proposition.

The exponential type of an entire function is also given by

$$\tau(f) = \limsup_{n\to\infty} |f^{(n)}(z_0)|^{1/n} \quad (z_0 \in \mathbb{C}).$$

The proof rests on Cauchy's estimate for the coefficients of the Taylor series and on Stirling's formula for $n!$.

Example:

$$(\mathrm{e}^{\tau z})^{(n)} = \tau^n \mathrm{e}^{\tau z}, \qquad \lim_{n\to\infty} |\tau^n \mathrm{e}^{\tau z}|^{1/n} = |\tau|.$$

The derivative of f has the same exponential type as f.

Cauchy's inequalities

Augustin-Louis Cauchy

(1789 – 1857)

Write
$$f(z) = \sum_{n \geq 0} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

For all $r > 0$, we have (Parseval)
$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + r\mathrm{e}^{i\theta})|^2 \mathrm{d}\theta = \sum_{n \geq 0} \frac{|f^{(n)}(z_0)|^2}{n!^2} r^{2n}.$$

From the upper bound
$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + r\mathrm{e}^{i\theta})|^2 \mathrm{d}\theta \leq |f|_{r+|z_0|}^2$$

we deduce, for all $n \geq 0$ and $r > 0$ (Cauchy's inequalities)
$$\frac{|f^{(n)}(z_0)|}{n!} r^n \leq |f|_{r+|z_0|}.$$

Stirling formula

James Stirling

(1692 – 1770)

$$n! \simeq n^n e^{-n} \sqrt{2\pi n} \text{ as } n \to \infty.$$

Stirling's approximation can be extended to the double inequality

$$\sqrt{2\pi} n^{n+1/2} e^{-n+\frac{1}{12n+1}} < n! < \sqrt{2\pi} n^{n+1/2} e^{-n+\frac{1}{12n}}.$$

Robbins, H. "A Remark of Stirling's Formula." Amer. Math. Monthly 62, 26-29, 1955.
Feller, W. "Stirling's Formula." § 2.9 in: An Introduction to Probability Theory and Its Applications, Vol. 1, 3rd ed. New York: Wiley, pp. 50-53, 1968.

http://mathworld.wolfram.com/StirlingsApproximation.html

Proof of the result on the type

Recall: the type of an entire function f is defined as

$$\tau(f) = \limsup_{r \to \infty} \frac{\log |f|_r}{r}.$$

Proposition.

For each $z_0 \in \mathbb{C}$*, we have*

$$\limsup_{n \to \infty} |f^{(n)}(z_0)|^{1/n} = \tau(f).$$

Zeros of a nonzero entire function

The set of zeros of a nonzero entire function f is a discrete subset of $\mathbb{C}$ (zeros are isolated).
If $\alpha_1, \ldots, \alpha_N$ are zeroes of f (counting multiplicities), then

$$f(z) \prod_{n=1}^{N} (z - \alpha_n)^{-1}$$

is an entire function.

Schwarz Lemma

Let f be analytic in an open set containing the closed disc $|z| \leq R$, with at least N zeroes (counting multiplicities) in a disc $|z| \leq r$ with $r \leq R$. Then

$$|f|_r \leq \left(\frac{R}{3r}\right)^{-N} |f|_R.$$

Herman Schwarz

(1843 - 1921)

Corollary.

There is no nonzero entire function of exponential type $< 1/3\mathrm{e}$ *vanishing on* $\mathbb{N}$.

Schwarz Lemma with Blaschke products

Exercise. Prove the stronger estimate

$$|f|_r \le \left(\frac{R^2 + r^2}{2rR}\right)^{-N} |f|_R$$

Wilhelm Blaschke

(1885 - 1962)

https://mathshistory.st-andrews.ac.uk/Biographies/Blaschke/

Corollary.

There is no nonzero entire function of exponential type $\le 1/2\mathrm{e}$ *vanishing on* $\mathbb{N}$.

Blaschke products

Exercise. Let $a \in \mathbb{C}$ and $R > 0$ satisfy $|a| \leq R$. For $z \in \mathbb{C}$ satisfying $a\overline{z} \neq R^2$, define

$$\varphi_a(z) = \frac{z - a}{R^2 - \overline{a}z}.$$

Then

$$|\varphi_a(z)| = \frac{1}{R} \quad \text{for} \quad |z| = R.$$

Further, for r in the range $|a| \leq r < R$, we have

$$\sup_{|z|=r} |\varphi_a(z)| = |\varphi_a(-ar/|a|)| = \frac{r + |a|}{R^2 + r|a|} \leq \frac{2r}{R^2 + r^2}.$$

Furthermore, for $|a| \leq r \leq R$ and $|z| = r$ with $z \neq -ar/|a|$, we have

$$|\varphi_a(z)| < \frac{r + |a|}{R^2 + r|a|}.$$

Jensen's Formula

Johan Jensen

(1859 – 1925)

For an analytic function in an open set containing the disc $|z| \leq r$, assuming $f(0) \neq 0$, we have

$$\log|f(0)| = -\sum_{f(\alpha_k)=0} \log\frac{r}{|\alpha_k|} + \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\mathrm{d}\theta.$$

Corollary.

There is no nonzero entire function of exponential type < 1 vanishing on $\mathbb{N}$.

`Hint` : an upper bound for the integral is $\log|f|_r$. Take $r = N$, $\alpha_k = k$, translate with ϵ so that $f(0) \neq 0$. The sum of $\log r$ gives $N \log N$, the sum of $\log|\alpha_k|$ gives $\log N!$.

Laplace Borel Transform

Pierre-Simon Laplace

(1749 – 1827)

Émile Borel

(1871 – 1956)

Fritz Carlson

(1888 – 1952)

Theorem (Carlson, 1914).

There is no nonzero entire function of exponential type $< \pi$ *vanishing on* $\mathbb{N}$.

The function $\sin(\pi z)$ has exponential type π and vanishes on $\mathbb{N}$.

Classical proof: rests on Phragmén–Lindelöf theorem.

Laplace Borel Transform

There is no nonzero entire function of exponential type $< \pi$ vanishing on $\mathbb{N}$.

`Sketch of proof`: assume let f be an entire function of exponential type $\tau(f) = \tau$ with $\tau < \pi$. Let Γ be the circle $|z| = r$ with r in the interval $\tau < r < \pi$. Then

$$g(w) = \sum_{n \geq 0} f(n) w^{-n-1}$$

define a holomorphic function outside $\exp\{|w| \leq \tau\}$ which satisfies

$$f(z) = \frac{1}{2i\pi} \int_{\exp \Gamma} w^z g(w) \mathrm{d}w.$$

J. Dufresnoy & Ch. Pisot *Prolongement analytique de la série de Taylor*. Ann. sci. É.N.S. 3e série, **68** (1951), 105–124.
http://www.numdam.org/item?id=ASENS_1951_3_68__105_0

Entire functions vanishing on $\mathbb{N}$ of finite exponential type

An entire function f vanishing on $\mathbb{N}$ of finite exponential type $\tau(f)$ can be written as a trigonometric sum

$$f(z) = a_1(z)\sin(\pi z) + a_2(z)\sin(2\pi z) + \cdots + a_n(z)\sin(n\pi z)$$

with $a_1, \ldots, a_n$ in $\mathbb{C}[z]$ and $n \leq \tau(f)/\pi$.

If $\tau(f) < \pi$, then $f = 0$.

If $\tau(f) < 2\pi$, then $f = a_1(z)\sin(\pi z)$.

If $a_n \neq 0$, then $\tau(f) = n\pi$.

Laplace transform

Bijective map between the sets of entire function of finite exponential type $\leq \tau$ and analytic functions on $|t| > \tau$ vanishing at infinity.

Let

$$f(z) = \sum_{n\geq 0} \frac{a_n}{n!} z^n$$

be an entire function of exponential type $\tau(f)$.

The Laplace transform of f, viz.

$$F(t) = \sum_{n\geq 0} a_n t^{-n-1},$$

is analytic in the domain $|t| > \tau(f)$.

Laplace transform and inverse Laplace transform

From Cauchy's residue Theorem, it follows that for $r > \tau(f)$ we have

$$\frac{1}{2\pi i}\int_{|t|=r} t^n F(t)\mathrm{d}t = a_n.$$

Hence (inverse Laplace transform)

$$f(z) = \frac{1}{2\pi i}\int_{|t|=r} \mathrm{e}^{tz} F(t)\mathrm{d}t$$

and

$$f^{(n)}(z) = \frac{1}{2\pi i}\int_{|t|=r} t^n \mathrm{e}^{tz} F(t)\mathrm{d}t.$$

Weierstrass and Hadamard infinite products

Weierstrass products:

$$z^m \prod_j \left(1 - \frac{z}{z_j}\right) \mathrm{e}^{p_{n_j}(z/z_j)},$$

$$p_n(z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots + \frac{z^n}{n}.$$

Canonical product:

$$f(z) = \mathrm{e}^{g(z)} z^m \prod_j \left(1 - \frac{z}{z_j}\right) \mathrm{e}^{p_n(z/z_j)}$$

with g an entire function and with a uniform n minimal.
Hadamard: for f of finite order ϱ, g is a polynomial with

$$\varrho - 1 \leq \max\{\deg g, n\} \leq \varrho.$$

http://www-users.math.umn.edu/~garrett/m/complex/hadamard_products.pdf

Hadamard canonical products

Jacques Hadamard

(1865 – 1963)

- For $\mathbb{N} = \{0, 1, 2, \dots\}$:

$$z \prod_{n \geq 1} \left(1 - \frac{z}{n}\right) e^{z/n} = \frac{-e^{\gamma z}}{\Gamma(-z)},$$

where

$$\Gamma(z) = \int_0^\infty e^{-t} t^z \frac{dt}{t}.$$

The function $1/\Gamma(z)$ is entire of order 1 and infinite type.

- For $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$:

$$z \prod_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \left(1 - \frac{z}{n}\right) e^{-z/n} = z \prod_{n \geq 1} \left(1 - \frac{z^2}{n^2}\right) = \frac{-1}{z\Gamma(z)\Gamma(-z)} = \frac{\sin \pi z}{\pi}.$$

Weierstrass sigma function

Karl Weierstrass

(1815 – 1897)

Let $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$.
The canonical product of Weierstraß associated with Ω is the sigma function σ_Ω defined by

$$\sigma_\Omega(z) = z \prod_{\omega \in \Omega \setminus \{0\}} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right)$$

This function has a simple zero at each point of Ω. It is an entire function of order 2: for $\omega \in \Omega$,

$$\sigma_\Omega(z+\omega) = \chi(\omega)\sigma_\Omega(z)\mathrm{e}^{\eta(\omega)(z+(\omega/2))}, \quad \chi(\omega) = \pm 1.$$

Newton interpolation

Isaac Newton

(1643 – 1727)

Let f be an entire function and $\alpha_1, \alpha_2, \ldots$ be complex numbers. From

$$f(z) = f(\alpha_1) + (z - \alpha_1) f_1(z), \quad f_1(z) = f_1(\alpha_2) + (z - \alpha_2) f_2(z), \ldots$$

we deduce

$$f(z) = b_0 + b_1(z - \alpha_1) + b_2(z - \alpha_1)(z - \alpha_2) + \cdots$$

with

$$b_0 = f(\alpha_1), \quad b_1 = f_1(\alpha_2), \ldots, \quad a_n = f_n(\alpha_{n+1}).$$

Divided differences

Given variables $x_0, x_1, \ldots, x_n$ and a function f of a single variable, we have

$$\begin{aligned} f(x_n) &= f(x_0) + (x_n - x_0) f[x_0, x_1] + (x_n - x_0)(x_n - x_1) f[x_0, x_1, x_2] \\ &\quad + \cdots + (x_n - x_0)(x_n - x_1) \cdots (x_n - x_{n-1}) f[x_0, x_1, \ldots, x_n] \end{aligned}$$

with

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0},$$

$$f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}.$$

Divided differences

Hint :

$$f[x_0, x_1, \ldots, x_n] = \sum_{j=0}^{n} \frac{f(x_j)}{\prod\limits_{\substack{0 \le k \le n \\ k \ne j}} (x_j - x_k)}.$$

Consequence : Given distinct points $x_0, x_1, \ldots, x_n$, a polynomial f of degree $\le n$ has an expansion

$$f(x) = f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] \\ + \cdots + (x - x_0)(x - x_1) \cdots (x - x_{n-1}) f[x_0, x_1, \ldots, x_n]$$

Calculus of finite differences (arithmetic progressions)

Given an arithmetic function $f : \mathbb{N} \to \mathbb{C}$, define another arithmetic function Δf by

$$\Delta f(n) = f(n+1) - f(n).$$

If f is a polynomial of degree $d \geq 1$, then Δf is a polynomial of degree $d-1$. Hence for the d–th iterate Δ^d the polynomial $\Delta^d f$ is constant, and $\Delta^{d+1} f = 0$ (the zero function).
Set $z^{(n)} = z(z-1)\cdots(z-n+1)$. Then $\Delta z^{(n)} = nz^{(n-1)}$.
Newton expansion of a polynomial f of degree d:

$$f(z) = f(0) + (\Delta f)(0) z^{(1)} + (\Delta^2 f)(0) \frac{z^{(2)}}{2!} + \cdots + (\Delta^d f)(0) \frac{z^{(d)}}{d!}.$$

http://nonagon.org/ExLibris/calculus-finite-differences

Hermite formula

Following C. Hermite, starting from

$$\frac{1}{x-z} = \frac{1}{x-\alpha} + \frac{z-\alpha}{x-\alpha} \cdot \frac{1}{x-z}.$$

and repeating yields

$$\frac{1}{x-z} = \frac{1}{x-\alpha_1} + \frac{z-\alpha_1}{x-\alpha_1} \cdot \left(\frac{1}{x-\alpha_2} + \frac{z-\alpha_2}{x-\alpha_2} \cdot \frac{1}{x-z} \right).$$

Inductively, we deduce the next formula due to Hermite:

$$\frac{1}{x-z} = \sum_{j=0}^{n-1} \frac{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_j)}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{j+1})} + \frac{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)} \cdot \frac{1}{x-z}.$$

Newton interpolation formula

Multiplying by $(1/2i\pi)f(z)$ and integrating yields Newton's interpolation formula:

$$f(z) = \sum_{j=0}^{n-1} b_j(z-\alpha_1)\cdots(z-\alpha_j) + R_n(z)$$

with

$$b_j = \frac{1}{2i\pi}\int_{\mathcal{C}} \frac{f(x)\mathrm{d}x}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{j+1})} \quad (0 \le j \le n-1)$$

and

$$R_n(z) = (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)\cdot \frac{1}{2i\pi}\int_{\mathcal{C}} \frac{f(x)\mathrm{d}x}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)(x-z)}.$$

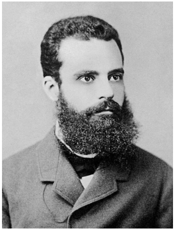
Abel series

Niels Abel

(1802 – 1829)

G.H. Halphén

(1844 – 1889)



V. Pareto

(1848 – 1923)

Abel's interpolation problem is to find an entire function f for which the values $f^{(n)}(n)$ are prescribed. It was studied by G. Halphén (1882), V. Pareto (1892), W. Gontcharoff (1930), R.C. Buck (1946).

https://www-history.mcs.st-andrews.ac.uk/Biographies/Halphen.html

https://fr.wikipedia.org/wiki/Vilfredo_Pareto

Abel's interpolation problem

The lack of unicity arises from nonzero entire functions f, like $\sin(\pi z/2)$, satisfying $f^{(n)}(n) = 0$ for $n \geq 0$.

Let us start with polynomials. Given a polynomial f, we are looking for a finite expansion

$$f(z) = \sum_{n \geq 0} f^{(n)}(n) P_n(z).$$

We need a sequence of polynomials $(P_n)_{n \geq 0}$ satisfying

$$P_n^{(k)}(k) = \delta_{kn} \quad \text{for} \quad k \geq 0 \quad \text{and} \quad n \geq 0.$$

Abel polynomials

Looking for a sequence of polynomials $(P_n)_{n\geq 0}$ satisfying

$$P_n^{(k)}(k) = \delta_{kn} \quad \text{for} \quad k \geq 0 \quad \text{and} \quad n \geq 0.$$

$n = 0$: $P_0 = 1$.
$n = 1$: $P_1(z) = z$.
$n = 2$: $P_2(0) = P_2'(1) = 0$, $P_2''(2) = 1$,

$$P_2(z) = \frac{1}{2} z(z-2).$$

These polynomials are defined inductively by $P_0 = 1$,

$$P_n'(z) = P_{n-1}(z-1) \quad (n \geq 1).$$

Abel polynomials

$$P_n'(z) = P_{n-1}(z-1) \quad (n \geq 1), \qquad P_0 = 1.$$

Solution (N. Abel, 1881)

$$P_n(z) = \frac{1}{n!} z(z-n)^{n-1} \quad (n \geq 1).$$

Using Stirling's formula one deduces, for $n \geq 0$,

$$|P_n|_r \leq \left(1 + \frac{r}{n}\right)^n \mathrm{e}^n.$$

Abel interpolation of entire functions

Let ω be the positive real number defined by $\omega e^{\omega+1} = 1$. The numerical value is $\omega = 0.278\,464\,542\ldots$

George Henri Halphen

(1844 – 1889)

Proposition (Halphén, 1882).

If f is an entire function of finite exponential type $< \omega$, then

$$f(z) = \sum_{n \geq 0} f^{(n)}(n) P_n(z)$$

where the series in the right hand side is absolutely and uniformly convergent on any compact of $\mathbb{C}$.

Generating series for $P_n(z)$

For $|t| < \omega$ and $z \in \mathbb{C}$, we have (Legendre, Abel)

$$\mathrm{e}^{tz} = \sum_{n \geq 0} t^n \mathrm{e}^{nt} P_n(z).$$

Corollary.

Let $\lambda \in \mathbb{C}$ satisfy $|\lambda| < 1/e$. Then the only solutions f of the equation

$$f'(z) = \lambda f(z-1)$$

which are entire functions of exponential type $< \omega$ are given by

$$f(z) = f(0)\mathrm{e}^{tz} \quad \textit{where} \quad t\mathrm{e}^{t} = \lambda.$$

John Macnaghten Whittaker

(1905 – 1984)

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Aleksandr Osipovich Gelfond

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Ralph Philip Boas Jr

(1912 – 1992)

Robert Creighton Buck

(1920 – 1998)

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