Updated: 2018

An elementary introduction to Cryptography

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http://www.math.jussieu.fr/~miw/

Data transmission, Cryptography and Arithmetic

Among the unexpected features of recent developments in technology are the connections between classical arithmetic on the one hand, and new methods for reaching a better security of data transmission on the other. We will illustrate this aspect of the subject by showing how modern cryptography is related to our knowledge of some properties of natural numbers. As an example, we explain how prime numbers play a key role in the process which enables you to withdraw safely your money from your bank account using your PIN (Personal Identification Number) secret code.

https://www.ias.ac.in/listing/articles/reso/023

A sketch of Modern Cryptology

The Art and Science of Secrecy Systems

by *Palash Sarkar*

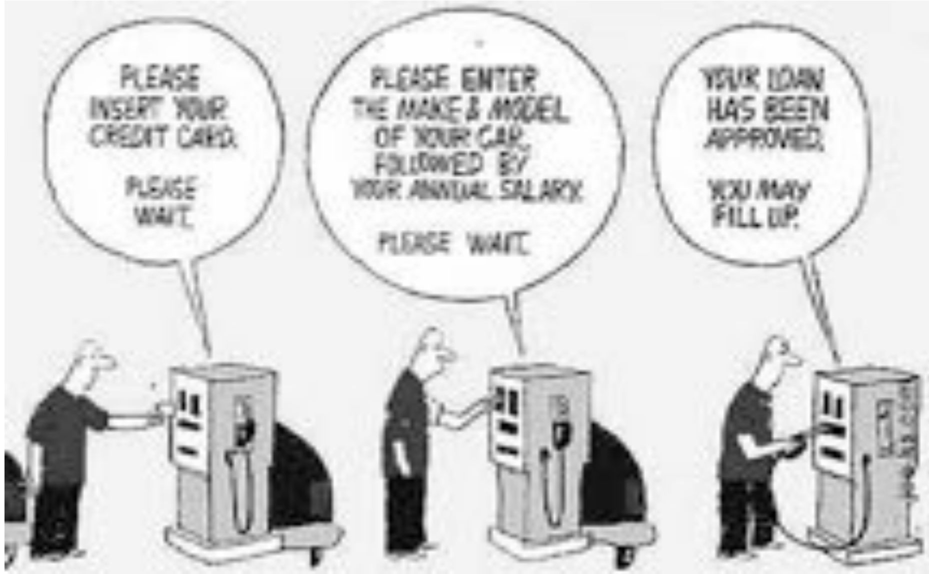
Resonance journal of science education

Volume 5 Number 9 (september 2000), p. 22-40

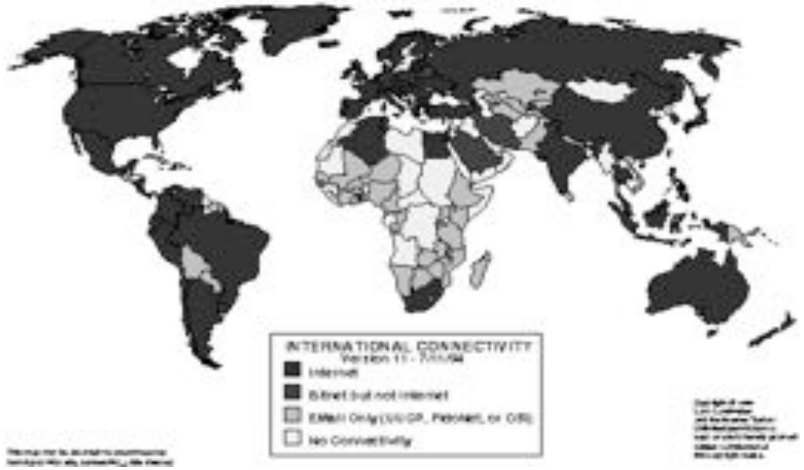
RSA SECURITY

Encryption for security

PGP



1994



Cryptology and the Internet: security norms, e-mail,
web communication (SSL: Secure Socket Layer),
IP protocol (IPSec), e-commerce…

Larry Landweber's International Connectivity maps

1997

Security of communication by cell phone,
Telecommunication, Pay TV, Encrypted television,…

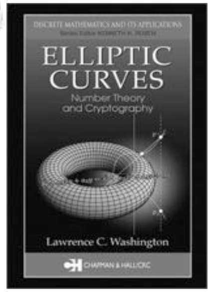
Activities to be implemented digitally and securely.

- Protect information — Code book, lock and key
- Identification — Driver's license, Social Security number, password, bioinformatics,
- Contract — Handwritten signature, notary
- Money transfer — Coin, bill, check, credit card
- Public auction — Sealed envelope
- Public election — Anonymous ballot
- Poker — Cards with concealed backs
- Public lottery — Dice, coins, rock-paper-scissors
- Anonymous communication — Pseudonym, ransom note

http://www.cs.princeton.edu/introcs/79crypto/

Mathematics in cryptography

- Algebra
- Arithmetic, number theory
- Geometry
- Topology
- Probability



Sending a suitcase

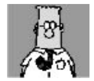
Alice

Charlie

Bob

- Assume Alice has a suitcase and a lock with the key; she wants to send the suitcase to Bob in a secure way so that nobody can see the content of the suitcase.
- Bob also has a lock and the corresponding key, but they are not compatible with Alice's ones.

The protocol of the suitcases

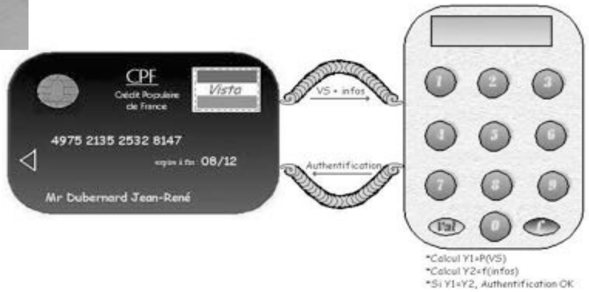


- Alice closes the suitcase with her lock and sends it to Bob.
- Bob puts his own lock and sends back to Alice the suitcase with two locks.
- Alice removes her lock and sends back the suitcase to Bob.
- Finally Bob is able to open the suitcase.

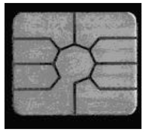
- *Later: a mathematical translation*.

Secret code of a bank card

ATM: Automated
Teller Machine



The memory electronic card (chip or smart card) was invented in the 70's by two french engineers, Roland Moreno and Michel Ugon.



- France adopted the card with a microprocessor as early as 1992.
- In 2005, more than 15 000 000 bank cards were smart cards in France.
- In European Union, more than 1/3 of all bank cards are smart cards.

`http://www.cartes-bancaires.com`

Secret code of a bank card

- You need to identify yourself to the bank. You know your secret code, but for security reason you are not going to send it to the bank. Everybody (including the bank) knows the public key. Only **you** know the secret key.

The memory electronic card (chip card) .

- The messages you send or receive should not reveal your secret key.
- Everybody (including the bank), who can read the messages back and forth, is able to check that the answer is correct, but is unable to deduce your secret code.

- *The bank sends you a random message.*
- *Using your secret code (also called secret key or password) you send an answer.*

Cryptography: a short history

Encryption using alphabetical transpositions and substitutions

- Julius Caesar: replaces each letter by another one in the same order (shift)

- For instance, (shift by 3) replace

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z
by
D E F G H I J K L M N O P Q R S T U V W X Y Z A B C

- *Example:*

CRYPTOGRAPHY becomes FUBSWRJUDSKB

- *More sophisticated examples:* use any permutation (does not preserve the order).

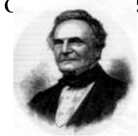
- 800-873, Abu Youssouf Ya qub Ishaq **Al Kindi**

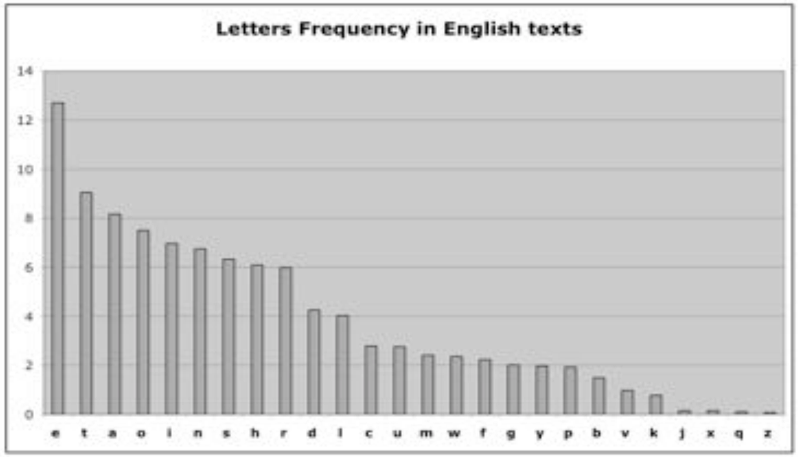
Manuscript on deciphering cryptographic messages.
Check the authenticity of sacred texts from Islam.

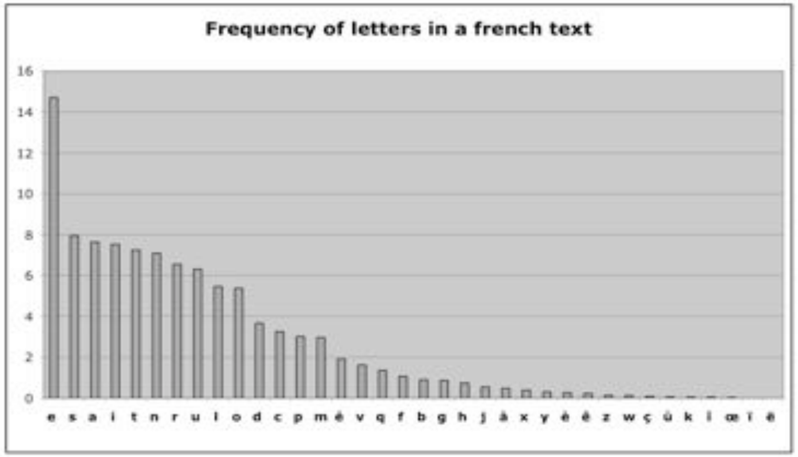
- XIIIth century, Roger Bacon: seven methods for encryption of messages.

- 1586, Blaise de Vigenère
(key: «table of Vigenère») 
Cryptograph, alchimist, writer, diplomat

- 1850, Charles Babbage (frequency of occurrences of letters)
Babbage machine (ancestor of computer)
Ada, countess of Lovelace: first programmer







International Morse code alphabet

Samuel Morse,
1791-1872

A	.-	N	-.	0	-----
B	-...	O	---	1	.----
C	-.-.	P	.--.	2	..---
D	-..	Q	--.-	3	...--
E	.	R	.-.	4	....-
F	..-.	S	...	5	.....
G	--.	T	-	6	-....
H	....	U	..-	7	--...
I	..	V	...-	8	---..
J	.---	W	.--	9	----.
K	-.-	X	-..-	Fullstop	.-.-.-
L	.-..	Y	-.--	Comma	--..--
M	--	Z	--..	Query	..--..

Interpretation of hieroglyphs

- Jean-François Champollion (1790-1832)
- Rosette stone (1799)

Data transmission

- ***Carrier-pigeons : first crusade - siege of Tyr, Sultan of Damascus***
- ***French-German war of 1870, siege of Paris***
- ***Military centers for study of carrier-pigeons created in Coëtquidan and Montoire.***

Data transmission

- ***James C. Maxwell***
  (1831-1879)

- ***Electromagnetism***
  Herz, Bose: radio

Auguste Kerckhoffs
«La cryptographie militaire»,
Journal des sciences militaires, vol. IX,
pp. 5–38, Janvier 1883,
pp. 161–191, Février 1883 .

Any secure encyphering method is supposed to be known by the enemy
The security of the system depends onl he choice of keys.

1917, Gilbert Vernam (**disposable mask)**
Example: the red phone Kremlin/White House
One time pad

Original message:	0 1 1 0 0 0 1 0 1 …
Key	0 0 1 1 0 1 0 0 1…
Message sent	0 1 0 1 0 1 1 0 0…

1950, *Claude Shannon* proves that the only secure secret key systems are those with a key at least as long as the message to be sent.

Alan Turing

Deciphering coded messages
(*Enigma*)

Computer science

Colossus

Max Newman,
the first programmable electronic computer (Bletchley Park before 1945)

Information theory

Claude Shannon
A mathematical theory of communication
Bell System Technical Journal, 1948.

Claude E. Shannon

" Communication Theory of Secrecy Systems ",

Bell System Technical Journal ,

28-4 (1949), 656 - 715.

Secure systems

Unconditional security: knowing the coded message does not yield any information on the source message: the only way is to try all possible secret keys.

In practice, all used systems do not satisfy this requirement.

Practical security: knowing the coded message does not suffice to recover the key nor the source message **within a reasonable time.**

DES:
Data Encryption Standard

In 1970, the NBS (*National Board of Standards*) put out a call in the *Federal Register* for an encryption algorithm

- with a high level of security which does not depend on the confidentiality of the algorithm but only on secret keys
- using secret keys which are not too large
- fast, strong, cheap
- easy to implement

DES was approved in 1978 by NBS

Algorithm DES:
combinations, substitutions and permutations between the text and the key

- The text is split in blocks of *64* bits
- The blocks are permuted
- They are cut in two parts, right and left
- Repetition *16* times of permutations and substitutions involving the secret key
- One joins the left and right parts and performs the inverse permutations.

Diffie-Hellman:
Cryptography with public key

- Whit Diffie and Martin E. Hellman,

New directions in cryptography,
IEEE Transactions on Information Theory,
22 (1976), 644-654

Symmetric versus **Assymmetric** cryptography

- **Symmetric** (secret key):
- Alice and Bob both have the key of the mailbox. Alice uses the key to put her letter in the mailbox. Bob uses his key to take this letter and read it.
- Only Alice and Bob can put letters in the mailbox and read the letters in it.

- **Assymmetric** (Public key):
- Alice finds Bob's address in a public list, and sends her letter in Bob's mailbox. Bob uses his secret key to read the letter.
- Anybody can send a message to Bob, only ***he*** can read it

RSA
(Rivest, Shamir, Adleman - 1978)

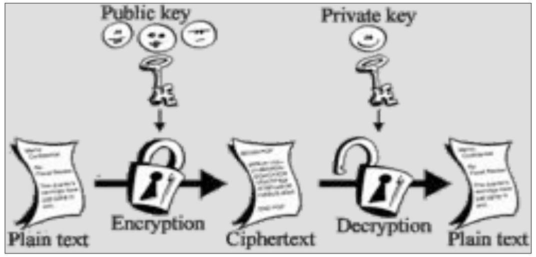
Adi Shamir　Ron Rivest　Len Adleman

R.L. **R**ivest, A. **S**hamir, and L.M. **A**dleman

A method for obtaining digital signatures and public-key cryptosystems,

Communications of the ACM

(2) **21** (1978), 120-126.



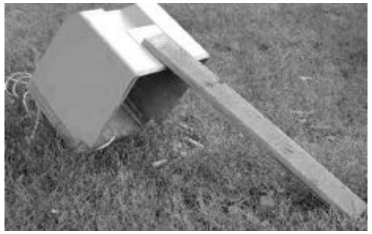
Trap functions

$x \rightarrow y$

is a *trap-door one-way function* if

- given x, it is easy to compute y
- given y , it is very difficult to find x, unless one knows a key.

Examples involve mathematical problems known to be difficult.



Example of a trapdoor one-way function: The discrete logarithm
(Simplified version)

Select a three digits number x.
Compute the cube: $x \times x \times x = x^3$.
Keep only the last three digits = remainder of the division by 1000: this is y.

- Starting from ***x,*** it is easy to find ***y.***
- If you know ***y,*** it is not easy to recover ***x.***

The discrete logarithm modulo 1000

- Example: assume the last three digits of x^3 are 631: we write $x^3 \equiv 631$ modulo 1000. ***Goal:*** *to find* x.
- Brute force: try all values of $x=001, 002, \ldots$
  you will find that $x=111$ is solution.
- *Check:* $111 \times 111 = 12\,321$
- Keep only the last three digits:

$$111^2 \equiv 321 \text{ modulo } 1000$$

- Next $111 \times 321 = 35\,631$
- Hence $111^3 \equiv 631$ modulo 1000.

Cube root modulo 1000

Solving $x^3 \equiv 631$ modulo 1000.

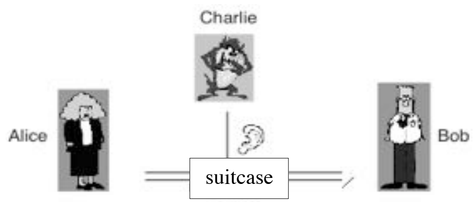
- **Other method:** *use a secret key*.
  The public key here is 3, since we compute x^3.
  A secret key is *67*.
- *This means that if you raise 631 to the power 67, you will find* x:

$$631^{67} \equiv x \text{ modulo } 1000.$$

Retreive x from x^7 modulo 1000

- With public key 3, a secret key is 67.
- Another example: public key 7, secret key is 43.
- If you know $x^7 \equiv 871$ modulo 1000
- Check $871^{43} \equiv 111$ modulo 1000
- Therefore $x = 111$.

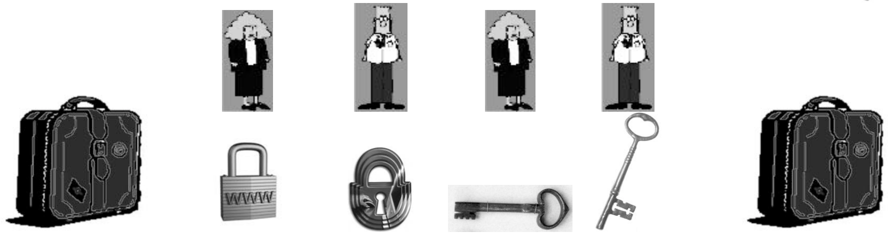

Sending a suitcase




- Assume Alice has a suitcase and a lock; she wants to send the suitcase to Bob in a secure way so that nobody can see the content of the suitcase.

- Bob also has a lock and the corresponding key, but they are not compatible with Alice's ones.

Sending a suitcase

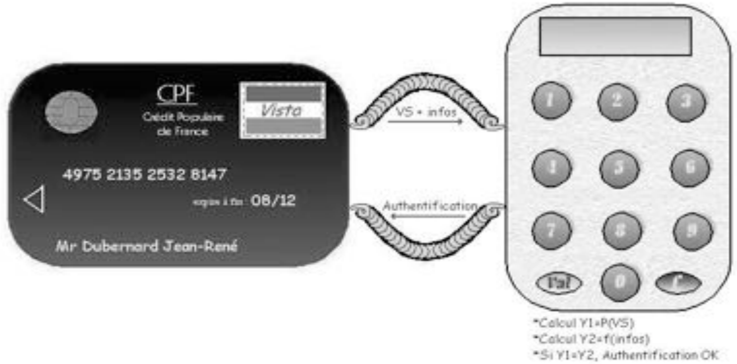


111 *7* *3* *43* *67* *111*

$111^7 \equiv 871$ $311^{43} \equiv 631$

$871^3 \equiv 311$ $631^{67} \equiv 111$

Security of bank cards



ATM

Random message	Pin Code	Public key
631	*67*	*3*
	$631^{67} \equiv 111$	$111^3 \equiv 631$

Everybody who knows your public key *3* and the message *631* of the bank, can check that your answer *111* is correct, but cannot find the result without knowing the pin code *67* (unless he uses the brute force method).

Message modulo n

- Fix a positive integer n *(in place of 1000)*: this is the size of the messages which are going to be sent.
- All computation will be done modulo n : we replace each integer by the remainder in its division by n.
- n will be a integer with some *300* digits.

It is easier to check a proof than to find it

Easy to multiply two numbers, even if they are large.

If you know only the product, it is difficult to find the two numbers.

Is *2047* the product of two smaller numbers?

Answer: yes $2047=23\times 89$

Example

p=1113954325148827987925490175477024844070922844843

q=1917481702524504439375786268230862180696934189293

pq=2135987035920910082395022704999628797051095341826417406442524165008583957746445088405009430865999

Size of n

We take for n the product of two prime numbers with some *150* digits each.

The product has some *300* digits: computers cannot find the two prime numbers.

Prime numbers, primality tests and factorization algorithms

- The numbers $2, 3, 5, 7, 11, 13, 17, 19, \ldots$ are prime.
- The numbers $4=2\times 2$, $6=2\times 3$, $8=2\times 2\times 2$, $9=3\times 3$, $10=2\times 5$, $2047=23\times 89$ … are composite.
- Any integer ≥ 2 is either a prime or a product of primes. For instance $12=2\times 2\times 3$.
- Given an integer, decide whether it is prime or not **(primality test).**
- Given a composite integer, give its decomposition into a product of prime numbers **(factorization algorithm).**

Primality tests

- Given an integer, decide whether it is the product of two smaller numbers or not.

Today's limit : more than 1000 digits

Factorization algorithms

- Given a composite integer, decompose it into a product of prime numbers

Today's limit : around 150 digits

Agrawal-Kayal-Saxena

- Manindra Agrawal, Neeraj Kayal and Nitin Saxena, ***PRIMES is in P*** (July 2002)

http://www.cse.iitk.ac.in/news/primality.html

Industrial primes

- **Probabilistic Tests** are not genuine primality tests: they do not garantee that the given number is prime. But they are useful whenever a small rate or error is allowed. They produce the **industrial primes.**

The four largest known primes:

January 3, 2018 $2^{77\,232\,917}-1$

23 249 425 digits

January 7, 2016 $2^{74\,207\,281}-1$

22 338 618 decimal digits

February 8, 2013 $2^{57\,885\,161}-1$

17 425 170 digits

August 23, 2008 $2^{43\,112\,609}-1$

12 978 189 digits

http://primes.utm.edu/largest.html

Electronic Frontier Foundation
Defending Freedom in the Digital World

Cooperative Computing Awards
» Rules » Frequently Asked Questions » Status » Resources

Through the EFF Cooperative Computing Awards,
EFF will confer prizes of:

* $100 000 (1 lakh) to the first individual or group who discovers a prime number with at least 10 000 000 decimal digits.

* $150 000 to the first individual or group who discovers a prime number with at least 100 000 000 decimal digits.

* $250 000 to the first individual or group who discovers a prime number with at least 1 000 000 000 decimal digits.

http://www.eff.org/awards/coop.php

Large primes

- *11* among the *12* largest known primes can be written as $2^p - 1$ (and we know *50* such primes)
- We know
  402 primes with more than *1 000 000* digits (*11* in *2007*),
  2201 primes with more than *500 000* digits (*55* in *2007*).
- The list of *5 000* largest known primes is available at

`http://primes.utm.edu/primes/`

Update: October 2018

Mersenne numbers (1588-1648)

- *Mersenne* numbers are numbers of the form $M_p = 2^p - 1$ with p prime.
- There are only 49 known Mersenne primes, the first ones are *3, 7, 31, 127* with $3 = M_2 = 2^2 - 1$, $7 = M_3 = 2^3 - 1$, $31 = M_5 = 2^5 - 1$, $127 = M_7 = 2^7 - 1$.
- 1536, Hudalricus Regius: $M_{11} = 2^{11} - 1$ is not prime: $2047 = 23 \times 89$.

Marin Mersenne (1588-1648), preface to *Cogitata Physica-Mathematica* (1644): *the numbers* $2^n - 1$ *are prime for*
$n = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127$ *and* 257
and composite for all other positive integers $n < 257$.

The correct list is:
2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107 and *127*.

http://www.mersenne.org/

A large composite Mersenne number

- $2^{2\,944\,999} - 1$ is composite: divisible by $3\,1458\,4703\,0730\,5708\,0643\,101377$

Perfect numbers

- An integer n is called *perfect* if n is the sum of the divisors of n distinct from n.
- The divisors of 6 distinct from 6 are $1, 2, 3$ and $6=1+2+3$.
- The divisors of 28 distinct from 28 are $1, 2, 4, 7, 14$ and $28=1+2+4+7+14$.
- *Notice that* $6=2\times 3$ *and* $28=4\times 7$

  while $3=M_2$ *and* $7=M_3$.
- Other perfect numbers are $496=16\times 31$, $8128=64\times 127, \ldots$

Even perfect numbers (Euclid)

- Even perfect numbers are numbers which can be written $2^{p-1}\times M_p$ with $M_p = 2^p - 1$ a Mersenne prime (hence p is prime).
- Are there infinitely many perfect numbers?
- Nobody knows whether there exists any odd perfect number.

Fermat numbers (1601-1665)

- A *Fermat* number is a number which can be written $F_n=2^{2^n}+1$.
- Construction with rule and compass of regular polygons.
- $F_0=3, F_1=5, F_2=17, F_3=257, F_4=65537$ are prime numbers.
- Fermat suggested in 1650 that all F_n are prime numbers.

Euler (1707-1783)

- $F_5 = 2^{32}+1$ is divisible by 641

$$4\ 294\ 967\ 297 = 641 \times 6\ 700\ 417$$

$$641 = 5^4 + 2^4 = 5 \times 2^7 + 1$$

- Are there infinitely many Fermat primes?
- Only *5* Fermat primes F_n are known:
  $F_0=3, F_1=5, F_2=17, F_3=257, F_4=65537.$

Factorization algorithms

- Given a composite integer, decompose it into a product of prime numbers
- Today's limit : around *150* decimal digits for a random number
- Most efficient algorithm: ***number field sieve*** Factorization of RSA-155 (*155* decimal digits) in *1999*
- Factorization of a divisor of $2^{953}+1$ with *158* decimal digits in *2002*.
- ***A number with 313 digits on May 21, 2007.***

http://www.loria.fr/~zimmerma/records/factor.html

https://en.wikipedia.org/wiki/Integer_factorization_records

RSA Laboratories

Challenge Number Prize $US

- RSA-576 $10,000 Factored December *2003*
- RSA-640 $20,000 Factored November *2005*
- RSA-704 $30,000 Not Factored
- RSA-768 $50,000 Factored December *2009*
- RSA-896 $75,000 Not Factored
- RSA-1024 $100,000 Not Factored
- RSA-1536 $150,000 Not Factored
- RSA-2048 $200,000 Not Factored

`http://www.rsasecurity.com/rsalabs/`

Closed in 2007

RSA Laboratories

RSA-768

Status: Factored December 12, *2009*

Decimal Digits: *232* Digit sum *1018*

1230186684530117755130494958384962720772853569595334792197322452151726400507263657518745202199786469389956474942774063845925192557326303453731548268507917026122142913461670429214311602221240479274737794080665351419597459856902143413

=

33478071698956898786044169848212690817704794983713768568912431388982883793878002287614711652531743087737814467999489

*

36746043666799590428244633799627952632279158164343087642676032283815739666511279233373417143396810270092798736308917

http://www.crypto-world.com/announcements/rsa768.txt

RSA Laboratories

RSA
The Security Division of EMC

RSA-704 Prize: $*30,000*

Status: Not Factored

Decimal Digits: *212*

- *7403756347956171282804679609742957314259318888923128908493623263897276503402826627689199641962511784399589433050212758537011896809828673317327310893090055250511687706329907239638078671008609696253793465056379635 9*

- Digit Sum: *1009*

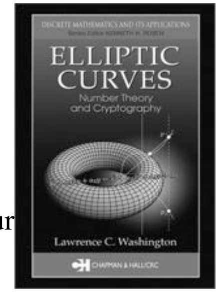
Other security problems of the modern business world

- Digital signatures
- Identification schemes
- Secret sharing schemes
- Zero knowledge proofs

Current trends in cryptography

- Computing modulo *n* means working in the multiplicative group of integers modulo *n*
- Specific attacks have been developed, hence a group of *large* size is required.
- We wish to replace this group by another one in which it is easy to compute, where the discrete logarithm is hard to solve.
- For smart cards, cell phones, … a *small* mathematical object is needed.
- A candidate is an elliptic curve over a finite field.

Research directions



To count efficiently the number of points on an elliptic cur over a finite field

To check the vulnerability to known attacks

To find new invariants in order to develop new attacks.

Discrete logarithm on the Jacobian of algebraic curves

Modern cryptography

- Quantum cryptography (Peter Shor) - magnetic nuclear resonance

Quizz: How to become a hacker?

Answer: Learn mathematics !

- **http://www.catb.org/~esr/faqs/hacker-howto.html**

$F_5 = 2^{32} + 1$ *is divisible by 641*

- $641 = 625 + 16 = 5^4 + 2^4$
- $641 = 5 \times 128 + 1 = 5 \times 2^7 + 1$
- *641* **divides** $2^{28} \times (5^4 + 2^4) = 5^4 \times 2^{28} + 2^{32}$
- $X^4 - 1 = (X+1)(X-1)(X^2+1)$

  641 **divides** $(5 \times 2^7)^4 - 1 = 5^4 \times 2^{28} - 1$
- **Hence** *641* **divides** $2^{32} + 1$