On Euler's Constant

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Abstract

To decide the arithmetic nature of a constant from analysis is almost always a difficult problem. Most often, the answer is not known. This is indeed the case for Euler's constant, the value of which is approximately

$$0.577\,215\,664\,901\,532\,860\,606\,512\,090\,082\,402\,431\,042\,1\ldots$$

However we know several properties of this number. We survey a few of them.

Reference

JEFFREY C. LAGARIAS
Euler's constant : Euler's work and modern developments
Bulletin Amer. Math. Soc. **50** (2013), No. 4, 527–628.

arXiv:1303.1856 [math.NT]
Bibliography : 314 references.

Archives Euler and index Eneström

http://eulerarchive.maa.org/

Gustaf Eneström (1852–1923)
Die Schriften Euler's chronologisch nach den Jahren geordnet, in denen sie verfasst worden sind
Jahresbericht der Deutschen Mathematiker–Vereinigung, 1913.

Gustaf Eneström.
Efter fotografi.

http://www.math.dartmouth.edu/~euler/index/enestrom.html

http://www.eulerarchive.org/

MAHA168

Judi Online, Slot Online, Casino Online

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Situs Judi Online, Slot Online dan Casino Online Maha168

Sebagai situs judi online terbaik Maha168 mulai beroperasi sejak tahun 2014 awal. Tahun dima na masih belum begitu banyak situs yang mengembangkan teknologi 1 ID untuk semua permainan. Maha168 merupakan salah satu penggagas teknologi ini untuk memudahkan member kami dalam bermain. Bayangkan, dengan bermodalkan 50 ribu Anda sudah bisa memainkan 1000 macam game slot online dan juga melihat dealer cantik di meja casino online kami. Tersedia juga taruhan judi bola, sabung ayam online dan judi poker online hingga tembak ikan.

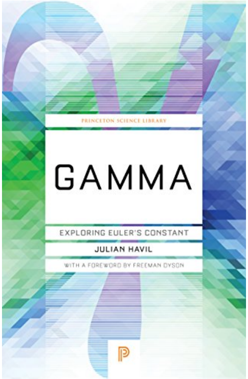
Kepopuleran Maha168 membawa setidaknya 25 ribu pemain slot dan casino setiap harinya berkunjung ke situs kami. Kepuasan atas pelayanan dari customer service kami dan minimal deposit yang terjangkau bagi pemain menjadi alasan utama member bergabung bersama kami. Kami memproses setidaknya 5000 ribu transaksi deposit dan withdraw setiap harinya. Rata - rata total penarikan harian di Maha168 mencapai lebih dari 1 miliar. Jumlahnya yang besar untuk membayar kemenangan member hingga membuat Maha168 layak disebut sebagai judi online terpercaya Indonesia. Setiap hari ada sedikitnya 550 member baru yang mendaftar di tempat kami. Member yang mendaftar karena mendapat referensi dari teman juga tidak sedikit. Terbukti member Maha168 dengan referral terbesar memiliki downline lebih dari 500 member.

Maha168 Situs Judi Online Terpercaya di Indonesia

Menjadi situs judi online Indonesia yang terpercaya merupakan kebanggaan Maha168 selama beroperasional selama hampir 7 tahun. Setiap bulannya ada ratusan member kami yang menjadi miliarder baru lahir dari Maha168. Kemenangan terbesar yang pernah dibayar oleh kami adalah sebesar 10 miliar, ketika salah satu pemain Maha168 mendapatkan jackpot progresive di judi slot online Maha168. Tak heran brand kami menjadi begitu populer dan terkenal Indonesia hingga Asia sebagai situs taruhan online. Terbukti menurut data dari Google, setiap bulannya terdapat lebih dari 100 ribu pencarian dengan kata kunci "Maha168" . Selain itu brand kami juga sangat mudah ditemukan dimana saja seperti media social Facebook, Line hingga di berbagai kata kunci di Google seperti casino online Maha168.

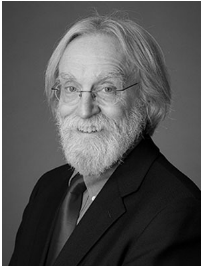
Exploring Euler's constant

Julian Havil
Gamma - exploring Euler's constant
Princeton University Press (2009)
Foreword by Freeman Dyson.



Further references

William Dunham
Euler and the Cubic Basel Problem
The American Mathematical Monthly
128 :4 (2021) 291–301.



Also :
https://en.wikipedia.org/wiki/Euler's_constant
References to works by A.I. Aptekarev, R. Murty, T. Rivoal, J. Sondow. . .

Harmonic numbers

$$H_1 = 1, \quad H_2 = 1 + \frac{1}{2} = \frac{3}{2}, \quad H_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6},$$

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{j=1}^{n} \frac{1}{j}.$$

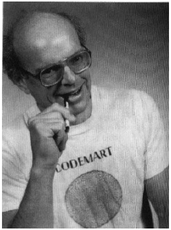
Sequence :

$$1, \quad \frac{3}{2}, \quad \frac{11}{6}, \quad \frac{25}{12}, \quad \frac{137}{60}, \quad \frac{49}{20}, \quad \frac{363}{140}, \quad \frac{761}{280}, \quad \frac{7129}{2520}, \ldots$$

The online encyclopaedia of integer sequences

https://oeis.org/

Neil J. A. Sloane



Numerators et denominators

Numerators : https://oeis.org/A001008

$$1, 3, 11, 25, 137, 49, 363, 761, 7129, 7381, 83711, 86021, 1145993,$$

$$1171733, 1195757, 2436559, 42142223, 14274301, 275295799,$$

$$55835135, 18858053, 19093197, 444316699, 1347822955, \ldots$$

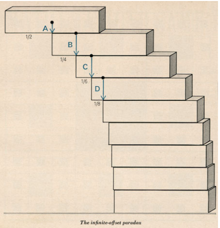
Denominators : https://oeis.org/A002805

$$1, 2, 6, 12, 60, 20, 140, 280, 2520, 2520, 27720, 27720, 360360,$$

$$360360, 360360, 720720, 12252240, 4084080, 77597520,$$

$$15519504, 5173168, 5173168, 118982864, 356948592, \ldots$$

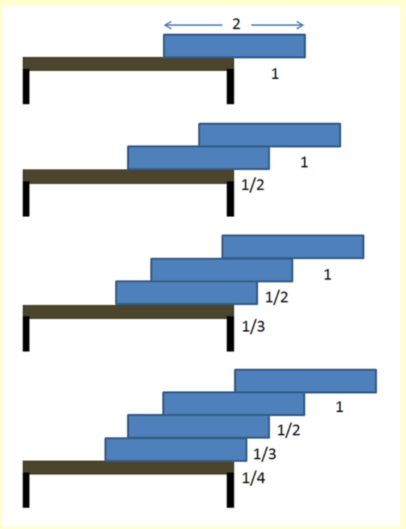
Divergence of the harmonic series



Proof by Nicolas Oresme, ∼1320-1322, 1382,
https://en.wikipedia.org/wiki/Nicole_Oresme

French philosopher of the later Middle Ages. He wrote influential works on economics, mathematics, physics, astrology and astronomy, philosophy, and theology ; was Bishop of Lisieux, a translator, a counselor of King Charles V of France, and one of the most original thinkers of 14th-century Europe.

The book stacking problem



Bridge collapsing

Navier: Blow-up and Collapse

Marco Cannone and Susan Friedlander

"All France knew of the disaster which happened in the heart of Paris to the first suspension bridge built by an engineer, a member of the Academy of Sciences; a melancholy collapse caused by blunders such as none of the ancient engineers—the man who cut the canal at Briare in Henri IV's time, or the monk who built the Pont Royal—would have made; but our administration consoled its engineer for his blunder by making him a member of the Council-general."

—Honoré de Balzac, from Le Curé de Village, *1841*

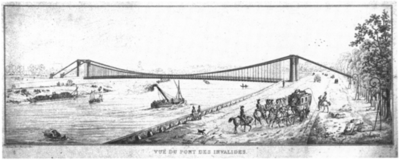


Figure 1. Pont des Invalides, taken from the original drawing by Navier in [13].

Claude Navier
1785 – 1836

Marco Cannone and Susan Friedlander,
Notices of the AMS, January 2003, 6 – 13.

Honoré de Balzac, Le curé de village, 1841

"All France knew of the disaster which happened in the heart of Paris to the first suspension bridge built by an engineer, a member of the Academy of Sciences; a melancholy collapse caused by blunders such as none of the ancient engineers—the man who cut the canal at Briare in Henri IV's time, or the monk who built the Pont Royal—would have made; but our administration consoled its engineer for his blunder by making him a member of the Council-general."

—Honoré de Balzac, from Le Curé de Village, *1841*

Marco Cannone and Susan Friedlander,
Notices of the AMS, January 2003, 6 – 13.

Divergence of the Harmonic series

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots$$

$$S_0 = \frac{S}{2} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots$$

$$S_1 = S - S_0 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots = S_0.$$

$$S_1 - S_0 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots > 0.$$

Divergence of the Harmonic series

Following Nicolas Oresme.

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$$

$$S = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}} + \cdots$$

$$\frac{1}{3} + \frac{1}{4} > \frac{2}{4} = \frac{1}{2}, \qquad \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{4}{8} = \frac{1}{2}, \ldots$$

$$H_n = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots + \tfrac{1}{n}$$

$$H_{2^k} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{} + \cdots + \frac{1}{2^k}.$$

$$\frac{1}{2} < \frac{1}{2^{k-1}+1} + \frac{1}{2^{k-1}+2} + \cdots + \frac{1}{2^k} < 1,$$

$$\frac{k}{2} + 1 < H_{2^k} < k + \frac{1}{2},$$

$$\frac{\log n}{2\log 2} < H_n < \frac{\log n}{\log 2}.$$

Estimate for H_n

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}.$$

$$\frac{H_n}{2} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}.$$

$$H_{2n} - \frac{H_n}{2} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{2n-1}.$$

$$H_{2n} - H_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots \to \log 2.$$

$$H_n \simeq \log n.$$

Euler (1731)

De progressionibus harmonicis observationes

The sequence

$$H_n - \log n$$

has a limit $\gamma = 0.577\,21\underline{8}\ldots$
when n tends to infinity.

Leonard Euler
1707 – 1783

Moreover,

$$\gamma = \sum_{m=2}^{\infty} (-1)^m \frac{\zeta(m)}{m}.$$

Riemann zeta function

Leonard Euler

1707 – 1783

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$$

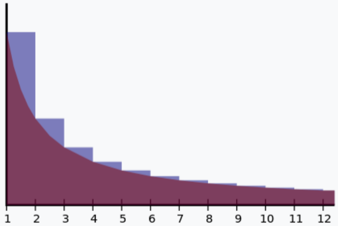
Bernhard Riemann

1826 – 1866

Euler : $s \in \mathbf{R}$.

Riemann : $s \in \mathbf{C}$.

Euler's constant



The area of the blue region converges to Euler's constant

Representations

Decimal	0.57721 56649 01532 86060 65120 90082 40243 10421 ...
Continued fraction (linear)	[0; 1, 1, 2, 1, 2, 1, 4, 3, 13, 5, 1, 1, 8, 1, 2, 4, 1, 1, ...][1] Unknown if periodic Unknown if finite
Binary	0.1001 0011 1100 0100 0110 0111 1110 0011 0111 1101 ...
Hexadecimal	0.93C4 67E3 7DB0 C7A4 D1BE 3F81 0152 CB56 A1CE CC3A ...

https://en.wikipedia.org/wiki/Euler's_constant

([1]) Sloane, N. J. A. (ed.). "Sequence A002852 (Continued fraction for Euler's constant)". The On-Line Encyclopedia of Integer Sequences. OEIS Foundation.

Numerical value of the Euler constant

$$\gamma = 0.577\,215\,664\,901\,532\,860\,606\,512\,090\,082\,402\,431\,042\,\ldots$$

The online encyclopaedia of integer sequences
https://oeis.org/A001620
Decimal expansion of Euler's constant
(or Euler–Mascheroni constant) gamma.

Yee (2010) computed $29\,844\,489\,545 > 2 \cdot 10^{10}$ decimal digits.

Seungmin Kim & Ian Cutress (2020) computed
$600\,000\,000\,100 > 6 \cdot 10^{11}$ digits
"Euler-Mascheroni Constant", Polymath Collector.

Nicholas Mercator (1668)

Nicholas Mercator (1620–1687)

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{k\geq 1} (-1)^{k+1} \frac{x^k}{k}.$$

Gerardus Mercator (1512–1594)

Nicholas is not Gerardus, the Mercator of the eponymus projection :

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Mercator_Gerardus.html

Computation of his constant by Euler in 1731

Euler replaces x by $1/m$ with $m = 1$, 2, 3, $4\ldots$ in Mercator's formula for $\log(1+x)$:

$$\log 2 = \frac{1}{1} - \frac{1}{2}\left(\frac{1}{1}\right)^2 + \frac{1}{3}\left(\frac{1}{1}\right)^3 - \cdots$$

$$\log \frac{3}{2} = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^3 - \cdots$$

$$\log \frac{4}{3} = \frac{1}{3} - \frac{1}{2}\left(\frac{1}{3}\right)^2 + \frac{1}{3}\left(\frac{1}{3}\right)^3 - \cdots$$

$$\log \frac{5}{4} = \frac{1}{4} - \frac{1}{2}\left(\frac{1}{4}\right)^2 + \frac{1}{3}\left(\frac{1}{4}\right)^3 - \cdots$$

Adding the first n terms of this sequence of formulae (telescoping series), Euler finds

$$\log(n+1) = H_n - \frac{1}{2}H_{n,2} + \frac{1}{3}H_{n,3} - \cdots$$

Euler's m–harmonic numbers

We have

$$\log(n+1) = H_n - \frac{1}{2}H_{n,2} + \frac{1}{3}H_{n,3} - \cdots$$

with

$$H_{n,m} = \sum_{j=1}^{n} \frac{1}{j^m}$$

for $n \geq 1$ and $m \geq 1$.

Hence, $H_{n,1} = H_n$ and, for $m \geq 2$,

$$\lim_{n\to\infty} H_{n,m} = \zeta(m).$$

Euler's proof (1731)

In the formula

$$H_n - \log(n+1) = \frac{1}{2} H_{n,2} - \frac{1}{3} H_{n,3} + \cdots,$$

when n tends to infinity,
the right hand side tends to

$$\sum_{m=2}^{\infty} (-1)^m \frac{\zeta(m)}{m}$$

Leonard Euler
1707 – 1783

which is the sum of an alternating series with a decreasing general term. Hence the left hand side has a limit, which is γ.

Lorenzo Mascheroni (1792)

He produced 32 decimals

$$\gamma = 0.577\,215\,664\,901\,532\,860\,6\underline{1}8\,112\,090\,082\,39$$

Lorenzo Mascheroni

1750 – 1800

the first 19 of them are correct ; the first 15 decimal were already found by Euler in 1755 and then in 1765.

Von Soldner (1809) : 22 decimals

$$\gamma = 0,577\,215\,664\,901\,532\,860\,606\,5$$

C.F. Gauss, F.G.B. Nicolai : 40 decimals

Computation of the decimal digits of Euler's constant

1872 :	J.W.L. Glaisher	100 decimals
1878 :	J.C. Adams	263 decimals
1952 :	J.W. Wrench Jr	328 decimals
1962 :	D. Knuth	1272 decimals
1963 :	D.W. Sweeney	3566 decimals
1964 :	W.A. Beyer and M.S. Waterman	7114 decimals (4879 correct)
1977 :	R.P. Brent	20 700 decimals
1980 :	R.P. Brent and E.M. McMillan	30 000 decimals
2010 :	Yee	29 844 489 545 decimals
2020 :	Kim and Cutress	600 000 000 100 decimals.

Euler Gamma function (1765)

De curva hypergeometrica hac aequationes expressa $y = 1 \cdot 2 \cdot 3 \cdots x$.

$$\begin{aligned}\Gamma(z) &= \int_0^\infty e^{-t} t^z \cdot \frac{dt}{t} \\ &= e^{-\gamma z} \frac{1}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}.\end{aligned}$$

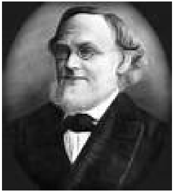
$$\Gamma'(1) = -\gamma = \int_0^\infty e^{-x} \log x \, dx.$$

$$\Gamma(z+1) = z\Gamma(z), \qquad \Gamma(n+1) = n!$$

Letter from Daniel Bernoulli to Christian Goldbach Octobre 6, 1729

http://fr.wikipedia.org/wiki/Fonction_gamma

Daniel Bernoulli
1700 – 1782



Christian Goldbach
1700 – 1782

Letter from Daniel Bernoulli to Christian Goldbach Octobre 6, 1729

St.-Pétersbourg ce 6 octobre 1729. **Dan. Bernoulli.**

P.S. Voici le terme général pour la suite $1+1.2+1.2.3+$etc. Soit x l'exposant du terme, et A un nombre infini, je dis que le terme général sera

$$\left(A+\frac{x}{2}\right)^{x-1}\left(\frac{2}{1+x}\cdot\frac{3}{2+x}\cdot\frac{4}{3+x}\cdots\cdot\frac{A}{A-1+x}\right)$$

Si au lieu de prendre A infiniment grand, on le fait $=$ à un nombre un peu grand, on aura le terme général à peu près. Si $x=\frac{3}{2}$ et qu'on fait $A=8$ on aura

$$\sqrt{\frac{17}{2}}\left(\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{8}{9}\cdot\frac{10}{11}\cdot\frac{12}{13}\cdot\frac{14}{15}\cdot\frac{16}{17}\right)=1,3005$$

par le moyen des logarithmes on approche très vitement. Si $x=3$ et $A=16$, au lieu de 6 on trouve

$$(6\cdot 17\tfrac{1}{2}\cdot 17\tfrac{1}{2}):17\cdot 18 = 6\tfrac{1}{204}.$$

Notation Γ : A.M. Legendre 1811

$$\Gamma(x+1) = \lim_{n\to\infty} \left(n+1+\frac{x}{2}\right)^{x-1} \prod_{i=1}^{n} \frac{i+1}{i+x}.$$

Adrien–Marie Legendre

1752–1834

Louis Legendre

1755–1797

This caricature by J-L Boilly is the only known portrait of Adrien-Marie Legendre.

Louis Legendre was an active participant in the French Revolution.

https://mathshistory.st-andrews.ac.uk/Biographies/Legendre/

Peter Duren. Changing Faces : The Mistaken Portrait of Legendre.

www.ams.org/notices/200911/rtx091101440p.pdf

Euler's formulae (1768)

$$\gamma = \int_0^\infty \left(\frac{e^{-t}}{1-e^{-t}} - \frac{e^{-t}}{t} \right) dt.$$

$$\gamma = \int_0^1 \left(\frac{1}{1-z} + \frac{1}{\log z} \right) dz.$$

Leonard Euler

1707 – 1783

$$\gamma = \sum_{n=2}^\infty \frac{n-1}{n} \left(\zeta(n) - 1\right).$$

$$\gamma = \frac{3}{4} - \frac{1}{2}\log 2 + \sum_{k=1}^\infty \left(1 - \frac{1}{2k+1}\right)\left(\zeta(2k+1) - 1\right).$$

Quoting Euler (1768)

"$\mathcal{O} = 0.577215664901532$5 qui numerus eo maiori attentione dignus videtur, quod eum, cum olim in hac investigatione multum studii consumsissem, nullo modo ad cognitum quantitatum genus reducere valui."

> *This number seems also the more noteworthy because even though I have spent much effort in investigating it, I have not been able to reduce it to a known kind of quantity.*

"Manet ergo quaestio magni momenti, cujusdam indolis sit numerus iste $\mathcal{O}$ et ad quodnam genus quantitatum sit referendus."

> *Therefore the question remains of great moment, of what character the number $\mathcal{O}$ is and among what species of quantities it can be classified.*

Jonathan Sondow http://home.earthlink.net/~sondow/

Jonathan Sondow

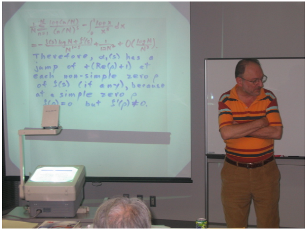
1943 – 2020

$$\gamma = \int_0^\infty \sum_{k=2}^\infty \frac{1}{k^2 \binom{t+k}{k}} \mathrm{d}t$$

$$\gamma = \lim_{s \to 1+} \sum_{n=1}^\infty \left(\frac{1}{n^s} - \frac{1}{s^n} \right)$$

$$\gamma = \int_1^\infty \frac{1}{2t(t+1)} \, {}_2F_3 \left(\begin{matrix} 1, & 2, & 2 \\ 3, & t+2 \end{matrix} \middle| \, 1 \right) \mathrm{d}t.$$

Jonathan Sondow and Wadim Zudilin



JONATHAN SONDOW & WADIM ZUDILIN, *Euler 's constant, q-logarithms, and formulas of Ramanujan and Gosper*, Ramanujan J. **12** (2006), 225–244.

Irrationality of Euler's constant

Conjecture. *Euler constant is irrational.*

Continued fraction expansion : $30\,000$ first terms have been computed.
http://oeis.org/A002852

$\gamma = [0, 1, 1, 2, 1, 2, 1, 4, 3, 13, 5, 1, 1, 8, 1, 2, 4, 1, 1, 40, 1, 11, 3, \dots]$

If $\gamma = a/b$ with positive integers a and b, then $b > 10^{242\ 080}$.
https://fr.wikipedia.org/wiki/Constante_d'Euler-Mascheroni

http://mathworld.wolfram.com/Euler-MascheroniConstant.html

The famous English mathematician G.H. Hardy is alleged to have offered to give up his Savilian Chair at Oxford to anyone who proved gamma to be irrational, although no written reference for this quote seems to be known. Hilbert mentioned the irrationality of gamma as an unsolved problem that seems "unapproachable" and in front of which mathematicians stand helpless. Conway and Guy (1996) are "prepared to bet that it is transcendental," although they do not expect a proof to be achieved within their lifetimes.

Hendrik W. Lenstra (1977)

At least one of the two numbers γ, e^γ is transcendental.

Euclidische getallenlichamen
Ph.D. thesis,
Mathematisch Centrum,
Universiteit van Amsterdam,
1977.

http://www.math.leidenuniv.nl/~hwl/PUBLICATIONS/1977c/art.pdf

Stellingen. Behorende bij het proefschrift van H.W. Lenstra Jr.

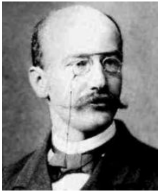
Theorems of Hermite and Lindemann

Ch. Hermite

1822 – 1901

Charles Hermite (1873) :
transcendence of e.

Ferdinand Lindemann (1882)
transcendence of π.



F. Lindemann

1852 – 1939

Hermite–Lindemann Theorem
For any non–zero complex number z, one at least of the two numbers z, e^z is transcendental.

Corollaries : transcendence of $\log\alpha$ and of e^β for α and β nonzero algebraic numbers with $\log\alpha \neq 0$.

e^γ

http://oeis.org/A073004

$e^\gamma = 1,781\,072\,417\,990\,197\,985\,236\,504\,103\,107\,179\,549\,169\ \ldots$

Conjecture. *The number* e^γ *is irrational.*

If $e^\gamma = p/q$, then $q > 10^{15\,000}$.
Continued fraction expansion of e^γ : $30\,000$ first terms computed.
http://oeis.org/A094644

$e^\gamma = [1, 1, 3, 1, 1, 3, 5, 4, 1, 1, 2, 2, 1, 7, 9, 1, 16, 1, 1, 1, 2, 6, 1, \ldots]$

Conjectures on the arithmetic nature of γ

Conjecture 1. *The Euler constant is irrational.*

Conjecture 2. *The Euler constant is transcendental.*

Conjecture 3. *The Euler constant is not a period in the sense of Kontsevich and Zagier.*

Periods : Maxim Kontsevich and Don Zagier

Periods,
Mathematics unlimited—2001 and beyond,
Springer 2001,
771–808.

A *period* is a complex number with real and imaginary parts given by absolutely convergent integrals of rational fractions with rational coefficients on domains of $\mathbf{R}^n$ defined by (in)equalities involving polynomials with rational coefficients.

Examples of periods

$$\sqrt{2} = \int_{2x^2 \leq 1} \mathrm{d}x$$

and all algebraic numbers are periods.

$$\log 2 = \int_{1<x<2} \frac{\mathrm{d}x}{x}$$

and all logarithms of algebraic numbers are periods.

$$\pi = \frac{1}{2i} \int_{|z|=1} \frac{\mathrm{d}z}{z} = 2 \int_0^\infty \frac{\mathrm{d}t}{1+t^2}.$$

The set of periods is a subalgebra of the field of complex numbers over the field of algebraic numbers ; it is expected that it is not a field.

Numbers which are not periods ?

Problem (Kontsevich – Zagier) : *Produce an explicit example of a number which is not a period.*

Several levels :

- *analog of Cantor* : the set of periods is countable.
- *analog of Liouville* : find a property which is satisfied by all periods and construct a number which does not satisfy it.
- *analog of Hermite* : prove that given constants arising from analysis are not periods.
  Candidates : $1/\pi$, e, γ, e^{γ}, $\Gamma(p/q)$, $\Gamma(1/2) = \sqrt{\pi}$, ...

Euler constant and arithmetic functions

The function sum of divisors

$$\sigma(n) = \sum_{d|n} d.$$

T.H. Grönwall
1877 – 1932

T.H. Grönwall (1913)

$$\limsup_{n\to\infty} \frac{\sigma(n)}{n \log\log n} = \mathrm{e}^{\gamma}.$$

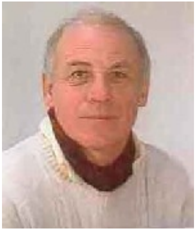
Euler's constant and Riemann hypothesis

Criterion of Guy Robin (1984) : *Riemann hypothesis is equivalent to*

$$\sigma(n) < e^{\gamma} n \log \log n$$

for all $n \geq 5\,041$.

Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann, J. Math. Pures Appl. **63** (1984), 187–213.



Vincel Hoang Ngoc Minh (2013)

http://hal.archives-ouvertes.fr/hal-00423455

On a conjecture by Pierre Cartier about a group of associators.
Acta Math. Vietnam (2013) **38** :339–398.

. . .we give a complete description of the kernel of polyzêta and draw some consequences about a structure of the algebra of convergent polyzêtas and about the arithmetical nature of the Euler constant.

Corollary 4.2 *If $\gamma \notin A$ then it is transcendental over the A-algebra generated by the convergent polyzetas.*

Or equivalently, by contraposition,

Corollary 4.3 *If there exists a polynomial relation with coefficients in A among the Euler constant, γ, and the convergent polyzetas then $\gamma \in A$.*

Therefore,

Corollary 4.4 *If the Euler constant, γ, does not belong to A then γ is not algebraic over A.*

Using Corollary 4.4, with $A = \mathbb{Q}$, it follows that

Corollary 4.5 *The Euler constant, γ, is not an algebraic irrational number.*

Corollary 4.6 *The Euler constant, γ, is a rational number.*

Proof Let us prove that in three steps:[27]

(1) Since γ verifies the equation $t^2 - \gamma^2 = 0$ then γ is algebraic over $\mathbb{Q}(\gamma^2)$.
(2) If γ is transcendental over $\mathbb{Q}$ then $\gamma \notin \mathbb{Q}(\gamma^2)$. Using Corollary 4.4, with $A = \mathbb{Q}(\gamma^2)$, γ is not algebraic over $A = \mathbb{Q}(\gamma^2)$. It contradicts the previous assertion (i.e. step (1)). Hence, γ is not transcendental over $\mathbb{Q}$.
(3) Thus, by Corollary 4.5, it remains that γ is rational over $\mathbb{Q}$.

□

[27] This part has been obtained after prolonged discussions with Michel Waldschmidt.

Irrationality

Lemma. *Let* γ *be a real number. Assume that for any subfield* K *of* $\mathbf{R}$*, the number* γ *is either in* K*, or else is transcendental over* K*. Then* γ *is a rational number.*

Proof. If the number γ is irrational, from the hypothesis it follows that it is transcendental over $\mathbf{Q}$. In this case γ is algebraic over the field $K = \mathbf{Q}(\gamma^2)$ and does not belong to K.

Divergent series

Euler (1760) : *On divergent series*. Four methods for evaluating

$$1 - 1 + 2 - 6 + 24 - 120 + \cdots$$

$$=$$

$$0! - 1! + 2! - 3! + 4! - 5! + \ldots$$

Wallis hypergeometric series

John Wallis
1616 – 1703

Hypergeometric series of Wallis

The divergent power series

$$0! - 1!x + 2!x^2 - 3!x^3 + 4!x^4 - 5!x^5 + \ldots$$

satisfies the linear differential equation

$$y' + \frac{1}{x^2}y = \frac{1}{x};$$

a solution which is convergent at $x = 1$ is given by the integral

$$\mathrm{e}^{\frac{1}{x}} \int_0^x \frac{1}{t}\mathrm{e}^{-\frac{1}{t}}\mathrm{d}t$$

which can be expanded into a continued fraction

$$[1, x, x, 2x, 2x, 3x, 3x, \ldots]$$

for which Euler gives the value at $x = 1$

$$\delta = 0.596\,347\,362\,\underline{1}23\,7\ldots$$

The Euler–Gompertz constant

$$0! - 1! + 2! - 3! + 4! - 5! + \cdots$$

$$\delta = \int_0^1 \frac{\mathrm{d}t}{1 - \log t} = \int_0^\infty \mathrm{e}^{-t} \log(t+1)\,\mathrm{d}t =$$

$$0.596\,347\,362\,323\,194\,074\,341\,078\,499\,369\,279\,376\,074\,177 \ \ldots$$

https://oeis.org/A073003

Benjamin Gompertz (1779–1865)

Benjamin Gompertz

Euler's constant

$$\gamma = -\int_0^\infty e^{-t} \log t \, dt$$

Gompertz's constant

$$\delta = \int_0^\infty e^{-t} \log(t+1) \, dt$$

Alexander Ivanovich Aptekarev (2007)

Letter of Ramanujan to Hardy (January 16, 1913)

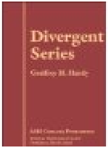
Srinivasa Ramanujan
(1887 – 1920)

Godfrey Harold Hardy
(1877 – 1947)

$$1 - 2 + 3 - 4 + \cdots = \frac{1}{4}$$

$$1 - 1! + 2! - 3! + \cdots = 0.596\ldots$$

G.H. Hardy : Divergent Series (1949)



Niels Henrik Abel
(1802 – 1829)

Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever.

Divergent series

Euler (1760) : *On divergent series*. Four methods to give a numerical value to

$$1 - 1 + 2 - 6 + 24 - 120 + \cdots$$

$$=$$

$$0! - 1! + 2! - 3! + 4! - 5! + \cdots$$

Wallis hypergeometric series

John Wallis
1616 – 1703)

Wallis hypergeometric series

The divergent formal power series

$$0!x - 1!x^2 + 2!x^3 - 3!x^4 + \cdots + (-1)^n n! x^{n+1} + \cdots$$

satisfies a linear differential equation

$$y' + \frac{1}{x^2} y = \frac{1}{x},$$

which has a solution converging at $x = 1$, given by the integral

$$e^{\frac{1}{x}} \int_0^x \frac{1}{t} e^{-\frac{1}{t}} dt$$

which can be expanded as a continued fraction

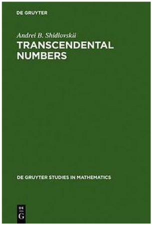
$$[1, x, x, 2x, 2x, 3x, 3x, \ldots]$$

of which Euler computes the value at $x = 1$

$$0,596\,347\,362\,\underline{1}23\,7\ldots$$

Andrei Borisovich Shidlovskii (1959)

One at least of the two numbers γ, δ is irrational.



A.B Shidlovskii

1915 – 2007

T. Rivoal, Kh. Pilehrood, T. Pilehrood (2012)

At least one of the two numbers γ, δ is transcendental.

Tanguy
Rivoal

Khodabakhsh
Hessami Pilehrood

Tatiana
Hessami Pilehrood

Tanguy Rivoal (2012)

Simultaneous rational approximations for the Euler constant and for the Euler–Gompertz constant.

$$\left|\gamma - \frac{p}{q}\right| + \left|\delta - \frac{r}{q}\right| > \frac{C(\epsilon)}{q^{3+\epsilon}}.$$

Kurt Mahler
1903–1988

Method of Mahler :
Two of the numbers e, γ, δ are algebraically independent.

The digamma function

For $z \in \mathbf{C} \setminus \{0, -1, -2, \ldots\}$,

$$\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

$$\psi(z) = -\gamma - \frac{1}{z} - \sum_{n=1}^{\infty} \left(\frac{1}{n+z} - \frac{1}{n} \right)$$

$$\psi(z+1) = -\gamma + \sum_{n=2}^{\infty} (-1)^n \zeta(n) z^{n-1} \qquad |z| < 1.$$

Euler and the digamma function (1765)

$\psi(n) = -\gamma + H_{n-1}$
for $n \geq 1$, with
$H_0 = H_{-1} = 0$.
For $n \geq 0$,

$$\psi\left(n + \frac{1}{2}\right) = -\gamma - 2\log 2 + 2H_{2n-1} - H_{n-1}.$$

Leonard Euler
1707 – 1783

The harmonic numbers

$$\frac{1-x^n}{1-x} = 1 + x + x^2 + \cdots, \qquad \int_0^1 x^j \mathrm{d}x = \frac{1}{j+1},$$

hence

$$H_n = \sum_{j=1}^{n} \frac{1}{j} = \int_0^1 \frac{1-x^n}{1-x} \mathrm{d}x.$$

L. Euler (1729) : for $z \geq 0$,

$$H_z = \int_0^1 \frac{1-x^z}{1-x} \mathrm{d}x.$$

$$H_{\frac{1}{2}} = 2 - 2\log 2 = 0.613\,705\,638\,880\ldots$$

The harmonic numbers and the digamma function

The function

$$H_z = \int_0^1 \frac{1 - x^z}{1 - x} \mathrm{d}x$$

which is defined for $z \geq 0$ and satisfies

$$H_n = \sum_{j=1}^{n} \frac{1}{j} \qquad \text{for } n \in \mathbf{Z},\ n \geq 0$$

is related with the digamma function

$$\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z} \log \Gamma(z)$$

by

$$\psi(z+1) = -\gamma + H_z.$$

Transcendence of the harmonic numbers

Ram Murty

Saradha Natarajan

For $q > 1$ and $1 \leq a \leq q-1$, the number $\psi(a/q) + \gamma$ is transcendental.
Hence, whenever q does not divide a, the harmonic number $H_{a/q}$ is transcendental.
(Ram Murty and N. Saradha, 2007).

Linear relations among the harmonic numbers

$r \in \mathbf{Q}$,

$$H_r = \int_0^1 \frac{1 - x^r}{1 - x} \mathrm{d}x = r \sum_{k=1}^{\infty} \frac{1}{k(r-k)} = \psi(r+2) + \gamma.$$

Tapas Chatterjee

Sonika Dhillon

Linear independence of harmonic numbers over the field of algebraic numbers
The Ramanujan Journal **51** (2020) 43–66

Euler–Briggs–Lehmer constants

William Briggs

1861 – 1932

Derrick Henry Lehmer

1905 – 1991

$$\gamma(h,k) = \lim_{x\to\infty} \left(\sum_{\substack{1\le n\le x \\ n\equiv h \mod k}} \frac{1}{n} - \frac{\log x}{k} \right) \qquad \gamma(2,4) = \frac{1}{4}\gamma$$

W. E. Briggs, The irrationality of γ or of sets of similar constants, 1961.
D. H. Lehmer, Euler constants for arithmetical progressions, 1975.

Euler–Briggs–Lehmer constants

Ram Murty

Saradha Natarajan

At most one of the numbers

$$\gamma(h,k), \quad 1 \leq h < k, \quad k \geq 2$$

is algebraic (Ram Murty and N. Saradha, 2010).

p–adic Euler–Briggs–Lehmer constants

Tapas Chatterjee

Sanoli Gun

At most one of the numbers in the set

$$\{\gamma_p\} \cup \{\gamma_p(r,q) \ \mid q \ \text{ prime }, \ 1 \leq r < q/2\}$$

is algebraic (2014).

Previous result by Murty and Saradha (2008) : for q fix.

Euler–Briggs–Lehmer constants

Ekata Saha

- Gun, Sanoli ; Saha, Ekata
A note on generalized Euler-Briggs constants.
Ramanujan Math. Soc. Lect. Notes Ser. 23, 93-104 (2016).
- Gun, Sanoli ; Saha, Ekata ; Sinha, Sneh Bala
A generalisation of an identity of Lehmer.
Acta Arith. 173, No. 2, 121-131 (2016).

- Gun, Sanoli ; Murty, V. Kumar ; Saha, Ekata
Linear and algebraic independence of generalized Euler-Briggs constants.
J. Number Theory 166, 117-136 (2016).
- Gun, Sanoli ; Saha, Ekata ; Sinha, Sneh Bala
Transcendence of generalized Euler–Lehmer constants.
J. Number Theory 145, 329-339 (2014).

$\zeta(1) = \gamma$?

We have

$$\Gamma(1+t) = \exp\left(-\gamma t + \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n} t^n\right).$$

We can write

$$\Gamma(1+t) = \exp\left(\sum_{n=1}^{\infty} (-1)^n \frac{\zeta(n)}{n} t^n\right).$$

provided that we set $\zeta(1) = \gamma$.
This normalisation is sometimes used in the study of *multizeta values*; another option is to replace $\zeta(1)$ by an unknown in the formulae involving $\zeta(n)$.

Thomas Johannes Stieltjes (1885)

The Laurent expansion of the Riemann zeta function at the pole $s = 1$ is

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (s-1)^n$$

with $\gamma_0 = \gamma$ and, for $n \geq 1$,

$$\gamma_n = \lim_{m\to\infty} \left(\sum_{k=1}^{m} \frac{(\log k)^n}{k} - \frac{(\log m)^{n+1}}{n+1} \right)$$

T. Stieltjes

1856 – 1894

Exponential periods

Paper by Kontsevich and Zagier :

The last chapter, which is at a more advanced level and also more speculative than the rest of the text, is by the first author only.

> *There have been some recent indications that one can extend the exponential motivic Galois group still further, adding as a new the Euler constant γ, which is, incidentally, the constant term of $\zeta(s)$ at $s = 1$. Then all classical constants are periods in an appropriate sense.*

Exponential periods

Lagarias quotes Kontsevich : the Euler constant is an exponential period :

$$\gamma = \int_0^1 \int_x^1 \frac{\mathrm{e}^{-x}}{y} \mathrm{d}y \mathrm{d}x - \int_1^\infty \int_1^x \frac{\mathrm{e}^{-x}}{y} \mathrm{d}y \mathrm{d}x.$$

Rests on

$$-\gamma = \int_0^\infty \mathrm{e}^{-x} \log x \mathrm{d}x.$$

The Gompertz constant is also an exponential period :

$$\delta = \int_0^\infty \frac{\mathrm{e}^{-t}}{1+t} \mathrm{d}t.$$

One conjectures that δ is not a period.

March 2, 2023

On Euler's Constant

Michel Waldschmidt

Professeur Émérite, Sorbonne Université,
Institut de Mathématiques de Jussieu, Paris
http://www.imj-prg.fr/~michel.waldschmidt/