Michel Waldschmidt

Exercices on the first course.

1. Let f be an entire function. Assume f is algebraic: there exists $P \in \mathbb{C}[X, Y]$, $P \neq 0$, such that $P(z, f(z)) = 0$. Prove that f is a polynomial: $f \in \mathbb{C}[z]$.

2. Given pairwise distinct complex numbers $\alpha_1, \ldots, \alpha_n$, positive integers $t_1, \ldots, t_n$ and complex numbers $\beta_{j,\tau}$ for $1 \leq j \leq n$, $0 \leq \tau < t_j$, show that there exists a unique polynomial f of degree $< t_1 + \cdots + t_n$ satisfying

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\tau} f(\alpha_j) = \beta_{j,\tau}$$

for $1 \leq j \leq n$ and $0 \leq \tau < t_j$.

3. Let f be a nonzero entire function of order $\leq \varrho$. For $r \geq 0$, denote by $n(f, r)$ the number of zeroes (counting multiplicities) of f in the disc $|z| \leq r$. Show that there exists a constant $c > 0$, depending only on f, such that, for $r \geq 1$,

$$n(f, r) \leq c r^{\varrho}.$$

4. Solve the exercise on Blaschke products p. 24.

5. From the definition of the Euler Gamma function by means of the canonical product:

$$\frac{1}{\Gamma(z)} = z \mathrm{e}^{\gamma z} \prod_{n \geq 1} \left(1 + \frac{z}{n}\right) \mathrm{e}^{-z/n},$$

deduce that $1/\Gamma(z)$ is an entire function of order 1 and infinite exponential type.

6. Check that Abel's polynomials

$$P_n(z) = \frac{1}{n!} z (z - n)^{n-1} \quad (n \geq 1)$$

satisfy, for $n \geq 1$,

$$|P_n|_r \leq \left(1 + \frac{r}{n}\right)^n \mathrm{e}^n.$$

7. Check the formula on divided differences p. 35.