Exercices on the second course.

1. Prove the two lemmas on entire functions p. 16.

2. Check $c_n'' = c_{n-1}$ for $n \geq 1$ p. 22.

3. Let S be a positive integer and let $z \in \mathbb{C}$. Using Cauchy's residue Theorem, compute the integral (see p. 26)

$$\frac{1}{2\pi i}\int_{|t|=(2S+1)\pi/2} t^{-2n-1}\frac{\mathrm{sh}(tz)}{\mathrm{sh}(t)}\mathrm{d}t.$$

4. Prove the proposition p. 31:
Let f be an entire function. The two following conditions are equivalent.
(i) $f^{(2k)}(0) = f^{(2k)}(1) = 0$ *for all* $k \geq 0$.
(ii) f *is the sum of a series*

$$\sum_{n\geq 1} a_n \sin(n\pi z)$$

which converges normally on any compact.
Prove also the following result:
Let f be an entire function. The two following conditions are equivalent.
(i) $f^{(2k+1)}(0) = f^{(2k)}(1) = 0$ *for all* $k \geq 0$.
(ii) f *is the sum of a series*

$$\sum_{n\geq 1} a_n \cos\left(\frac{(2n+1)\pi}{2} z\right).$$

which converges normally on any compact.

5. Complete the three proofs of the Lemma p. 33.

6. Let $(M_n(z))_{n\geq 0}$ and $(\widetilde{M}_n(z))_{n\geq 0}$ be two sequences of polynomials such that any polynomial $f \in \mathbb{C}[z]$ has a finite expansion

$$f(z) = \sum_{n=0}^{\infty}\left(f^{(2n)}(1)M_n(z) + f^{(2n+1)}(0)\widetilde{M}_n(z)\right),$$

with only finitely many nonzero terms in the series (see p. 34). Check

$$\widetilde{M}_n(z) = -M'_{n+1}(1-z)$$

for $n \geq 0$.
`Hint:` Consider $f'(1-z)$.

7. Let S be a positive integer and let $z \in \mathbb{C}$. Using Cauchy's residue Theorem, compute the integral (see p. 39)

$$\frac{1}{2\pi i}\int_{|t|=S\pi} t^{-2n-1}\frac{\mathrm{ch}(tz)}{\mathrm{ch}(t)}\mathrm{d}t.$$

8. Give examples of complete, redundant and indeterminate systems in Whittaker classification p. 43.