Michel Waldschmidt

Exercices: hints, solutions, comments

Fourth course

1. Answer the quizz p. 29.

1. The definition of u^z when u is a positive real number and z a complex number is $u^z = \exp(z \log u)$ with the real logarithm of u.When $u \neq 1$, this is an entire function of z of order 1 and exponential type $|\log u|$. When u is a nonzero complex number which is not real > 0, for instance a negative real number, the definition of u^z depends on the choice of a logarithm of u, namely a complex number, say $\log u$, such that $\exp(\log u) = u$. There are infinitely many of them, we selects one. Then the exponential type of this function $u^z = \exp(z \log u)$ is $|\log u|$.

Since the Golden ratio $\phi = \frac{1+\sqrt{5}}{2}$ is > 0, the function ϕ^z is well defined by $\phi^z = \exp(z \log \phi)$, it has exponential type $\log \phi = 0.481\ldots$

However since $\tilde{\phi} = -\phi^{-1} = -0.618\cdots < 0$, the definition of $\tilde{\phi}^z$ depends on a choice of the logarithm of the negative number $\tilde{\phi}$. The minimal modulus of such a logarithm is

$$\tau = \Big((\log|\tilde{\phi}|)^2 + \pi^2\Big)^{1/2} = 3.178\ldots$$

when $\log \tilde{\phi} = \log|\tilde{\phi}| \pm i\pi$. With such a choice, the type of $\tilde{\phi}^z = \exp(z \log \tilde{\phi})$ is τ.

2. Show that there exist entire functions of arbitrarily large order giving counterexamples to Bieberbach's claim p. 44.

2. For $k \geq 1$, set $a(z) = \frac{1}{2} z(z-1)(z-2)\cdots(z-4k+1)$. The function $f(z) = \mathrm{e}^{a(z)}$ has order $4k$ and type $\tau_{4k}(f) = 1/2$.

There are $2k$ even factors and $2k$ odd factors, hence modulo $2\mathbb{Z}[z]$ the polynomial $a(z)$ is congruent to $z^{2k}(z^{2k}-1)$. The coefficients of z^{2i+1} are even, hence $a'(z) \in \mathbb{Z}[z]$. We deduce that $f(z) = \mathrm{e}^{a(z)}$ is a k–point Hurwitz entire function.

For $k = 1$, this reduces to the example p. 45.

3. Let f be an entire function. Let $A \geq 0$. Assume

$$\limsup_{r\to\infty} \mathrm{e}^{-r}\sqrt{r}|f|_r < \frac{\mathrm{e}^{-A}}{\sqrt{2\pi}}\cdot$$

(a) Prove that there exists $n_0 > 0$ such that, for $n \geq n_0$ and for all $z \in \mathbb{C}$ in the disc $|z| \leq A$, we have

$$|f^{(n)}(z)| < 1.$$

(b) Assume that f is transcendental. Deduce that the set

$$\big\{(n, z_0) \in \mathbb{N} \times \mathbb{C} \;\mid\; |z_0| \leq A,\; f^{(n)}(z_0) \in \mathbb{Z} \setminus \{0\}\big\}$$

is finite.

3.

(a) By assumption, there exists $\eta > 0$ such that, for n sufficiently large, we have

$$|f|_n < (1-\eta)\frac{\mathrm{e}^{n-A}}{\sqrt{2\pi n}}\cdot$$

We use Cauchy's inequalities
$$\frac{|f^{(n)}(z_0)|}{n!}r^n \le |f|_{r+|z_0|},$$
(which are valid for all $z_0 \in \mathbb{C}$, $n \ge 0$ and $r > 0$) with $r = n - A$: for $|z| \le A$, we have
$$|f^{(n)}(z)| \le \frac{n!}{(n-A)^n}|f|_n.$$
Hence Stirling's inequality
$$n! \ge n^n \mathrm{e}^{-n}\sqrt{2\pi n}$$
yields
$$|f^{(n)}(z)| \le (1-\eta)\mathrm{e}^{-A+1/(12n)}\left(1-\frac{A}{n}\right)^{-n}.$$
For n sufficiently large, the right hand side is < 1.
(b) *We need to assume that f is transcendental : indeed, if f is a polynomial with leading term $a_0 z^d$ where $d!a_0 \in \mathbb{Z}\setminus\{0\}$, then $f^{(d)}(z_0) = d!a_0 \in \mathbb{Z}\setminus\{0\}$ for all z_0 with $|z_0| \le A$, and hence the set is infinite.*

The condition $f^{(n)}(z_0) \in \mathbb{Z}\setminus\{0\}$ implies $|f^{(n)}(z_0)| \ge 1$. From (a) we deduce that there exists n_0 such that the conditions $(n, z_0) \in \mathbb{N}\times\mathbb{C}$, $|z_0| \le A$ and $f^{(n)}(z_0) \in \mathbb{Z}\setminus\{0\}\}$ imply $n \le n_0$. Fix $n \le n_0$. The function f is bounded on the disc $|z| \le A$, say $|f(z)| \le B$ for $|z| \le A$. Let $b \in \mathbb{Z}\setminus\{0\}$, $|b| \le B$. Since $f^{(n)}$ is not constant, the function $f^{(n)}(z) - b$ is not zero, it has only finitely many zeroes in the disc $|z| \le A$ and therefore the set of z_0 with $|z_0| \le A$ such that $f^{(n)}(z_0) = b$ is finite.

4. Let $(e_n)_{n\ge1}$ be a sequence of elements in $\{1, -1\}$. Check that the function
$$f(z) = \sum_{n\ge0}\frac{e_n}{2^n!}z^{2^n}$$
is a transcendental entire functions which satisfies
$$\limsup_{r\to\infty}\sqrt{r}\mathrm{e}^{-r}|f|_r = \frac{1}{\sqrt{2\pi}}.$$

4. Let $\epsilon > 0$ and let r tend to infinity. Let N be the integer such that
$$2^{N-\frac{1}{2}} \le r < 2^{N+\frac{1}{2}}.$$
For $|z| = r$, we split the sum defining $f(z)$ in three subsums. Set
$$S_1 = \sum_{n<N}\frac{1}{2^n!}r^{2^n}, \quad S_2 = \frac{1}{2^N!}r^{2^N}, \quad S_3 = \sum_{n>N}\frac{1}{2^n!}r^{2^n}.$$
We claim
$$\max\{S_1, S_3\} \le \epsilon\frac{\mathrm{e}^r}{\sqrt{2\pi r}}$$
and
$$S_2 \le (1+\epsilon)\frac{\mathrm{e}^r}{\sqrt{2\pi r}}.$$
This will prove
$$\limsup_{r\to\infty}\sqrt{r}\mathrm{e}^{-r}|f|_r \le \frac{1}{\sqrt{2\pi}}.$$

With $r = 2^N$ we get equality.

Set $M = 2^N$, so that $\frac{M}{\sqrt{2}} \le r < M\sqrt{2}$.. Write $M = \alpha r$ with $\frac{1}{\sqrt{2}} \le \alpha < \sqrt{2}$. Stirling's formula yields

$$\frac{r^M}{M!} = \left(\frac{\mathrm{e}^\alpha}{\alpha^\alpha}\right)^r \frac{1}{\sqrt{2\pi\alpha r}}(1 + o(r)).$$

The function $x - x\log x$ for $x > 0$ has it maximum at $x = 1$, this maximum is 1. Hence $\frac{\mathrm{e}^\alpha}{\alpha^\alpha} \le e$ with equality at $\alpha = 1$. We deduce

$$\frac{r^M}{M!} \le \frac{e^r}{\sqrt{2\pi r}}(1 + o(r)).$$

The upper bound for S_2 follows. For S_1 and S_3, use

$$S_1 \le \frac{N}{2^{N-1}!} r^{2^{N-1}} \quad \text{and} \quad S_3 \le \frac{2}{(2M)!} r^{2M}$$

and apply Stirling's formula as above.

5. Let s_0 and s_1 be two complex numbers and f an entire function satisfying $f^{(2n)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$ for all sufficiently large n. Assume the exponential type $\tau(f)$ satisfies

$$\tau(f) < \min\left\{1, \frac{\pi}{|s_0 - s_1|}\right\}.$$

Prove that f is a polynomial.

Prove that the assumption on $\tau(f)$ is optimal.

5. (a) Let f satisfy the assumptions. Using exercise 3 above, we deduce from the assumption $\tau(f) < 1$ that the sets

$$\{n \ge 0 \mid f^{(2n)}(s_0) \ne 0\} \text{ and } \{n \ge 0 \mid f^{(2n)}(s_1) \ne 0\}$$

are finite. Define, for $n \ge 0$,

$$\widehat{\Lambda}_n(z) = (s_1 - s_0)^{2n} \Lambda_n\left(\frac{z}{s_1 - s_0}\right).$$

Hence

$$P(z) = \sum_{n=0}^{\infty} \left(f^{(2n)}(s_1)\widehat{\Lambda}_n(z - s_0) - f^{(2n)}(s_0)\widehat{\Lambda}_n(z - s_1)\right)$$

is a polynomial satisfying

$$P^{(2n)}(s_0) = f^{(2n)}(s_0) \text{ and } P^{(2n)}(s_1) = f^{(2n)}(s_1) \text{ for all } n \ge 0.$$

The function $\tilde{f}(z) = f(z) - P(z)$ has the same exponential type as f and satisfies

$$\tilde{f}^{(2n)}(s_0) = \tilde{f}^{(2n)}(s_1) = 0 \text{ for all } n \ge 0.$$

Set

$$\hat{f}(z) = \tilde{f}\big(s_0 + z(s_1 - s_0)\big),$$

so that

$$\hat{f}^{(2n)}(0) = \hat{f}^{(2n)}(1) = 0 \text{ for all } n \ge 0.$$

The exponential types of f and $\hat{f}$ are related by

$$\tau(\hat{f}) = |s_1 - s_0|\tau(f).$$

From the assumption on the upper bound for $\tau(f)$ we deduce $\tau(\hat{f}) < \pi$. From Poritsky's Theorem (course 2 p. 29) we deduce that $\hat{f}(z)$ is a polynomial, hence f also.
(b) The function

$$f(z) = \frac{\mathrm{sh}(z - s_1)}{\mathrm{sh}(s_0 - s_1)}$$

has exponential type 1 and satisfies $f(s_0) = 1$, $f(s_1) = 0$ and $f'' = f$, hence $f^{(2n)}(s_0) = 1$ and $f^{(2n)}(s_1) = 0$ for all $n \geq 0$.

The function

$$f(z) = \sin\left(\pi \frac{z - s_0}{s_1 - s_0}\right)$$

has exponential type $\frac{\pi}{|s_1 - s_0|}$ and satisfies $f^{(2n)}(s_0) = f^{(2n)}(s_1) = 0$ for all $n \geq 0$.

6. Let s_0 and s_1 be two complex numbers and f an entire function satisfying $f^{(2n+1)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$ for all sufficiently large n. Assume the exponential type $\tau(f)$ satisfies

$$\tau(f) < \min\left\{1, \frac{\pi}{2|s_0 - s_1|}\right\}.$$

Prove that f is a polynomial.
Prove that the assumption on $\tau(f)$ is optimal.

6. (a) Let f satisfy the assumptions. Using exercise 3 above, we deduce from the assumption $\tau(f) < 1$ that the sets

$$\{n \geq 0 \mid f^{(2n+1)}(s_0) \neq 0\} \text{ and } \{n \geq 0 \mid f^{(2n)}(s_1) \neq 0\}$$

are finite. Define, for $n \geq 0$,

$$\widehat{M}_n(z) = (s_1 - s_0)^{2n} M_n\left(\frac{z}{s_1 - s_0}\right).$$

Hence

$$P(z) = \sum_{n=0}^{\infty} \left(f^{(2n)}(s_1)\widehat{M}_n(z - s_0) + f^{(2n+1)}(s_0)\widehat{M}'_{n+1}(z - s_1)\right)$$

is a polynomial satisfying

$$P^{(2n+1)}(s_0) = f^{(2n+1)}(s_0) \text{ and } P^{(2n)}(s_1) = f^{(2n)}(s_1) \text{ for all } n \geq 0.$$

The function $\tilde{f}(z) = f(z) - P(z)$ has the same exponential type as f and satisfies

$$\tilde{f}^{(2n+1)}(s_0) = \tilde{f}^{(2n)}(s_1) = 0 \text{ for all } n \geq 0.$$

Set

$$\hat{f}(z) = \tilde{f}\big(s_0 + z(s_1 - s_0)\big),$$

so that

$$\hat{f}^{(2n+1)}(0) = \hat{f}^{(2n)}(1) = 0 \text{ for all } n \geq 0.$$

The exponential types of f and $\hat{f}$ are related by

$$\tau(\hat{f}) = |s_1 - s_0|\tau(f).$$

From the assumption on the upper bound for $\tau(f)$ we deduce $\tau(\hat{f}) < \pi/2$. From Whittaker's Theorem (course 2 p. 37) we deduce that $\hat{f}(z)$ is a polynomial, hence f also.
(b) The function

$$f(z) = \frac{\mathrm{sh}(z - s_1)}{\mathrm{ch}(s_0 - s_1)}$$

has exponential type 1 and satisfies $f'(s_0) = 1$, $f(s_1) = 0$ and $f'' = f$, hence $f^{(2n+1)}(s_0) = 1$ and $f^{(2n)}(s_1) = 0$ for all $n \geq 0$.

The function

$$f(z) = \cos\left(\frac{\pi}{2} \cdot \frac{z - s_0}{s_1 - s_0}\right)$$

has exponential type $\frac{\pi}{2|s_1 - s_0|}$ and satisfies $f^{(2n+1)}(s_0) = f^{(2n)}(s_1) = 0$ for all $n \geq 0$.

7. Recall Abel's polynomials $P_0(z) = 1$,

$$P_n(z) = \frac{1}{n!} z(z - n)^{n-1} \quad (n \geq 1).$$

Let ω be the positive real number defined by $\omega \mathrm{e}^{\omega+1} = 1$. The numerical value is $\omega = 0.278\,464\,542\ldots$
(a) For $t \in \mathbb{C}$, $|t| < \omega$ and $z \in \mathbb{C}$, check

$$\mathrm{e}^{tz} = \sum_{n\geq 0} t^n \mathrm{e}^{nt} P_n(z),$$

where the series in the right hand side is absolutely and uniformly convergent on any compact of $\mathbb{C}$.
`Hint.` *Let $t \in \mathbb{R}$ satisfy $0 < t < \omega$ and let $z \in \mathbb{R}$. For $n \geq 0$, define*

$$R_n(z) = \mathrm{e}^{tz} - \sum_{k=0}^{n-1} t^k \mathrm{e}^{kt} P_k(z).$$

Check $R_n(0) = 0$, $R_n'(z) = R_{n-1}(z - 1)$, so that

$$R_n(z) = t\mathrm{e}^t \int_0^z R_{n-1}(w - 1)\mathrm{d}w = (t\mathrm{e}^t)^n \int_0^z \mathrm{d}w_1 \int_1^{w_1} \mathrm{d}w_2 \cdots \int_{n-1}^{w_{n-1}} R_0(w_n - 1)\mathrm{d}w_n.$$

Deduce

$$|R_n(z)| \leq (t\mathrm{e}^t)^n \frac{(|z| + n)^n}{n!} \mathrm{e}^{t|z|}$$

(see [Gontcharoff 1930, p. 11-12] and [Whittaker, 1933, Chap. III, (8.7)]).
(b) Let f be an entire function of finite exponential type $< \omega$. Prove

$$f(z) = \sum_{n\geq 0} f^{(n)}(n) P_n(z),$$

where the series in the right hand side is absolutely and uniformly convergent on any compact of $\mathbb{C}$.
(c) Prove that there is no nonzero entire function f of exponential type $< \omega$ satisfying $f^{(n)}(n) = 0$ for all $n \geq 0$. Give an example of a nonzero entire function f of finite exponential type satisfying $f^{(n)}(n) = 0$ for all $n \geq 0$.
(d) Let $t \in \mathbb{C}$ satisfy $|t| < \omega$. Set $\lambda = t\mathrm{e}^t$. Let f be an entire function of exponential type $< \omega$ which satisfies

$$f'(z) = \lambda f(z - 1).$$

Prove

$$f(z) = f(0)\mathrm{e}^{tz}.$$

7. (a) By analytic continuation it suffices to prove the formula for $0 < t < \omega$ and $z \in \mathbb{R}$. Fix such a t. For $n \geq 0$ and $z \in \mathbb{R}$, define

$$R_n(z) = e^{tz} - \sum_{k=0}^{n-1} t^k e^{kt} P_k(z).$$

We have $R_0(z) = e^{tz} - 1$ and for $n \geq 1$

$$R'_n(z) = te^t R_{n-1}(z-1)$$

with $R_n(0) = 0$, so that

$$R_n(z) = te^t \int_0^z R_{n-1}(w-1)\mathrm{d}w = (te^t)^n \int_0^z \mathrm{d}w_1 \int_1^{w_1} \mathrm{d}w_2 \cdots \int_{n-1}^{w_{n-1}} e^{tw_n}\mathrm{d}w_n.$$

We deduce, for $z \in \mathbb{R}$ and $n \geq 0$,

$$|R_n(z)| \leq (te^t)^n \frac{(|z|+n)^n}{n!} e^{t|z|}$$

(for the details, see (see [Gontcharoff 1930, p. 11-12] and [Whittaker, 1933, Chap. III, (8.7)]). Stirling's formula shows that the assumption $te^{t+1} < 1$ implies $R_n(z) \to 0$ as $n \to \infty$.
(b) Let

$$f(z) = \sum_{n\geq 0} \frac{a_n}{n!} z^n$$

be an entire function of exponential type $\tau(f)$. The Laplace transform of f, viz.

$$F(t) = \sum_{n\geq 0} a_n t^{-n-1},$$

is analytic in the domain $|t| > \tau(f)$. From Cauchy's residue Theorem, it follows that for $r > \tau(f)$ we have

$$f(z) = \frac{1}{2\pi i} \int_{|t|=r} e^{tz} F(t)\mathrm{d}t.$$

We replace e^{tz} by the series in the formula proved in (a) and get, for $\tau(f) < r < \omega$,

$$f(z) = \frac{1}{2\pi i} \int_{|t|=r} \sum_{n\geq 0} t^n e^{nt} P_n(z) F(t)\mathrm{d}t = \sum_{n\geq 0} P_n(z) \frac{1}{2\pi i} \int_{|t|=r} t^n e^{nt} F(t)\mathrm{d}t.$$

For $n \geq 0$ we have

$$f^{(n)}(z) = \frac{1}{2\pi i} \int_{|t|=r} t^n e^{tz} F(t)\mathrm{d}t,$$

hence

$$f^{(n)}(n) = \frac{1}{2\pi i} \int_{|t|=r} t^n e^{nt} F(t)\mathrm{d}t,$$

so that

$$f(z) = \sum_{n\geq 0} f^{(n)}(n) P_n(z).$$

(c) From (b), one deduces that an entire function f of exponential type $< \omega$ satisfying $f^{(n)}(n) = 0$ for all $n \geq 0$ is the zero function.

The function $\sin(\pi z/2)$ has type $\pi/2$ and satisfies $f^{(n)}(n) = 0$ for all $n \geq 0$. Notice that $\omega < \pi/2 = 1.570\ldots$
(d) The function $g(z) = f(z) - f(0)\mathrm{e}^{tz}$ satisfies $g(0) = 0$ and $g'(z) = \lambda g(z-1)$, hence $g^{(n)}(n) = 0$ for all $n \geq 0$. Since g has an exponential type $< \omega$, we deduce from (c) that $g = 0$.

Références

[Gontcharoff 1930] W. Gontcharoff, "Recherches sur les dérivées successives des fonctions analytiques. Généralisation de la série d'Abel", *Ann. Sci. École Norm. Sup.* (3) **47** (1930), 1–78. MR Zbl

[Whittaker, 1933] Whittaker, J. M. (1933). On Lidstone's series and two-point expansions of analytic functions. *Proc. Lond. Math. Soc. (2)*, 36 :451–469. `https://doi.org/10.1112/plms/s2-36.1.451`