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Sectional Meeting of the AMS
Special session on Algebraic Number Theory,
Diophantine Equations and Related Topics

Families of Diophantine equations with only trivial S-integral points

(joint work with Claude Levesque)

Michel Waldschmidt

Institut de Mathématiques de Jussieu & CIMPA
http://www.math.jussieu.fr/∼miw/

Abstract

So far, a rather small number of families of Diophantine Thue equations having only trivial solutions have been exhibited – explicit families of Thue–Mahler equations having this property were not known. We produce a large collection of examples. Newcastle : 15/03 4.00pm Michel Waldschmidt – Keynote Speaker - Some families of curves with finitely many integer points
So far, a rather small number of families of Thue curves having only trivial points have been exhibited. In a joint work with Claude Levesque, for each number field of degree at least 3, we produce families of curves related to the units of the number field having only trivial points. Further, we exhibit a large collection of examples of explicit families of Thue–Mahler equations having only trivial solutions

Families of Thue equations

The first families of Thue equations having only trivial solutions were introduced by A. Thue himself.

$$(a+1)X^n - aY^n = 1.$$

E. Thomas in 1990 studied

$$X^3 - (a-1)X^2Y - (a+2)XY^2 - Y^3 = 1.$$

Further work on this equation are due to E. Thomas, M. Mignote, F. Lemmermeyer.

Families of Thue equations (continued)

E. Lee, M. Mignotte and N. Tzanakis studied

$$X^3 - aX^2Y - (a+1)XY^2 - Y^3 = 1.$$

The left hand side is $X(X+Y)(X-(a+1)Y) - Y^3$.
I. Wakabayashi studied

$$X^3 - a^2XY^2 + Y^3 = 1.$$

A. Togbé considered

$$X^3 - (n^3 - 2n^2 + 3n - 3)X^2Y - n^2XY^2 - Y^3 = \pm 1.$$

I. Wakabayashi used Padé approximation for solving the Diophantine inequality

$$|X^3 + aXY^2 + bY^3| \leq a + |b| + 1.$$

Families of Thue equations (continued)

E. Thomas considered some families of Diophantine equations

$$X^3 - bX^2Y + cXY^2 - Y^3 = 1$$

for restricted values of b and c.

Further work by J.H. Chen, I. Gaál, C. Heuberger, B. Jadrijević, G. Lettl, C. Levesque, M. Mignotte, A. Pethő, R. Roth, R. Tichy, E. Thomas, A. Togbé, P. Voutier, I. Wakabayashi, P. Yuan, V. Ziegler...

Surveys by I. Wakabayashi (2002) and C. Heuberger (2005).

New families of Thue equations

Let K be a number field. For each $\varepsilon \in \mathbf{Z}_K^\times$, let $f_\varepsilon(X) \in \mathbf{Z}[X]$ be the irreducible polynomial of ε over $\mathbf{Q}$. Denote by $d = [\mathbf{Q}(\varepsilon) : \mathbf{Q}]$ its degree.

Set $F_\varepsilon(X,Y) = Y^d f_\varepsilon(X/Y)$. Hence $F_\varepsilon(X,Y) \in \mathbf{Z}[X,Y]$ is an irreducible binary form with integer coefficients.

A corollary of our main result is the following :

Corollary

Let K be a number field and let $m \in K$, $m \neq 0$. Then the set

$$\left\{(x,y,\varepsilon) \in \mathbf{Z}^2 \times \mathbf{Z}_K^\times \mid xy \neq 0,\ [\mathbf{Q}(\varepsilon):\mathbf{Q}] \geq 3,\ F_\varepsilon(x,y) = m\right\}$$

is finite.

Families of Thue–Mahler equations

A more general corollary of our main result is the following :

Corollary

Further, let $p_1, \dots, p_s$ be finitely many primes. Then the set of $(x,y,z_1,\dots,z_s,\varepsilon) \in \mathbf{Z}^{2+s} \times \mathbf{Z}_K^\times$ with $z_j \geq 0$ for $j = 1,\dots,s$, $xy \neq 0$ and $\gcd(xy, p_1 \cdots p_s) = 1$ such that $[\mathbf{Q}(\varepsilon):\mathbf{Q}] \geq 3$ and

$$F_\varepsilon(x,y) = m p_1^{z_1} \cdots p_s^{z_s}$$

is finite.

The general equation

Let K be a number field, S a finite set of places of K containing the infinite places, μ, $\alpha_1, \alpha_2, \alpha_3$ nonzero elements in K. Consider the equation

$$(X - \alpha_1 E_1 Y)(X - \alpha_2 E_2 Y)(X - \alpha_3 E_3 Y) Z = \mu E,$$

where the variables take for values

$$(x, y, z, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon) \in \mathcal{O}_S^3 \times (\mathcal{O}_S^\times)^4.$$

Trivial solutions are solutions with $xy = 0$.
Two nontrivial solutions $(x, y, z, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon)$ and $(x', y', z', \varepsilon'_1, \varepsilon'_2, \varepsilon'_3, \varepsilon')$ are called S^3*–dependent* if there exist S–units η_1, η_2 and η_3 in $\mathcal{O}_S^\times$ such that

$$x' = x\eta_1,\ y' = y\eta_1\eta_3^{-1},\ z' = z\eta_2,\ \varepsilon'_i = \varepsilon_i\eta_3,\ \varepsilon' = \varepsilon\eta_1^3\eta_2.$$

The main result

Theorem

The set of classes of S^3–dependence of the nontrivial solutions

$$(x, y, z, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon) \in \mathcal{O}_S^3 \times (\mathcal{O}_S^\times)^4$$

of the equation

$$(X - \alpha_1 E_1 Y)(X - \alpha_2 E_2 Y)(X - \alpha_3 E_3 Y)Z = \mu E$$

satisfying $\mathrm{Card}\{\alpha_1\varepsilon_1, \alpha_2\varepsilon_2, \alpha_3\varepsilon_3\} = 3$ *is finite*

The number of these classes is bounded by an explicit constant depending only on K, μ, $\alpha_1, \alpha_2, \alpha_3$ and the rank s of the group $\mathcal{O}_S^\times$.

A "special" case

It turns out that the special case of the equation

$$(X - Y)(X - E_1 Y)(X - E_2 Y) = E$$

is equivalent to the general case.

Two solutions $(x, y, \varepsilon_1, \varepsilon_2, \varepsilon)$ and $(x', y', \varepsilon_1', \varepsilon_2', \varepsilon')$ in $\mathcal{O}_S^2 \times (\mathcal{O}_S^\times)^3$ of this equation are called S*–dependent* if there exists $\eta \in \mathcal{O}_S^\times$such that

$$x' = x\eta,\ y' = y\eta,\ \varepsilon_1' = \varepsilon_1,\ \varepsilon_2' = \varepsilon_2,\ \varepsilon' = \varepsilon\eta^3.$$

Theorem

The number of classes of S–dependence of the solutions $(x, y, \varepsilon_1, \varepsilon_2, \varepsilon)$ with $\varepsilon_1 \neq 1$, $\varepsilon_2 \neq 1$, $\varepsilon_1 \neq \varepsilon_2$ of the equation $(X - Y)(X - E_1 Y)(X - E_2 Y) = E$ is finite.

Effectivity

Explicit upper bounds for the number of solutions or for the number of classes of solutions are obtained by means of quantitative versions of the Subspace Theorem of W.M. Schmidt, but effective bounds for the solutions or for the heights of the solutions are not available in general.

In a few special cases we are able to produce effective results (work in progress).

The norm N_S

Let $\alpha \in K^\times$. Write the fractional ideal (α) of K with respect to $\mathbf{Z}_K$ as $\mathfrak{A}\mathfrak{B}$, where $\mathfrak{A}$ is a product of prime ideals which are not above the finite places of S, while $\mathfrak{B}$ is a product of prime ideals which are above the finite places of S. Denote by $N_S(\alpha)$ the norm of the ideal $\mathfrak{A}$.

Hence, for $\alpha \in K$, we have $\alpha \in \mathcal{O}_S$ if and only if $\mathfrak{A}$ is an integral ideal of $\mathbf{Z}_K$.

Remarks on the norm N_S

- The S–units of K are the elements ε in $\mathcal{O}_S$ such that

$$\mathrm{N}_S(\varepsilon) = 1.$$

- In case $S = S_\infty$, then $\mathcal{O}_S = \mathbf{Z}_K$, $\mathcal{O}_S^\times = \mathbf{Z}_K^\times$ and N_S is the absolute value of the usual norm $|\mathrm{N}_{K/\mathbf{Q}}|$.

- In case $K = \mathbf{Q}$, for $x \in \mathbf{Q}^\times$, we have

$$\mathrm{N}_S(x) = \prod_{p \notin S} p^{v_p(x)} \quad \text{for} \quad x = \pm \prod_p p^{v_p(x)}.$$

Thue–Mahler inequality

Let m be a positive integer and $F \in K[X, Y]$ a binary form. Two solutions (x, y), (x', y') of the inequality

$$0 < \mathrm{N}_S(F(x, y)) \leq m$$

are called dependent if there exists $\eta \in K^\times$ such that $x' = \eta x$ and $y' = \eta y$.

This inequality is nothing else but a finite collection of Thue–Mahler equations.

Equivalent binary forms

Two binary forms $F(X, Y)$ and $G(X, Y)$ in $\mathcal{O}_S[X, Y]$ are called S*–equivalent* if there exist $\alpha, \beta, \gamma, \delta$ in $\mathcal{O}_S$ and η in $\mathcal{O}_S^\times$ satisfying $\alpha\delta - \beta\gamma \in \mathcal{O}_S^\times$ and

$$G(X, Y) = \eta F(\alpha X + \beta Y, \gamma X + \delta Y).$$

Evertse – Győry

Let K be a number field, S a finite set of places of K containing the archimedean places and n an integer ≥ 3. Denote by $\mathcal{F}(n, K, S)$ the set of binary forms $F \in \mathcal{O}_S[X, Y]$ of degree n which split in $K[X, Y]$ and for which the decomposition into linear factors contains at least three distinct linear factors.

Then, for any $m > 0$, there are only finitely many S–equivalence classes of binary forms F in $\mathcal{F}(n, K, S)$ such that there are more than two independent solutions $(x, y) \in \mathcal{O}_S \times \mathcal{O}_S$ to the inequality

$$0 < \mathrm{N}_S(F(x, y)) \leq m.$$

The forms $F_{\underline{\varepsilon}}$ and the set $\mathcal{E}$

Let K be a number field, S a finite set of places of K containing the archimedean places, n an integer ≥ 3, $\alpha_1, \dots, \alpha_n$ elements in $K^\times$ and $F \in K[X, Y]$ the binary form

$$F(X, Y) = (X - \alpha_1 Y)(X - \alpha_2 Y) \cdots (X - \alpha_n Y).$$

For $\underline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n) \in (\mathcal{O}_S^\times)^n$, denote by $F_{\underline{\varepsilon}} \in K[X, Y]$ the binary form

$$F_{\underline{\varepsilon}}(X, Y) = (X - \alpha_1\varepsilon_1 Y)(X - \alpha_2\varepsilon_2 Y) \cdots (X - \alpha_n\varepsilon_n Y).$$

Denote by $\mathcal{E}$ the set of $\underline{\varepsilon}$ in $(\mathcal{O}_S^\times)^n$ such that $\varepsilon_1 = 1$ and

$$\mathrm{Card}\{\alpha_1\varepsilon_1, \alpha_2\varepsilon_2, \dots, \alpha_n\varepsilon_n\} \geq 3.$$

$$F_{\underline{\varepsilon}}(X, Y) = (X - \alpha_1\varepsilon_1 Y)(X - \alpha_2\varepsilon_2 Y) \cdots (X - \alpha_n\varepsilon_n Y)$$

Proposition

There exists a finite subset $\mathcal{E}^\star$ of $\mathcal{E}$ such that, for any $\underline{\varepsilon} \in \mathcal{E} \setminus \mathcal{E}^\star$ and any $(x, y) \in \mathcal{O}_S \times \mathcal{O}_S$, the condition

$$F_{\underline{\varepsilon}}(x, y) \in \mathcal{O}_S^\times$$

implies $xy = 0$.

Connection with the result of Evertse–Győry

Let $\underline{\varepsilon} \in \mathcal{E} \setminus \mathcal{E}^\star$. There are only finitely many $\underline{\varepsilon}' \in \mathcal{E} \setminus \mathcal{E}^\star$ such that the two binary forms $F_{\underline{\varepsilon}}$ and $F_{\underline{\varepsilon}'}$ are S–equivalent.

From our main result we deduce :
For each $m \in \mathbf{N}$ and each $\underline{\varepsilon} \in \mathcal{E}$ outside a finite set (depending on m), the inequality

$$0 < \mathrm{N}_S\big(F_{\underline{\varepsilon}}(x, y)\big) \leq m$$

has no solution $(x, y) \in \mathcal{O}_S \times \mathcal{O}_S$ with $xy \neq 0$.

Therefore we obtain an infinite number of S–equivalence classes of binary forms which produce Thue–Mahler inequalities $0 < \mathrm{N}_S(F(x, y)) \leq m$ having only trivial solutions.

Connection with a result of P. Vojta

Let D be a divisor of $\mathbf{P}^n$ with at least $n+2$ distinct components. Then any set of D–integral points on $\mathbf{P}^n$ is degenerate (namely : is contained in a proper Zarisky closed set).

With $n = 4$, with projective coordinates $(X : Y : Z : E_1 : E_2)$ and with the divisor

$$Z\, E_1\, E_2\, (X - Y)(XZ - E_1 Y)(XZ - E_2 Y) = 0$$

on $\mathbf{P}^4$, one deduces that the set of solutions of the equation

$$(X - Y)(X - E_1 Y)(X - E_2 Y) = E$$

is degenerate.

Sketch of proof of the main theorem

Let α_1, α_2, α_3, μ be nonzero elements of the number field K. Consider a solution $(x, y, z, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon)$ in $\mathcal{O}_S^3 \times (\mathcal{O}_S^\times)^4$ of the equation

$$(X - \alpha_1 E_1 Y)(X - \alpha_2 E_2 Y)(X - \alpha_3 E_3 Y)Z = \mu E$$

satisfying $xy \neq 0$ and $\mathrm{Card}\{\alpha_1\varepsilon_1, \alpha_2\varepsilon_2, \alpha_3\varepsilon_3\} = 3$:

$$(x - \alpha_1\varepsilon_1 y)(x - \alpha_2\varepsilon_2 y)(x - \alpha_3\varepsilon_3 y)z = \mu\varepsilon.$$

Sketch of proof of the main theorem (continued)

Set $\beta_j = x - \alpha_j\varepsilon_j y$ ($j = 1, 2, 3$), so that $\beta_1\beta_2\beta_3 z = \mu\varepsilon$.
À la Siegel, eliminate x and y among the three equations

$$\beta_1 = x - \alpha_1\varepsilon_1 y, \; \beta_2 = x - \alpha_2\varepsilon_2 y, \; \beta_3 = x - \alpha_3\varepsilon_3 y.$$

We deduce

$$u_{12} - u_{13} + u_{23} - u_{21} + u_{31} - u_{32} = 0,$$

where

$$u_{ij} = \alpha_i\epsilon_i\beta_j, \quad (i, j = 1, 2, 3, \; i \neq j).$$

This is a generalized S–unit equation with six terms. But nontrivial subsums may vanish...

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