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Representation Theory. College of Science, University of Sulaimani, Sulaymaniyah. CIMPA West Asian Mathematical School.

# Introduction to Galois Theory

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## Abstract

Field extensions. Degree of extension. Algebraic numbers. Geometric constructions with ruler and compasses.The Galois group of an extension. The Galois correspondence between subgroups and intermediate fields. Splitting field for a polynomial. Transitivity of the Galois group on the zeros of an irreducible polynomial in a normal extension. Properties equivalent to normality. Galois groups of normal separable extensions. Properties of Galois correspondence for normal separable extensions. Normal subgroups and normal intermediate extensions. The Fundamental Theorem of Galois Theory.



Fields : $\mathbb{C}$, $\overline{\mathbb{Q}}$, $\mathbb{R}$,

$\mathbb{Q}$

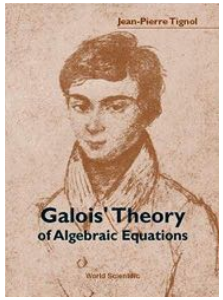
$\mathbb{F}_q$

Function fields

Differential fields



## Galois theory of algebraic equations




Jean–Pierre Tignol

https://perso.uclouvain.be/jean-pierre.tignol/Page_personnelle_de_Jean-Pierre_Tignol.html

## Quadratic equations

Simon Stevin
1548–1620

François Viète
1540–1603

http://www-history.mcs.st-and.ac.uk/Biographies/Stevin.html
http://www-history.mcs.st-and.ac.uk/Biographies/Viete.html



## $x^2 - x = b$

Plimpton 322
1800BC

Emmanuel Peyre. Les Points Rationnels. Gazette SMF N° 159, janvier 2019, 13–22.

https://smf.emath.fr/publications/la-gazette-des-mathematiciens-159-janvier-2019



## $x + y = a$, $xy = b$

Euclid of Alexandria
325 BC–265 BC

http://www-history.mcs.st-and.ac.uk/Biographies/Euclid.html



## $x^2 + ax = b$, $x^2 + b = ax$, $x^2 = ax + b$

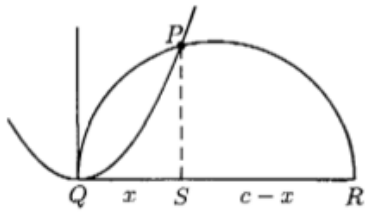

Abu Ja'far Muhammad ibn Musa Al-Khwarizmi
780–850

Hisab al-jabr w'al-muqabala

http://www-history.mcs.st-and.ac.uk/Biographies/Al-Khwarizmi.htm

## Geometric solutions to cubic equations

Omar Khayyam
1048–1131




$x^3 + b^2x = b^2c$

$x^2 = by$, the triangles $QSP$ and $PSR$ are similar,

$$\frac{x}{PS} = \frac{PS}{c-x}, \quad \frac{b}{x} = \frac{x}{y} = \frac{y}{c-x} == \frac{x^2}{b(c-x)}.$$

http://www-history.mcs.st-and.ac.uk/Biographies/Khayyam.html



## Leonardo da Pisa (Fibonacci)

Fibonacci
1170–1250

Liber Abaci (1202), Flos (1225)

In Flos, Fibonacci proves that the root of the equation $10x + 2x^2 + x^3 = 20$ (from Omar Khayyam's algebra book) is neither an integer nor a fraction, nor the square root of a fraction and gives the approximation $1.368\,808\,107\,5$, which is correct to nine decimal places.

http://www-history.mcs.st-and.ac.uk/Biographies/Fibonacci.html



## "The solutions of $x^3 + mx = n$ and $x^3 + n = mx$ are as impossible as the quadrature of the circle"

Luca Pacioli
1445–1517

http://www-history.mcs.st-and.ac.uk/Biographies/Pacioli.html



## $x^3 + mx = n$ – unpublished

Scipione del Ferro
1465–1526

http://www-history.mcs.st-and.ac.uk/Biographies/Ferro.html

## Nicolo Fontana alias Tartaglia

Nicolo Tartaglia
1500–1557

http://www-history.mcs.st-and.ac.uk/Biographies/Tartaglia.html



## $x^3 + mx = n$

Girolamo Cardano
1501–1576

$x^3 + mx = n$, $t - u = n$, $tu = (m/3)^3$, $x = \sqrt[3]{t} - \sqrt[3]{u}$.
$x^3 = 15x + 4$ : $x = 4$. Introduction complex numbers.
$x^2 + 2x = 48$ : 1 quad p :2 pos aeq 48

http://www-history.mcs.st-and.ac.uk/Biographies/Cardan.html



## Algebra

Rafael Bombelli
1526–1572

http://www-history.mcs.st-and.ac.uk/Biographies/Bombelli.html



## Solution of quartic equations

Lodovico Ferrari
1522–1565

Published by Cardano : “Ars Magna”.
Resolvent cubic equation.

http://www-history.mcs.st-and.ac.uk/Biographies/Ferrari.html

## Negative numbers

Simon Stevin
1548–1620

http://www-history.mcs.st-and.ac.uk/Biographies/Stevin.html



## Letters for unknown quantities (positive numbers)

François Viète
1540–1603

http://www-history.mcs.st-and.ac.uk/Biographies/Viete.html



## La Géométrie, 1637

René Descartes
1596–1650

1567 : $X - a$ divides $P(X)$ if and only if $P(a) = 0$.

$X^4 + pX^2 + qX + r = (X^2 + aX + b)(X^2 + cX + d)$

$a^6 + 2pa^4 + (p^2 - 4r)a^2 - q^2 = 0$

http://www-history.mcs.st-and.ac.uk/Biographies/Descartes.html



## 1629 : invention nouvelle en algèbre

Albert Girard
1595–1632

The number of roots is the degree (including impossible solutions).
Relation between roots and coefficients.

http://www-history.mcs.st-and.ac.uk/Biographies/Girard_Albert.html

Sir Isaac Newton
1643–1727

$\sum {\alpha_i}^k$.

Numerical methods.

http://www-history.mcs.st-and.ac.uk/Biographies/Newton.html



## Tschirnhaus

Ehrenfried Walter von Tschirnhaus
1651–1708

$X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$
$Y = X^m + b_{m-1}X^{m-1} + \cdots + b_1X + b_0$

http://www-history.mcs.st-and.ac.uk/Biographies/Tschirnhaus.html



## How to solve Tschirnhaus system ?

Gottfried Wilhelm von Leibniz
1646–1716

http://www-history.mcs.st-and.ac.uk/Biographies/Leibniz.html



## Réflexions sur la résolution algébrique des équations : 1772

Joseph-Louis Lagrange
1736–1813

Permutations of the roots. Number of permutations leaving invariant the polynomial. Lagrange theorem on the order of a subgroup. Lagrange resolvent.

http://www-history.mcs.st-and.ac.uk/Biographies/Lagrange.html

## The fondamental theorem of algebra

Albert Girard
1595–1632

1629 : invention nouvelle en algèbre

http://www-history.mcs.st-and.ac.uk/Biographies/Girard_Albert.html



## The fondamental theorem of algebra : 1746

Jean Le Rond d'Alembert
1717–1813

Analytic proof

http://www-history.mcs.st-and.ac.uk/Biographies/DAlembert.html



## The fondamental theorem of algebra : 1749

Leonhard Euler
1707–1783

Any irreducible polynomial over $\mathbb{R}$ has degree $1$ or $2$.

http://www-history.mcs.st-and.ac.uk/Biographies/Euler.html



## Réflexions sur la résolution algébrique des équations : 1772

Joseph-Louis Lagrange
1736–1813

Assumes there are $d$ "imaginary" roots to a polynomial of degree $d$, proves that they are in $\mathbb{C}$.

http://www-history.mcs.st-and.ac.uk/Biographies/Lagrange.html

## The fondamental theorem of algebra : 1799, 1815

Johann Carl Friedrich Gauss
1777-1855

Corrects the proofs of d'Alembert and Lagrange, later produces two other proofs.

http://www-history.mcs.st-and.ac.uk/Biographies/Gauss.html



## Mémoire sur la résolution des équations 1774

Alexandre-Théophile Vandermonde
1735–1796

Abbé Henri Grégoire
1750–1831

Determinant
Cyclotomic polynomials

http://www-history.mcs.st-and.ac.uk/Biographies/Vandermonde.html
https://fr.wikipedia.org/wiki/Henri_Gregoire



## Cyclotomic polynomials 1799

Johann Carl Friedrich Gauss
1777-1855

http://www-history.mcs.st-and.ac.uk/Biographies/Gauss.html



## 1837

Pierre Laurent Wantzel
1814–1848

http://www-history.mcs.st-and.ac.uk/Biographies/Wantzel.html

## Fermat primes

Pierre de Fermat
1601 ?–1665

$F_n = 2^{2^n} + 1$ :
$F_0 = 3$, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$, $F_4 = 65\,537$

$F_5 = 2^{32} + 1 \equiv 0 \pmod{641}$.

http://www-history.mcs.st-and.ac.uk/Biographies/Fermat.html



## 1799 : general equation of degree 5

Paolo Ruffini
1765–1822

516 pages - non solvability of the general equation of degree 5.
Gauss : "It appears more and more likely that this resolution is impossible".

http://www-history.mcs.st-and.ac.uk/Biographies/Ruffini.html



## 1824 : non solvability of equations of degree 5

Niels Henrik Abel
1802–1829

http://www-history.mcs.st-and.ac.uk/Biographies/Abel.html



Évariste Galois
1811–1832

Necessary and sufficient condition for an equation to be solvable by radicals.

http://www-history.mcs.st-and.ac.uk/Biographies/Galois.html

## Referee of Galois's note

Jean Baptiste Joseph Fourier
1768–1830

http://www-history.mcs.st-and.ac.uk/Biographies/Fourier.html



## 1846 : publication of Galois's work

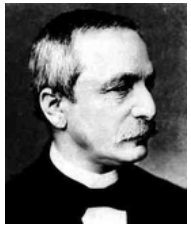
Joseph Liouville
1809–1882

Permutations which preserves the relations among the roots.
Behavior under the extension of the base field.

http://www-history.mcs.st-and.ac.uk/Biographies/Liouville.html



## Field theory : constructivist



Leopold Kronecker
1823–1891

http://www-history.mcs.st-and.ac.uk/Biographies/Kronecker.html



## Field theory : axiomatic

Julius Wilhelm Richard Dedekind
1831–1916

http://www-history.mcs.st-and.ac.uk/Biographies/Dedekind.html

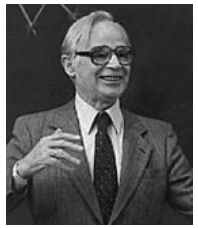
## Galois correspondence - published 1942

Emil Artin
1898–1962

http://www-history.mcs.st-and.ac.uk/Biographies/Artin.html



## Moderne Algebra 1930



Bartel Leendert van der Waerden
1903–1996

http://www-history.mcs.st-and.ac.uk/Biographies/Van_der_Waerden.html



## Field theory

Field extensions $K/k$.

Algebraic elements, transcendental elements.

Algebraic extensions , finite extensions , degree of an extension $[K : k]$. Number field.
A finite extension is algebraic.
The set of algebraic numbers (over $\mathbb{Q}$) is a field : $\overline{\mathbb{Q}}$.

Stem field of a polynomial. Splitting field for a polynomial.



## Conjugates, normal extensions

Conjugate of an element. Unicity of the stem field up to isomorphism.

Normal extensions. Properties equivalent to normality : finite normal extension = splitting field of a polynomial.

Separable irreducible polynomial. Separable polynomial. Separable algebraic extension.

## Galois extensions

The subgroup $G(K/k)$ of $\mathrm{Aut}(K)$. Fixed field $K^H$ of a subgroup $H$ of $G(K/k)$.
Galois extension = finite, normal, separable.

Transitivity of the Galois group on the zeros of an irreducible polynomial in a normal extension.

**Proposition.** *If $K/k$ is Galois, then $G(K/k)$ is a finite group of order $[K:k]$ and $k$ is the fixed field of $G(K/k)$.*

**Theorem.** *If $H$ is a finite subgroup of $\mathrm{Aut}(K)$ and $k = K^H$ the fixed field of $H$, then $K/k$ is Galois and $H = G(K/k)$.*



## The Fundamental Theorem of Galois Theory

The Galois correspondence between subgroups and intermediate fields.

$K/k$ a Galois extension, $G = G(K/k)$

$$\begin{aligned} S(K/k) &= \{\text{subfields } E \text{ of } K \text{ containing } k\} & k \subset E \subset K \\ S(G) &= \{\text{subgroups } H \text{ of } G\} & H \subset G \end{aligned}$$

$$\begin{array}{ccc} S(K/k) & \to & S(G) \\ E & \mapsto & G(K/E) \end{array} \qquad \begin{array}{ccc} S(G) & \to & S(K/k) \\ H & \mapsto & K^H \end{array}$$



## Galois correspondence

$H \in S(G) \Rightarrow$
$K^H \in S(K/k)$

$$G \left\{ \begin{array}{c} K \\ | \\ K^H \\ | \\ k \end{array} \right. \begin{array}{l} \\ H \\ \\ \\ \end{array}$$

$E \in S(K/k) \Rightarrow$
$G(K/E) \in S(G)$

$$G \left\{ \begin{array}{c} K \\ | \\ E \\ | \\ k \end{array} \right. \begin{array}{l} \\ G(K/E) \\ \\ \\ \end{array}$$



## Normal subgroups and normal intermediate extensions

$$G \left\{ \begin{array}{c} K \\ | \\ K^H \\ | \\ k \end{array} \right. \begin{array}{l} \\ H \\ \\ G/H \\ \end{array}$$

$H$ is normal in $G$ if and only if $K^H/k$ is a Galois extension.
In this case,
$G(K^H/k) = G/H$.

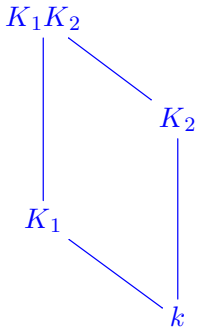
## Compositum

Assume $K_1/k$ is Galois. Then $K_1K_2/K_2$ is Galois and

$$G(K_1K_2/K_2) \subset G(K_1/k)$$

hence

$[K_1K_2 : K_2]$ divides $[K_1 : k]$.



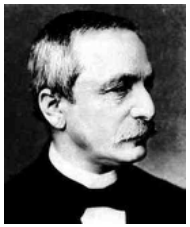

Remark : $k = \mathbb{Q}$, $K_1 = \mathbb{Q}(\sqrt[3]{2})$, $K_2 = \mathbb{Q}(j\sqrt[3]{2})$, $[K_1 : \mathbb{Q}] = 3$, $[K_1K_2 : K_2] = 2$.



## Cyclotomic fields. Abelian extensions of $\mathbb{Q}$

Let $\mu_n$ be the cyclic group of $n$–th roos of unity in $\mathbb{C}$. The cyclotomic field $\mathbb{Q}(\mu_n)$ is a Galois extension of $\mathbb{Q}$ with Galois group $(\mathbb{Z}/n\mathbb{Z})^\times$.

**Kronecker–Weber Theorem.** *Every finite abelian extension of $\mathbb{Q}$ is contained in some cyclotomic field.*



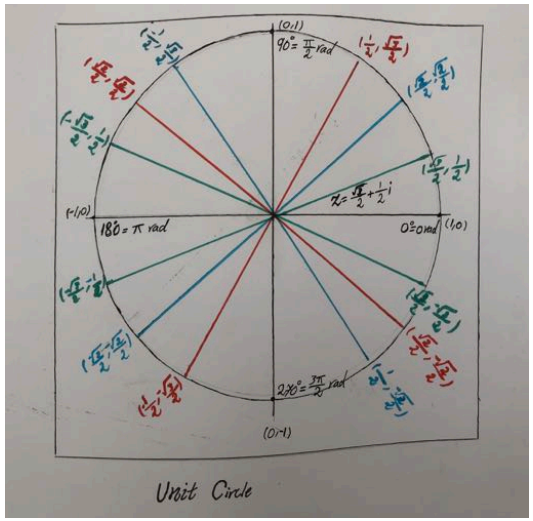
Leopold Kronecker
1823–1891

Heinrich Weber
1842–1913

http://www-history.mcs.st-and.ac.uk/Biographies/Kronecker.html
http://www-history.mcs.st-and.ac.uk/Biographies/Weber_Heinrich.h

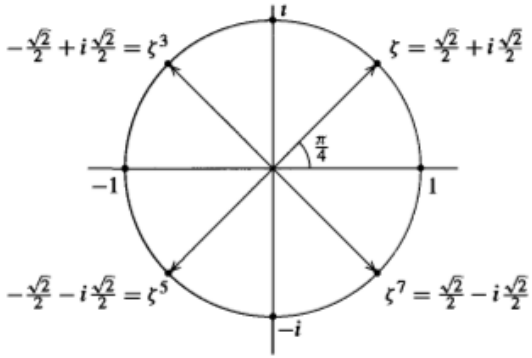


## Sixteenth roots of unity : $\Phi_{16}(X) = X^8 + 1$, $\zeta^{16} = 1$





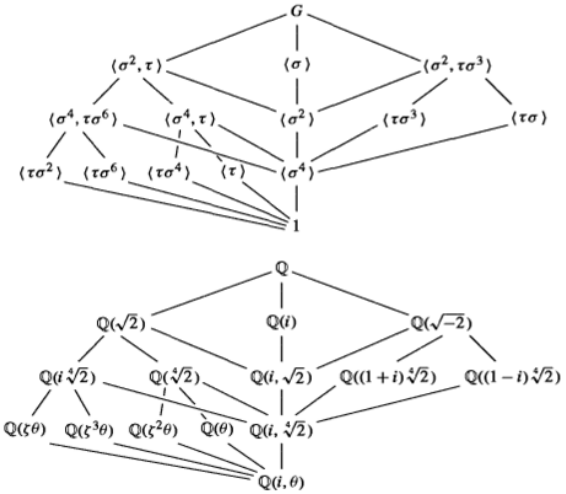

## Eighth roots of unity : $\Phi_8(X) = X^4 + 1$, $\zeta^8 = 1$




$[\mathbb{Q}(\zeta_8) : \mathbb{Q}] = \varphi(8) = 4$, $G(\mathbb{Q}(\zeta_8)/\mathbb{Q}) = (\mathbb{Z}/8\mathbb{Z})^\times \simeq (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.

David S. Dummit & Richard M. Foote, Abstract Algebra, Prentice Hall, 1991
2nd Ed. 1999.

## Splitting field of $X^8 - 2$ and subfields $\theta = \sqrt[8]{2}$




David S. Dummit & Richard M. Foote, Abstract Algebra, Prentice Hall, 1991
2nd Ed. 1999.



## Galois theory of finite fields

If $F$ is a finite extension of $\mathbb{F}_q$, then $F/\mathbb{F}_q$ is a Galois extension and $G(F/\mathbb{F}_q)$ is a cyclic group generated by the Frobenius $\mathrm{Frob}_q : x \to x^q$.

Ferdinand Georg Frobenius
1849–1917

http://www-history.mcs.st-and.ac.uk/Biographies/Frobenius.html



## Solvability of algebraic equations by radicals

Extension solvable by radicals.

In characteristic $\neq 2$ and $\neq 3$, an extension of degree $\leq 4$ is solvable by radicals.

Existence of extensions of degree $p$ with Galois group $\mathfrak{S}_p$.

Example : $X^5 + 2X^3 - 24X - 2$



## Geometric constructions with ruler and compasses

Stable subsets of $\mathbb{R} \times \mathbb{R}$. Stable closure $S(E)$ of a subset $E$ of $\mathbb{R} \times \mathbb{R}$. Stable closure $S(k)$ of a subfield $k$ of $\mathbb{R}$.

Let $k$ be a subfield of $\mathbb{R}$.
An element $x \in \mathbb{R}$ is in $C(k)$ if and only if there exists a Galois extension $K/k$ such that $[K : k] = 2^m$ and $x \in K$.

- Trisection of an angle
- Squaring the circle
- Doubling the cube
- Construction of regular polygons

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