Eighth course: september 25, 2007. [11]

2.2.4 Algebraic preliminaries: algebraic and transcendental elements, algebraic independence

Content

Rings: domains (no zero divisor), Euclidean rings, examples ($\mathbb{Z}$, $k[X]$, $\mathbb{Z}[i]$), PID (domain where any ideal is principal), UFD (unique factorization domain), further example ($\mathbb{Z}[X]$, $k[X_1,\dots,X_n]$).
Fields. Vector spaces, modules. Example: $\mathbb{Z}$–module = abelian group.
Extensions of fields. Subrings, subfields. Intersection of subrings, subfields, submodules, vector subspaces. Subrings or subfield generated by a subset: $A[E]$, $k(E)$ (and modules or vector spaces spanned by a subset). Special cases where $E = \{\alpha_1,\dots,\alpha_n\}$: finitely generated ring or field extension: $A[\alpha_1,\dots,\alpha_n]$, $k(\alpha_1,\dots,\alpha_n)$. Simple extension ($n = 1$).
Finite extension, example: $\mathbb{Q}(i)/\mathbb{Q}$. Degree $[K : k]$ of a finite extension K/k. Multiplicativity of the degree for $K_1 \subset K_2 \subset K_3$.
Algebraic element over a field k: equivalent properties. Transcendental element, examples: e is transcendental over $\mathbb{Q}$; also X in the field of rational fractions over the complex field is transcendental over $\mathbb{C}$. Irreducible (monic) polynomial of an algebraic element. For α in $\mathbb{C}$, we have $k(\alpha) = k[\alpha]$ iff α is algebraic; computing the inverse of an algebraic element using Euclidean division and Bézout's Theorem.
Sum and product of algebraic elements: prove that they are algebraic either by linear algebra or by means of the theorem of the elementary symmetric functions (see § 2.2.5). Field $\overline{\mathbb{Q}}$ of algebraic numbers, algebraic closure.
Example of an algebraic extension which is not finite: $\overline{\mathbb{Q}}/\mathbb{Q}$. Algebraic extension, any finite extension is algebraic.
Algebraically independent or dependent elements (algebraically free subset). Examples: numbers, functions. Transcendence degree, transcendence basis of a finitely generated extension. Properties: additivity of the transcendence degree for $K_1 \subset K_2 \subset K_3$. Transcendence degree 0 means algebraic extension.
Corollary: algebraic independence over $\mathbb{Q}$ is equivalent to algebraic independence over $\overline{\mathbb{Q}}$.
The transcendence degree of $k(E)/k$ is $\geq n$ if and only if there exists in E a set of at least n algebraically independent elements over k.

Exercise 2.21. *Show that the field $\overline{\mathbb{Q}}$ is an algebraic extension of $\mathbb{Q}$ which is not finite.*

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Hint. *One of many solutions is to check that for any* $n \geq 1$ *the polynomial* $X^n - 2$ *is irreducible.*
A more challenging solution is to check that the numbers $\sqrt{m}$*, where* m *ranges over the squarefree integers, are linearly independent over* $\mathbb{Q}$*. One may show by induction that if* $a_1, \ldots, a_n$ *are positive squarefree integers* > 1 *which are pairwise relatively prime, then the* 2^n *numbers*

$$\sqrt{\prod_{i \in I} a_i}, \quad I \subset \{1, \ldots, n\},$$

are linearly independent over $\mathbb{Q}$*. Hence the field* $\mathbb{Q}(\sqrt{a_1}, \ldots, \sqrt{a_n})$ *has degree* 2^n *over* $\mathbb{Q}$*.*

Exercise 2.22. *Let* R *and* S *be two rational fractions in* $k(T)$*. Show that there exists a non–zero polynomial* $F \in k[X, Y]$ *such that* $F(R, S) = 0$*. Deduce that any set of at least two elements in* $k(T)$ *consists of algebraically dependent elements, hence* $k(T)$ *has transcendence degree* 1 *over* k*.*
Generalize to $k(T_1, \ldots, T_n)$

Exercise 2.23. *Let* $t_1, \ldots, t_n$ *be algebraically independent complex numbers. Check that any subset of* $\{t_1, \ldots, t_n\}$ *consists of algebraically independent element. Check that for any* P *and* Q *in* $\overline{\mathbb{Q}}[X_1, \ldots, X_n]$ *for which* $Q(t_1, \ldots, t_n) \neq 0$ *and such that the rational fraction* $R = P/Q$ *is not constant, the number* $R(t_1, \ldots, t_n)$ *is transcendental.*

Exercise 2.24. *Check that an entire function* (which means a complex function which is analytic in all of $\mathbb{C}$) *is transcendental if and only if it is not a polynomial. Check that a meromorphic function in* $\mathbb{C}$ *is transcendental if and only if it is not a rational function.*

References

[1] Bùi Xuân Hải – *Lý Thuyết Trường & Galois*, Nha xuat ban Dai hoc Quoc gia NXB ĐHQG Tp HCM 2007.

There is a also good collection of Lecture Notes which are available on the internet. A list can be found at the URL
http://www.numbertheory.org/ntw/lecture_notes.html.
See for instance
Algebraic Number Theory and commutative algebra, Lecture Notes by Robert Ash
and
Course Notes by Jim Milne: *Algebraic number theory.*