On the numbers e^e, e^{e^2} and e^{π^2}

by

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Abstract. We give two measures of simultaneous approximation by algebraic numbers, the first one for the triple (e, e^e, e^{e^2}) and the second one for (π, e, e^{π^2}). We deduce from these measures two transcendence results which had been proved in the early 70's by W.D. Brownawell and the author.

Introduction

In 1949, A.O. Gel'fond introduced a new method for algebraic independence, which enabled him to prove that the two numbers $2^{\sqrt[3]{2}}$ and $2^{\sqrt[3]{4}}$ are algebraically independent. At the same time, he proved that one at least of the three numbers e^e, e^{e^2}, e^{e^3} is transcendental (see[G] Chap. III,).

At the end of his book [S] on transcendental numbers, Th. Schneider suggested that one at least of the two numbers e^e, e^{e^2} is transcendental; this was the last of a list of eight problems, and the first to be solved, in 1973, by W.D. Brownawell [B] and M.Waldschmidt [W 1], independently and simultaneously. For this result they shared the Distinguished Award of the Hardy-Ramanujan Society in 1986. Another consequence of their main result is that one at least of the two following statements holds true:

(i) The numbers **e** and π are algebraically independent
(ii)The number e^{π^2} is transcendental

Our goal is to shed a new light on these results. It is hoped that our approach will yield further progress towards a solution of the following open problems:

(?) Two at least of the three numbers e, e^e, e^{e^2} are algebraically independent.

(?) Two at least of the three numbers π, e, e^{π^2} are algebraically independent.

Further conjectures are as follows:
(?) Each of the numbers e^e, e^{e^2} e^{π^2} is transcendental

(?) The numbers **e** and π are algebraically independent.

We conclude this note by showing how stronger statements are consequences of Schanuel's conjecture.

1. Heights

Let γ be a complex algebraic number. The *minimal polynomial* of γ over Z Z is the unique polynomial

$$f(X) = a_0 X^d + a_1 X^{d-1} + \ldots + a_d \epsilon \mathbb{Z}[X]$$

which vanishes at the point γ, is irreducible in the factorial ring $\mathbb{Z}[X]$ and has leading coefficient $a_0 > 0$. The integer $d = \deg f$ is the *degree* of γ, denoted by $[Q(\gamma) : Q]$. The *usual height* $H(\gamma)$ of γ is defined by

$$H(\gamma) = \max\{|a_0|, ..., |a_d|\}.$$

It will be convenient to use also the so-called *Mahler's measure* of γ, which can be defined in three equivalent ways. The first one is

$$M(\gamma) = \exp\left(\int_0^1 \log|f(e^{2i\pi t})|dt\right).$$

For the second one, let $\gamma_1, ...\gamma_d$ denote the complex roots of **f**, so that

$$f(X) = a_0 \prod_{i=1}^{d} (X - \gamma_i).$$

Then, according to Jensen's formula, we have

$$M(\gamma) = |a_0| \prod_{i=1}^{d} \max\{1, |\gamma_i|\}.$$

For the third one, let **K** be a number field (that is a subfield of Cl which is a Q/ -vector space of finite dimension [K:Q/]) containing γ, and let M_K be the set of (normalized) absolute values of **K**. Then

$$M(\gamma) = \prod_{v \in M_K} \max\{1, |\gamma|_v\}^{[K_v:\mathbb{Q}]}$$

where K_v is the completion of **K** for the absolute value **v** and $\mathbb{Q}_v$ the topological closure of $\mathbb{Q}$ in K_v and $[K_v : \mathbb{Q}_v]$ the local degree.

Mahler's measure is related to the usual height by

$$2^{-d}H(\gamma) \le M(\gamma) \le \sqrt{d+1}.H(\gamma).$$

From this point of view it does not make too much difference to use **H** or **M**, but one should be careful that **d** denotes the exact degree of γ, not an upper bound. We shall deal below with algebraic numbers of degree **d** bounded by some parameter **D**.

Definition. For an algebraic number γ of degree **d** and Mahler's measure $M(\gamma)$, we define the *absolute logarithmic height* $h(\gamma)$ by

$$h(\gamma) = \frac{1}{d}\log M(\gamma).$$

2.Simultaneous Approximation

We state two results dealing with simultaneous Diophantine approximation. Both of them are consequences of the main result in [W 2]. Details of the proof will appear in the forthcoming book [W 3].

2.1. Simultaneous Approximation to e, e^e and e^{e^2}

Theorem 1. *There exists a positive absolute constant* c_1 *such that, if* $\gamma_0, \gamma_1, \gamma_2$ *are algebraic numbers in a field of degree D, then*

$$|e-\gamma_0|+|e^e-\gamma_1|+|e^{e^2}-\gamma_2| > \exp\{-c_1 D^2(h_0+h_1+h_2)^{1/2}(h_1+h_2)^{1/2}(h_0+\log D)(\log D)^{-1}\}$$

where $h_i = \max\{e, h(\gamma_i)\}\ (i=0,1,2)$.

2.2. Simultaneous Approximation to π, e and e^{π^2}

Theorem 2. *There exists a positive absolute constant* c_2 *such that, if* $\gamma_0, \gamma_1, \gamma_2$ *are algebraic numbers in a field of degree* ***D****, then*

$$|\pi-\gamma_0|+|e^e-\gamma_1|+e^{e^2}-\gamma_2| > \exp\{-c_2 D^2(h_0+\log(Dh_1\ h_2))^{\frac{1}{2}}h_1^{\frac{1}{2}}h_2^{\frac{1}{2}}(\log\ D)^{-1}$$

where $h_i = \max\{e, h(\gamma_i)\}\ (i=0,1,2)$.

3. Transcendence Criterion

3.1 Algebraic Approximations to a Given Transcendental Number

The following result is Théoréme 3.2 of [R-W 1]; see also Theorem 1.1 of [R-W 2].

Theorem 3. *Let* $\theta \in C$ *be a complex number. The two following conditions are equivalent:*
(i) the number θ *is transcendental.*
(ii) For any real number $h \geq 10^7$*, there are infinitely many integers* $d \geq 1$ *for which there exists an algebraic number* γ *of degree* ***d*** *and absolute logarithmic height* $h(\gamma) \leq h$ *which satisfies*

$$0 < |\theta-\gamma| \leq \exp(-10^{-7}hd^2).$$

Notice that the proof of $(ii) \Rightarrow (i)$ is an easy consequence of Liouville's inequality.

3.2. Application to e^e and e^{e^2}

Corollary to Theorem 1. *One at least of the two numbers* $e^e,\ e^{e^2}$ *is transcendental.*

Proof of the corollary. Assume that the two numbers $e^e\ e^{e^2}$ are algebraic, say γ_1 and γ_2. Then, according to Theorem 1, there exists a constant $c_3 > 1$ such that, for any algebraic number γ of degree $\leq D$ and height $h(\gamma) \leq h$ with $h \geq e$,

$$|e-\gamma| > \exp\{-c_3 D^2 h^{1/2}(h+\log D)(\log D)^{-1}\}.$$

We now use Theorem 3 for $\theta = e$ with $h = 10^{15}c_3^2$ and derive a contradiction.

4. Algebraic Independence

4.1 Simultaneous Approximation

The proof of the following result is given in [R-W 1], Théoréme 3.1, as a consequence of Theorem 3 (see also [R-W 2] Corollary 1.2).

Corollary to Theorem 3. *Let* $\theta_1, \ldots, \theta_m$ *be complex numbers such that the field* $Q(\theta_1, .., \theta_m)$ *has tyranscendence degree 1 over Q. There exists a constant* ***c > 0*** *such that, for any real number* $h \geq c$*, there are infinitely many integers* ***D*** *for which there exists a tuple* $(\gamma_1, \ldots, \gamma_m)$ *of algebraic numbers satisfying*

$$[Q(\gamma_1, \ldots, \gamma_m) : Q] \leq D, \quad \max_{1 \leq i \leq m} h(\gamma_i) \leq h$$

and

$$\max_{1 \leq i \leq m} \{|\theta_i - \gamma_i|\} \leq \exp(-c^{-1} h D^2).$$

4.2. Application to π, e and e^{π^2}

Corollary to Theorem 2. *One at least of the two following statements is true:*
(i) The numbers ***e*** *and* π *are algebraiclly independent.*
(ii) The number e^{π^2} *is transcendental.*

Remark. This corollary can be stated in an equivalent way as follows:

For any non constant polynomial $P \in Z[X]$, the complex number

$$e^{\pi^2} + iP(e, \pi)$$

is transcendental.

The idea behind this remark originates in [R].

Proof of the Corollary. Assume that the number e^{π^2} is algebraic. Theorem 2 with $\gamma_2 = e^{\pi^2}$ shows that there exists a constant $c_4 > 0$ such that, for any pair (γ_0, γ_1) of algebraic numbers, if we set

$$D = [Q(\gamma_0, \gamma_1) : Q] \quad and \quad h = \max\{e, h(\gamma_0), h(\gamma_1)\},$$

then

$$|\pi - \gamma_0| + |e - \gamma_1| > \exp\{-c_4 D^2 (h + \log D)^{1/2} h^{1/2} (\log D)^{-1}\}$$

Therefore we dedeuce from the Corollary to Theorem 3 that the field $Q(\pi, e)$ has transcendence degree 2.

5. Schanuel's Conjecture

The following conjecture is stated in [L] Chap. III p. 30: (The resulsts of this section are based on the conjecture to be stated).

Schanuel's Conjecture. *Let* $x_1, \ldots, x_n$ *be Q-linearly independent complex numbers. Then, among the* ***2n***

numbers

$$x_1, ..., x_n, e^{x_1}, ..., e^{x_n},$$

*at least **n** are algebraically independent.*

Let us deduce from Schanuel's Conjecture the following statement (which is an open problem):

(?) The 7 numbers

$$e,\ \pi,\ e^e,\ e^{e^2},\ e^{\pi^2},\ 2^{\sqrt[3]{2}},\ 2^{\sqrt[3]{2}}$$

are algebraically independent.

We shall use Schanuel's cinjecture twice. We start with the numbers $1, \log 2,$ and $i\pi$ which are linearly independent over **Q** because $\log 2$ is irrational. Therefore, according to Schanuel's conjecture, three at least of the numbers

$$1,\ \log\ 2,\ i\pi,\ \ e,\ 2,\ -1$$

are algebraically independent. This means that the three numbers $\log 2,$ π and **e** are algebraically independent. Therefore the 8 numbers

$$1,\ i\pi,\ \pi^2,\ e,\ e^2,\ \log 2,\ 2^{\frac{1}{3}} \log\ 2,\ 4^{\frac{1}{3}} \log\ 2$$

are **Q**-linearly independent. Again, Schanuel's conjecture implies that 8 at least of the numbers

$$1,\ i\pi,\ \pi^2,\ e,\ e^2,\ \log^2,\ 2^{\frac{1}{3}} \log\ 2,\ 4^{\frac{1}{3}} \log\ 2\ e,\ -1,\ e^{\pi^2},\ e^e,\ e^{e^2},\ 2, 2^{\sqrt[3]{2}},\ 2^{\sqrt[3]{4}}$$

are algebraically independent, and this means that the 8 numbers

$$e,\ \pi,\ e^e,\ e^{e^2},\ e^{\pi^2},\ 2^{\sqrt[3]{2}},\ 2^{\sqrt[3]{4}},\ \log\ 2$$

are algebraically independent.

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