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Transcendental Number Theory: Schanuel's Conjecture

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Abstract

One of the main open problems in transcendental number theory is Schanuel's Conjecture which was stated in the 1960's :

> *If $x_1, \ldots, x_n$ are $\mathbf{Q}$–linearly independent complex numbers, then among the $2n$ numbers $x_1, \ldots, x_n$, $e^{x_1}, \ldots, e^{x_n}$, at least n are algebraically independent.*

We first give a list of consequences of this statement ; next we describe the state of the art by giving special cases of the conjecture which have been proved, and finally we introduce a promising approach which has been initiated in 1999 by D. Roy.

Algebraic and transcendental numbers

Algebraic numbers : Roots of polynomials with rational coefficients. Algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$:

$$\mathbf{Q} \subset \overline{\mathbf{Q}} \subset \mathbf{C}$$

Transcendental numbers : Complex numbers that are not algebraic.

The set of algebraic numbers behaves well with respect to addition, multiplication, division : *It is a field.*

The set of transcendental numbers is the complement in the field $\mathbf{C}$ of the field $\overline{\mathbf{Q}}$. The sum of transcendental numbers may be rational, algebraic or transcendental. The same for the product.

However, the sum of a transcendental number and an algebraic number is transcendental, and the product of a transcendental number and a non–zero algebraic number is transcendental.

The exponential function

For $z \in \mathbf{C}$,

$$e^z = \exp(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots = \sum_{n \geq 0} \frac{z^n}{n!}.$$

Addition Theorem : $e^{z_1+z_2} = e^{z_1}e^{z_2}$, $e^{2i\pi} = 1$.

$$\exp : \mathbf{C} \longrightarrow \mathbf{C}^\times, \qquad \ker(\exp) = 2i\pi\mathbf{Z}, \qquad \mathbf{C}/2i\pi\mathbf{Z} \simeq \mathbf{C}^\times.$$

Differential equation :

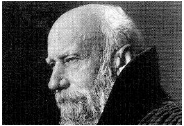
$$\frac{d}{dz}e^z = e^z.$$

First examples of transcendental numbers

$\sum_{n\geq 1} 2^{-n!}$: Liouville, 1844

e : Hermite, 1872

π : Lindemann, 1881



Ch. Hermite – F. Lindemann – C. Weierstraß

$\log 2$, $e^{\sqrt{2}}$: Theorem of Hermite–Lindemann

$e^{\sqrt{2}} + \sqrt{3}e^{\sqrt{6}}$: Lindemann–Weierstraß, 1888

Question of Euler and Hilbert

Leonhard Euler (1707 – 1783)
Introductio in analysin infinitorum (1737)
Transcendence of $2^{\sqrt{2}}$

David Hilbert (1862 - 1943)
ICM 1900 : 7–th Problem
Transcendence of $\log a / \log 2$.

A.O. Gel'fond – Th. Schneider – A. Baker

e^{π} : Gel'fond, 1929

$2^{\sqrt{2}}$, $\log 2/\log 3$: Gel'fond and Schneider, 1934

$\log 2 + \sqrt{3}\log 3$, $e^{\sqrt{2}}2^{\sqrt{3}}5^{\sqrt{7}}$: Baker, 1968.

Some open problems

For each of the following numbers, it is expected that it is transcendental, but it is not even known whether it is rational or not.

$$e+\pi, \quad e\pi, \quad \pi^e, \quad e^e, e^{e^2}, \dots, \quad e^{e^e}, \dots, \quad \pi^\pi, \pi^{\pi^2}, \dots \quad \pi^{\pi^\pi} \dots$$

$$\log\pi, \quad \log(\log 2), \quad \pi\log 2, \quad (\log 2)(\log 3), \quad 2^{\log 2}, \quad (\log 2)^{\log 3} \dots$$

In other words we do not know whether a degree 1 polynomial could vanish at the corresponding point, but we expect that no non–zero polynomial of any degree vanishes at this point.

Algebraic independence

Algebraicity and transcendence deal with a single complex number and one variable polynomials.
Algebraic dependence or independence is the same but for tuples and multivariate polynomials.

Let $K \supset k$ be an extension of fields and $(\theta_1, \ldots, \theta_m)$ be a m–tuple of elements in K. We say that $\theta_1, \ldots, \theta_m$ are *algebraically dependent over k* if there exists a non–zero polynomial $f \in k[X_1, \ldots, X_m]$ which vanishes at the point $(\theta_1, \ldots, \theta_m) \in K^m$.
Otherwise we say that $\theta_1, \ldots, \theta_m$ are *algebraically independent over k*.

Special cases

In the case $m = 1$, to say that θ_1 is algebraically independent over k just means that it is transcendental over k.

Dealing with complex numbers $K = \mathbf{C}$, the words algebraic, transcendental, algebraically dependent, algebraically independent refer to the case $k = \mathbf{Q}$.

Examples

The numbers $\sqrt{2}$ and π are algebraically dependent :
The polynomial $X^2 - 2 \in \mathbf{Z}[X, Y]$ vanishes at $(\sqrt{2}, \pi)$.

The numbers π and $\sqrt{\pi^2 + 1}$ are algebraically dependent (and both are transcendental numbers) :
The polynomial $Y - X^2 - 1$ vanishes at $(\pi, \sqrt{\pi^2 + 1})$.

The two numbers e and $e^{\sqrt{2}}$ are algebraically independent, which means that for any non–zero polynomial $f \in \mathbf{Z}[X, Y]$ with rational integer coefficients, the number $f(e, e^{\sqrt{2}})$ is not zero.
Special case of the Lindemann-Weierstraß Theorem.

Transcendence degree

Let K/k be a field extension. The maximal number of elements in K which are algebraically independent over k is called the transcendence degree of K over k and denoted $\operatorname{tr}\deg_k K$.

Let $t = \operatorname{tr}\deg_k K$. A subset $\{\theta_1, \dots, \theta_t\}$ of K with t elements which are algebraically independent is called a transcendence basis of K over k. It is the same as a maximal subset of k–algebraically independent elements in K.
Hence K is an algebraic extension of $k(\theta_1, \dots, \theta_t)$.

Transcendence degree of extensions

Assume
$$k \subset K \subset L.$$

The union of a transcendence basis of K over k and of a transcendence basis of L over K produces a transcendence basis of L over k.
Hence
$$\operatorname{tr\,deg}_k L = \operatorname{tr\,deg}_k K + \operatorname{tr\,deg}_K L.$$

An algebraic extension K/k is an extension of transcendence degree 0 : This means that there is no transcendental element in K over k (any element in K is algebraic over k).

Algebraic independence of complex numbers

For complex numbers, algebraic independence over $\mathbf{Q}$ or over $\overline{\mathbf{Q}}$ is the same. In particular if $\theta_1, \dots, \theta_m$ are algebraically independent complex numbers, then for any non-constant polynomial f with algebraic coefficients the number $f(\theta_1, \dots, \theta_m)$ is transcendental.

For instance, the two numbers e and $e^{\sqrt{2}}$ are algebraically independent. As a consequence for any non–constant polynomial $f \in \overline{\mathbf{Q}}[X, Y]$ with algebraic coefficients, the number $f(e, e^{\sqrt{2}})$ is transcendental.

Schanuel's Conjecture

Let $x_1, \dots, x_n$ be $\mathbf{Q}$-linearly independent complex numbers. Then at least n of the $2n$ numbers $x_1, \dots, x_n,\ e^{x_1}, \dots, e^{x_n}$ are algebraically independent.

Since there are $2n$ numbers only, the transcendence degree over $\mathbf{Q}$ of the field

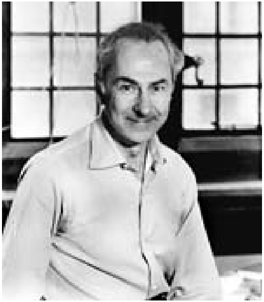
$$\mathbf{Q}\bigl(x_1, \dots, x_n,\ e^{x_1}, \dots, e^{x_n}\bigr)$$

is at most $2n$. The conjecture is that this transcendence degree is always $\geq n$:

$$n \leq^{?} \operatorname{tr}\deg_{\mathbf{Q}} \mathbf{Q}\bigl(x_1, \dots, x_n,\ e^{x_1}, \dots, e^{x_n}\bigr) \leq 2n.$$

Origin of Schanuel's Conjecture

Course given by Serge Lang (1927–2005) at Yale in the 60's



S. Lang – *Introduction to transcendental numbers*, Addison-Wesley 1966.

also attended by M. Nagata (1927–2008)
(14*th* Problem of Hilbert).

Nagata's Conjecture solved by E. Bombieri.

Schanuel's Conjecture

Let $x_1, \ldots, x_n$ be $\mathbf{Q}$-linearly independent complex numbers. Then

$$\operatorname{tr}\deg_{\mathbf{Q}} \mathbf{Q}\bigl(x_1, \ldots, x_n,\ e^{x_1}, \ldots, e^{x_n}\bigr) \geq n.$$

`Remark` : For almost all tuples (with respect to the Lebesgue measure) the transcendence degree is $2n$.

A.O. Gel'fond CRAS 1934

ARITHMÉTIQUE. — *Sur quelques résultats nouveaux dans la théorie des nombres transcendants.* Note de M. A. GELFOND, présentée par M. Hadamard.

J'ai démontré (¹) que le nombre ω^r, où $\omega \neq 0, 1$ est un nombre algébrique et r un nombre algébrique irrationnel, doit être transcendant.

Par une généralisation de la méthode qui sert pour la démonstration du théorème énoncé, j'ai démontré les théorèmes plus généraux suivants :

I. THÉORÈME. — *Soient* $P(x_1, x_2, \ldots, x_n, y_1, \ldots, y_m)$ *un polynome à coefficients entiers rationnels et* $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_m$ *des nombres algébriques,* $\beta_i = 0, 1$.

L'égalité

$$P(e^{\alpha_1}, e^{\alpha_2}, \ldots, e^{\alpha_n}, \ln\beta_1, \ln\beta_2, \ldots, \ln\beta_m) = 0$$

est impossible; les nombres, $\alpha_1, \alpha_2, \ldots, \alpha_n$, *et aussi les nombres* $\ln\beta_1, \ln\beta_2, \ln\beta_3, \ldots, \ln\beta_m$ *sont linéairement indépendants dans le corps des nombres rationnels.*

Ce théorème contient, comme cas particuliers, le théorème de Hermite et Lindemann, la résolution complète du problème de Hilbert, la transcendance des nombres $e^{\omega_1 e^{\omega_2}}$ (où ω_1 et ω_2 sont des nombres algébriques), le théorème sur la transcendance relative des nombres e et π.

II. THÉORÈME. — *Les nombres*

$$e^{\omega_1 e^{\omega_2 e^{\cdot^{\cdot^{\cdot \omega_{n-1} e^{\omega_n}}}}}} \quad \text{et} \quad \alpha_1^{\alpha_2^{\alpha_3^{\cdot^{\cdot^{\cdot \alpha_m}}}}},$$

où $\omega_1 \neq 0, \omega_2, \ldots, \omega_n$ et $\alpha_1 \neq 0, 1, \alpha_2 \neq 0, 1, \alpha_3 \neq 0, \alpha_4, \ldots, \alpha_m$ *sont des nombres algébriques, sont des nombres transcendants et entre les nombres de cette forme n'existent pas de relations algébriques, à coefficients entiers rationnels* (non triviales).

La démonstration de ces résultats et de quelques autres résultats sur les nombres transcendants sera donnée dans un autre Recueil.

(¹) *Sur le septième problème de D. Hilbert* (*C. R. de l'Acad. des Sciences de l'U. R. S. S.*, 2, I, 1er avril 1934, et *Bull. de l'Acad. des Sciences de l'U. R. S. S.*, 7e série, 4, 1934, p. 623).

Statement by Gel'fond (1934)

Let $\beta_1, \ldots, \beta_n$ be $\mathbf{Q}$-linearly independent algebraic numbers and let $\log \alpha_1, \ldots, \log \alpha_m$ be $\mathbf{Q}$-linearly independent logarithms of algebraic numbers. Then the numbers

$$e^{\beta_1}, \ldots, e^{\beta_n}, \ \log \alpha_1, \ldots, \log \alpha_m$$

are algebraically independent over $\mathbf{Q}$.

Further statement by Gel'fond

Let $\beta_1, \ldots, \beta_n$ be algebraic numbers with $\beta_1 \neq 0$ and let $\alpha_1, \ldots, \alpha_m$ be algebraic numbers with $\alpha_1 \neq 0, 1$, $\alpha_2 \neq 0, 1$, $\alpha_i \neq 0$. Then the numbers

$$e^{\beta_1 e^{\beta_2 e^{\cdot^{\cdot^{\cdot^{\beta_{n-1} e^{\beta_n}}}}}}} \qquad \text{and} \qquad \alpha_1^{\alpha_2^{\cdot^{\cdot^{\cdot^{\alpha_m}}}}}$$

are transcendental, and there is no nontrivial algebraic relation between such numbers.

`Remark` : The condition on α_2 should be that it is irrational.

Easy consequence of Schanuel's Conjecture

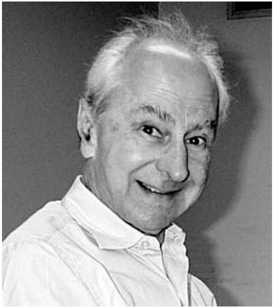
According to Schanuel's Conjecture, the following numbers are algebraically independent :

$$e+\pi, \quad e\pi, \quad \pi^e, \quad e^e, e^{e^2}, \dots, \quad e^{e^e}, \dots, \quad \pi^\pi, \pi^{\pi^2}, \dots \quad \pi^{\pi^\pi} \dots$$

$$\log \pi, \quad \log(\log 2), \quad \pi \log 2, \quad (\log 2)(\log 3), \quad 2^{\log 2}, \quad (\log 2)^{\log 3} \dots$$

Proof : This is an easy exercise.

Lang's exercise



Define $E_0 = \mathbf{Q}$. Inductively, for $n \geq 1$, define E_n as the algebraic closure of the field generated over E_{n-1} by the numbers $\exp(x) = e^x$, where x ranges over E_{n-1}. Let E be the union of E_n, $n \geq 0$. Then Schanuel's Conjecture implies that the number π does not belong to E.

More precisely : Schanuel's Conjecture implies that the numbers $\pi, \log \pi, \log \log \pi, \log \log \log \pi, \ldots$ are algebraically independent over E.

A variant of Lang's exercise

Define $L_0 = \mathbf{Q}$. Inductively, for $n \geq 1$, define L_n as the algebraic closure of the field generated over L_{n-1} by the numbers y, where y ranges over the set of complex numbers such that $e^y \in L_{n-1}$. Let L be the union of L_n, $n \geq 0$. Then Schanuel's Conjecture implies that the number e does not belong to L.

More precisely : Schanuel's Conjecture implies that the numbers $e, e^e, e^{e^e}, e^{e^{e^e}} \ldots$ are algebraically independent over L.

Arizona Winter School AWS2008, Tucson

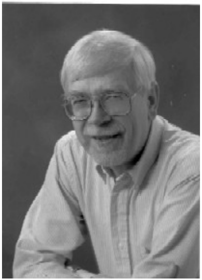
Theorem [Jonathan Bober, Chuangxun Cheng, Brian Dietel, Mathilde Herblot, Jingjing Huang, Holly Krieger, Diego Marques, Jonathan Mason, Martin Mereb and Robert Wilson.] *Schanuel's Conjecture implies that the fields E and L are linearly disjoint over $\overline{\mathbf{Q}}$.*

Definition Given a field extension F/K and two subextensions $F_1, F_2 \subseteq F$, we say F_1, F_2 are linearly disjoint over K when the following holds : Any set $\{x_1, \ldots, x_n\} \subseteq F_1$ of $K-$ linearly independent elements is linearly independent over F_2.

Reference : arXiv.0804.3550 [math.NT] 2008.

Formal analogs

W.D. Brownawell
(was a student of Schanuel)



J. Ax's Theorem (1968) :
Version of Schanuel's Conjecture for power series over $\mathbf{C}$
(and R. Coleman for power series over $\overline{\mathbf{Q}}$)

Work by W.D. Brownawell and K. Kubota on the elliptic analog of Ax's Theorem.

Conjectures by A. Grothendieck and Y. André

Generalized Conjecture on Periods by Grothendieck : Dimension of the Mumford–Tate group of a smooth projective variety.

Generalization by Y. André to motives.

Case of 1–motives : Elliptico-Toric Conjecture of C. Bertolin.

Ubiquity of Schanuel's Conjecture

Other contexts : p–adic numbers, Leopoldt's Conjecture on the p–adic rank of the units of an algebraic number field
Non-vanishing of Regulators
Non–degenerescence of heights
Conjecture of B. Mazur on rational points
Diophantine approximation on tori

Dipendra Prasad

Gopal Prasad

Methods from logic

Ehud Hrushovski

Boris Zilber

Jonathan Kirby

Calculus of "predimension functions" (E. Hrushovski)

Zilber's construction of a "pseudoexponentiation"

Also : A. Macintyre, D.E. Marker, G. Terzo, A.J. Wilkie, D. Bertrand...

Methods from logic : Model theory

Exponential algebraicity in exponential fields
by
Jonathan Kirby

The dimension of the exponential algebraic closure operator in an exponential field satisfies a weak Schanuel property.

A corollary is that there are at most countably many essential counterexamples to Schanuel's conjecture.

`arXiv :0810.4285v2`

Known

Lindemann–Weierstraß Theorem = case where $x_1, \ldots, x_n$ are algebraic.

Let $\beta_1, \ldots, \beta_n$ be algebraic numbers which are linearly independent over $\mathbf{Q}$. Then the numbers $e^{\beta_1}, \ldots, e^{\beta_n}$ are algebraically independent over $\mathbf{Q}$.

Problem of Gel'fond and Schneider

Raised by A.O. Gel'fond in 1948 and Th. Schneider in 1952.

Conjecture : *If α is an algebraic number, $\alpha \neq 0$, $\alpha \neq 1$ and if β is an irrational algebraic number of degree d, then the $d-1$ numbers*

$$\alpha^{\beta},\ \alpha^{\beta^2},\ \dots,\alpha^{\beta^{d-1}}$$

are algebraically independent.

Special case of Schanuel's Conjecture : Take $x_i = \beta^{i-1}\log\alpha$, $n = d$, so that $\{x_1,\dots,x_n,\ e^{x_1},\dots,e^{x_n}\}$ is

$$\{\log\alpha,\ \beta\log\alpha,\ \dots,\ \beta^{d-1}\log\alpha, \quad \alpha,\ \alpha^{\beta},\ \dots,\ \alpha^{\beta^{d-1}}\}.$$

The conclusion of Schanuel's Conjecture is

$$\operatorname{tr}\deg_{\mathbf{Q}}\mathbf{Q}\bigl(\log\alpha,\alpha^{\beta},\ \alpha^{\beta^2},\ \dots,\alpha^{\beta^{d-1}}\bigr) = d.$$

Algebraic independence method of Gel'fond

A.O. Gel'fond (1948)
The two numbers $2^{\sqrt[3]{2}}$ and $2^{\sqrt[3]{4}}$ are algebraically independent.
More generally, if α is an algebraic number, $\alpha \neq 0$, $\alpha \neq 1$ and if β is a algebraic number of degree $d \geq 3$, then two at least of the numbers

$$\alpha^{\beta},\ \alpha^{\beta^2},\ \dots, \alpha^{\beta^{d-1}}$$

are algebraically independent.

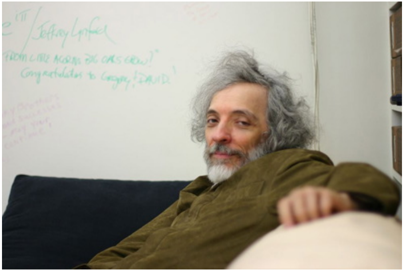
Tools

Transcendence criterion : Replaces Liouville's inequality in transcendence proofs.
Liouville : A non–zero rational integer $n \in \mathbf{Z}$ satisfies $|n| \geq 1$.
Gel'fond : Needs to give a lower bound for $|P(\theta)|$ with $P \in \mathbf{Z}[X] \setminus \{0\}$ when θ is transcendental.

Zero estimate for exponential polynomials :
C. Hermite, P. Turan, K. Mahler, R. Tijdeman,...

Small transcendence degree :
A.O. Gel'fond, A.A. Smelev, R. Tijdeman, W.D. Brownawell...

Large transcendence degree



G.V. Chudnovsky (1976)
Among the numbers

$$\alpha^{\beta},\ \alpha^{\beta^2},\ \dots,\alpha^{\beta^{d-1}}$$

at least $[\log_2 d]$ are algebraically independent.

G.V. Chudnovsky – *On the path to Schanuel's conjecture. Algebraic curves close to a point.*
I. General theory of colored sequences.
II. Fields of finite transcendence type and colored sequences. Resultants.
Studia Sci. Math. Hungar. **12** (1977), 125–157 (1980).

Partial result on the problem of Gel'fond and Schneider

A.O. Gel'fond, G.V. Chudnovskii, P. Philippon, Yu.V. Nesterenko.




G. Diaz : If α is an algebraic number, $\alpha \neq 0$, $\alpha \neq 1$ and if β is an irrational algebraic number of degree d, then

$$\mathrm{tr}\deg_{\mathbf{Q}} \mathbf{Q}\left(\alpha^{\beta},\ \alpha^{\beta^2},\ \ldots,\alpha^{\beta^{d-1}}\right) \geq \left[\frac{d+1}{2}\right].$$

Conjecture of algebraic independence of logarithms of algebraic numbers

Denote by $\mathcal{L}$ the set of complex numbers λ for which e^λ is algebraic :

$$\mathcal{L} = \{\log \alpha \; ; \; \alpha \in \overline{\mathbf{Q}}^\times\}.$$

Hence $\mathcal{L}$ is a $\mathbf{Q}$-vector subspace of $\mathbf{C}$.
The most important special case of Schanuel's Conjecture is :

Conjecture. *Let $\lambda_1, \ldots, \lambda_n$ be $\mathbf{Q}$-linearly independent elements in $\mathcal{L}$. Then the numbers $\lambda_1, \ldots, \lambda_n$ are algebraically independent over $\mathbf{Q}$.*

Not yet known that the transcendence degree is ≥ 2 :
Open problem : *Among all logarithms of algebraic numbers, one at least is transcendental over $\mathbf{Q}(\pi)$.*

Structural rank of a matrix

Let K be a field, k a subfield and $\mathtt{M}$ a matrix with entries in K. Consider the k-vector subspace $\mathcal{E}$ of K spanned by the entries of $\mathtt{M}$. Choose an injective morphism φ of $\mathcal{E}$ into a k-vector space $kX_1 + \cdots + kX_n$. The image $\varphi(\mathtt{M})$ of $\mathtt{M}$ is a matrix whose entries are in the field $k(X_1, \ldots, X_n)$ of rational fractions. Its rank does not depend on the choice of φ.

This is the structural rank of $\mathtt{M}$ with respect to k.

Example

Let

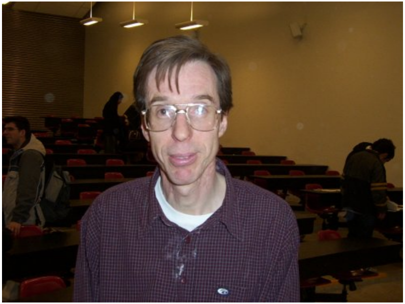
$$\mathtt{M} = \left(b_{ij} + \lambda_{ij}\right)_{\substack{1 \le i \le d \\ 1 \le j \le \ell}}$$

be a matrix with coefficients in $\mathbf{Q} + \mathcal{L}$. Consider a basis of the $\mathbf{Q}$–vector spanned by the entries, and replace the elements in this basis by unknowns : This gives a new matrix $\widetilde{\mathtt{M}}$ with coefficients in a field of rational fractions, the rank of which is the structural rank of $\mathtt{M}$ (with respect to $\mathbf{Q}$).

As a consequence of the conjecture of algebraic independence of logarithms of algebraic numbers, the rank of $\widetilde{\mathtt{M}}$ should be the same as the rank of its specialization $\mathtt{M}$.

Equivalence between the two conjectures

Following D. Roy, the conjecture on algebraic independence of logarithms of algebraic numbers is equivalent to :



Conjecture. *Any matrix*

$$\left(b_{ij} + \lambda_{ij}\right)_{\substack{1\le i\le d\\ 1\le j\le \ell}}$$

with $b_{ij} \in \mathbf{Q}$ *and* $\lambda_{ij} \in \mathcal{L}$ *has a rank equal to its structural rank.*

Any Polynomial is the Determinant of a Matrix

The proof of the equivalence uses the nice auxiliary result :

For any $P \in k[X_1, \ldots, X_n]$ *there exists a square matrix with entries in the* k*-vector space* $k + kX_1 + \cdots + kX_n$ *whose determinant is* P*.*

Half of the Conjecture is solved

From a certain point of view, half of the conjecture of algebraic independence of logarithms of algebraic numbers is solved :

Theorem [D. Roy]. *The rank of any matrix*

$$\left(b_{ij} + \lambda_{ij}\right)_{\substack{1 \le i \le d \\ 1 \le j \le \ell}}$$

with $b_{ij} \in \mathbf{Q}$ *and* $\lambda_{ij} \in \mathcal{L}$ *is at least half its structural rank.*

Reformulation by D. Roy

Instead of taking logarithms of algebraic numbers and looking for the algebraic independence relations, D. Roy fixes a polynomial and looks at the points, with coordinates logarithms of algebraic numbers, on the corresponding hypersurface.

Reformulation by D. Roy

Roy's reformulation of the conjecture of algebraic independence of logarithms is :

Conjecture. *For any algebraic subvariety V of $\mathbf{C}^n$ defined over the field $\overline{\mathbf{Q}}$ of algebraic numbers, the set $V \cap \mathcal{L}^n$ is the union of the sets $E \cap \mathcal{L}^n$, where E ranges over the set of vector subspaces of $\mathbf{C}^n$ which are defined over $\mathbf{Q}$ and contained in V.*

Trivial : Any element in $E \cap \mathcal{L}^n$, where E is a vector subspace of $\mathbf{C}^n$ defined over $\mathbf{Q}$ and contained in V, belongs to $V \cap \mathcal{L}^n$.

Example : The Four Exponentials Conjecture

Take for V the hypersurface of $\mathbf{C}^4$ defined by the equation

$$z_1 z_4 = z_2 z_3.$$

The maximal $\mathbf{C}$–vector subspaces of $\mathbf{C}^4$ defined over $\mathbf{Q}$ and contained in V are the planes

$$az_1 = bz_2, \quad bz_4 = az_3$$

and the planes

$$az_1 = bz_3, \quad bz_4 = az_2$$

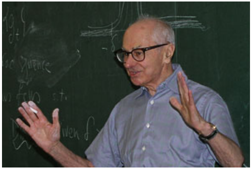
with $(a,b) \in \mathbf{Q}^2 \setminus \{(0,0)\}$.
Hence Schanuel's Conjecture implies that if $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are logarithms of algebraic numbers satisfying

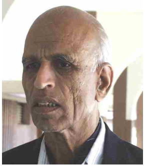
$$\lambda_1 \lambda_4 = \lambda_2 \lambda_3,$$

then either λ_1, λ_2 are linearly dependent over $\mathbf{Q}$, or else λ_1, λ_3 are linearly dependent over $\mathbf{Q}$.

Six Exponentials Theorem and Four Exponentials Conjecture

A. Selberg C.L. Siegel S. Lang K. Ramachandra





The Four Exponentials Conjecture

Conjecture (Th. Schneider, S. Lang, K. Ramachandra). *If x_1, x_2 are $\mathbf{Q}$–linearly independent complex numbers and y_1, y_2 are $\mathbf{Q}$–linearly independent complex numbers, then one at least of the four numbers*

$$e^{x_1 y_1},\ e^{x_1 y_2},\ e^{x_2 y_1},\ e^{x_2 y_2}$$

is transcendental.

Equivalent statement : *Any 2×2 matrix*

$$\begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix}$$

with entries in $\mathcal{L}$ and with $\mathbf{Q}$–linearly independent rows and $\mathbf{Q}$–linearly independent columns has maximal rank 2.

How could we attack Schanuel's Conjecture ?

Let $x_1, \ldots, x_n$ be $\mathbf{Q}$–linearly independent complex numbers. Following the transcendence methods of Hermite, Gel'fond, Schneider..., one may start by introducing an auxiliary function

$$F(z) = P(z, e^z)$$

where $P \in \mathbf{Z}[X_0, X_1]$ is a non–zero polynomial. One considers the derivatives

$$\left(\frac{d}{dz}\right)^k F = (\mathcal{D}^k P)(z, e^z)$$

at the points

$$m_1 x_1 + \cdots + m_n x_n$$

for various values of $(m_1, \ldots, m_n) \in \mathbf{Z}^n$.

Auxiliary function

Let $\mathcal{D}$ denote the derivation

$$\mathcal{D} = \frac{\partial}{\partial X_0} + X_1 \frac{\partial}{\partial X_1}$$

over the ring $\mathbf{C}[X_0, X_1]$. Recall that $x_1, \dots, x_n$ are $\mathbf{Q}$–linearly independent complex numbers. Let $\alpha_1, \dots, \alpha_n$ be non–zero complex numbers.

The transcendence machinery produces a sequence $(P_N)_{N \geq 0}$ of polynomials with integer coefficients satisfying

$$\left| \left(\mathcal{D}^k P_N\right) \left(\sum_{j=1}^{n} m_j x_j, \prod_{j=1}^{n} \alpha_j^{m_j} \right) \right| \leq \exp(-N^u)$$

for any non-negative integers k, $m_1, \dots, m_n$ with $k \leq N^{s_0}$ and $\max\{m_1, \dots, m_n\} \leq N^{s_1}$.

Roy's approach to Schanuel's Conjecture (1999)

If the number of equations we produce is too small, such a set of relations does not contain any information : The existence of a sequence of non–trivial polynomials $(P_N)_{N\geq 0}$ follows from linear algebra.

On the other hand, following D. Roy, one may expect that the existence of such a sequence $(P_N)_{N\geq 0}$ producing sufficiently many such equations will yield the desired conclusion :

$$\operatorname{tr}\deg_{\mathbf{Q}}\mathbf{Q}\big(x_1,\ldots,x_n,\alpha_1,\ldots,\alpha_n\big)\geq n.$$

Roy's Conjecture (1999)

Let s_0, s_1, t_0, t_1, u positive real numbers satisfying

$$\max\{1, t_0, 2t_1\} < \min\{s_0, 2s_1\}$$

and

$$\max\{s_0, s_1 + t_1\} < u < \frac{1}{2}(1 + t_0 + t_1).$$

Assume that, for any sufficiently large positive integer N, there exists a non–zero polynomial $P_N \in \mathbf{Z}[X_0, X_1]$ with partial degree $\leq N^{t_0}$ in X_0, partial degree $\leq N^{t_1}$ in X_1 and height $\leq e^N$ which satisfies

$$\left| \left(\mathcal{D}^k P_N\right)\left(\sum_{j=1}^{n} m_j x_j, \prod_{j=1}^{n} \alpha_j^{m_j}\right) \right| \leq \exp(-N^u)$$

for any non-negative integers k, $m_1, \ldots, m_n$ with $k \leq N^{s_0}$ and $\max\{m_1, \ldots, m_n\} \leq N^{s_1}$. Then

$$\operatorname{tr}\deg_{\mathbf{Q}} \mathbf{Q}(x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_n) \geq n.$$

Roy's Theorem (1999)

Roy's conjecture is equivalent to Schanuel's Conjecture.

Equivalence between Schanuel and Roy

Let $(x, \alpha) \in \mathbf{C} \times \mathbf{C}^\times$, and let s_0, s_1, t_0, t_1, u be positive real numbers satisfying the inequalities of Roy's Conjecture. *Then the following conditions are equivalent :*

(a) The number αe^{-x} *is a root of unity.*

(b) For any sufficiently large positive integer N*, there exists a non–zero polynomial* $Q_N \in \mathbf{Z}[X_0, X_1]$ *with partial degree* $\le N^{t_0}$ *in* X_0*, partial degree* $\le N^{t_1}$ *in* X_1 *and height* $\mathrm{H}(Q_N) \le e^N$ *such that*

$$\left|(\mathcal{D}^k Q_N)(mx, \alpha^m)\right| \le \exp(-N^u)$$

for any $k, m \in \mathbf{N}$ *with* $k \le N^{s_0}$ *and* $m \le N^{s_1}$*.*

Gel'fond's transcendence criterion

Simple form : *Given a complex number ϑ, if there exists a sequence $(P_n)_{n\geq 1}$ of non–zero polynomials in $\mathbf{Z}[X]$, with P_n of degree $\leq n$ and height $\leq e^n$, such that*

$$|P_n(\vartheta)| \leq e^{-7n^2}$$

for all $n \geq 1$, then ϑ is algebraic and $P_n(\vartheta) = 0$ for all $n \geq 1$.

First extension : Replace the bound for the degree by $\leq d_n$, the bound for the height by e^{b_n}, and the bound for $|P_n(\vartheta)|$ by $e^{-cd_nb_n}$ with some constant $c > 0$ independent of n.

Some mild conditions are required on the sequences $(d_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$.

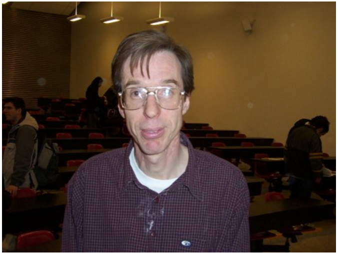
Transcendence criterion with multiplicities

With derivatives : *Given a complex number ϑ, if there exists a sequence $(P_n)_{n\geq 1}$ of non–zero polynomials in $\mathbf{Z}[X]$, with P_n of degree $\leq d_n$ and height $\leq e^{b_n}$, such that*

$$\left|P_n^{(j)}(\vartheta)\right| \leq e^{-cd_nb_n/t_n}$$

for all j in the range $0 \leq j < t_n$ and all $n \geq 1$, then ϑ is algebraic.

Due to M. Laurent and D. Roy, applications to algebraic independence.



Criterion with several points

Goal : *Given a sequence of complex numbers $(\vartheta_i)_{i\geq 1}$, if there exists a sequence $(P_n)_{n\geq 1}$ of non–zero polynomials in $\mathbf{Z}[X]$, with P_n of degree $\leq d_n$ and height $\leq e^{b_n}$, such that*

$$\left|P_n^{(j)}(\vartheta_i)\right| \leq e^{-cd_n b_n/t_n s_n}$$

for $0 \leq j < t_n$, $1 \leq i \leq s_n$ and all $n \geq 1$, then the numbers ϑ_i are algebraic.

D. Roy : Not true in general, but true in some special cases with a structure on the sequence $(\vartheta_i)_{i\geq 1}$.
Combines the elimination arguments used for criteria of algebraic independence and for zero estimates.

Small value estimates for the additive group

D. Roy. *Small value estimates for the additive group*. Intern. J. Number Theory, to appear.

Let ξ be a transcendental complex number, let β, σ, τ and ν be non-negative real numbers, let n_0 be a positive integer, and let $(P_n)_{n\geq n_0}$ be a sequence of non–zero polynomials in $\mathbf{Z}[T]$ satisfying $\deg(P_n) \leq n$ and $H(P_n) \leq \exp(n^\beta)$ for each $n \geq n_0$. Suppose that $\beta > 1$, $(3/4)\sigma + \tau < 1$ and $\nu > 1 + \beta - (3/4)\sigma - \tau$. Then for infinitely many n, we have

$$\max\left\{|P_n^{[j]}(i\xi)|\,;\, 0 \leq i \leq n^\sigma,\, 0 \leq j \leq n^\tau\right\} > \exp(-n^\nu).$$

Small value estimates for the multiplicative group

D. Roy. *Small value estimates for the multiplicative group*. Acta Arith., to appear.

Let $\xi_1, \ldots, \xi_m$ be multiplicatively independent complex numbers in a field of transcendence degree 1. Under suitable assumptions on the parameters β, σ, τ, ν, for infinitely many positive integers n, there exists no non–zero polynomial $P \in \mathbf{Z}[T]$ satisfying $\deg(P) \le n$, $H(P) \le \exp(n^\beta)$ and

$$\max\left\{ |P^{[j]}(\xi_1^{i_1} \cdots \xi_m^{i_m})| \,;\, 0 \le i_1, \ldots, i_m \le n^\sigma,\, 0 \le j \le n^\tau \right\} > \exp(-n^\nu).$$

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Transcendental Number Theory: Schanuel's Conjecture

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