International Conference on Algebra and Related Topics (ICART 2008) *May 29, 2008*

http ://www.math.sc.chula.ac.th/∼icart2008/

On the Markoff Equation

$$x^2 + y^2 + z^2 = 3xyz$$

Michel Waldschmidt

http://www.math.jussieu.fr/∼miw/

Abstract

It is easy to check that the equation $x^2 + y^2 + z^2 = 3xyz$, where the three unknowns x, y, z are positive integers, has infinitely many solutions. There is a simple algorithm which produces all of them. However, this does not answer to all questions on this equation : in particular Frobenius asked whether it is true that for each integer $z > 0$, there is at most one pair (x, y) such that $x < y < z$ and (x, y, z) is a solution. This question is an active research topic nowadays.

Abstract (continued)

Markoff's equation occurred initially in the study of minima of quadratic forms at the end of the XIX–th century and the beginning of the XX–th century. It was investigated by many a mathematician, including Lagrange, Hermite, Korkine, Zolotarev, Markoff, Frobenius, Hurwitz, Cassels. The solutions are related with the *Lagrange-Markoff spectrum*, which consists of those quadratic numbers which are badly approximable by rational numbers. It occurs also in other parts of mathematics, in particular free groups, Fuchsian groups and hyperbolic Riemann surfaces (Ford, Lehner, Cohn, Rankin, Conway, Coxeter, Hirzebruch and Zagier...).

We discuss some aspects of this topic without trying to cover all of them.

The sequence of Markoff numbers

A *Markoff number* is a positive integer z such that there exist two positive integers x and y satisfying

$$x^2 + y^2 + z^2 = 3xyz.$$

For instance 1 is a Markoff number, since $(x, y, z) = (1, 1, 1)$ is a solution.

Andrei Andreyevich Markoff (1856–1922)

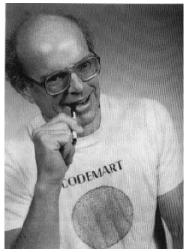
Photos : http ://www-history.mcs.st-andrews.ac.uk/history/

The On-Line Encyclopedia of Integer Sequences

1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610, 985, 1325, 1597, 2897, 4181, 5741, 6466, 7561, 9077, 10946, 14701, 28657, 33461, 37666, 43261, 51641, 62210, 75025, 96557, 135137, 195025, 196418, 294685, ...

The sequence of Markoff numbers is available on the web
The On-Line Encyclopedia of Integer Sequences

Neil J. A. Sloane



http ://www.research.att.com/∼njas/sequences/A002559

Integer points on a surface

Given a Markoff number z, there exist infinitely many pairs of positive integers x and y satisfying

$$x^2 + y^2 + z^2 = 3xyz.$$

This is a cubic equation in the 3 variables (x, y, z), of which we know a solution $(1, 1, 1)$.

There is an algorithm producing all integer solutions.

Markoff's cubic variety

The surface defined by Markoff's equation

$$x^2 + y^2 + z^2 = 3xyz.$$

is an algebraic variety with many automorphisms : permutations of the variables, changes of signs and

$$(x, y, z) \mapsto (3yz - x, y, z).$$

A.A. Markoff (1856–1922)

Algorithm producing all solutions

Let (m, m_1, m_2) be a solution of Markoff's equation :

$$m^2 + m_1^2 + m_2^2 = 3mm_1m_2.$$

Fix two coordinates of this solution, say m_1 and m_2. We get a quadratic equation in the third coordinate m, of which we know a solution, hence the equation

$$x^2 + m_1^2 + m_2^2 = 3xm_1m_2.$$

has two solutions, $x = m$ and, say, $x = m'$, with $m + m' = 3m_1m_2$ and $mm' = m_1^2 + m_2^2$. This is the *cord and tangente process.*

Hence *another* solution is (m', m_1, m_2) with $m' = 3m_1m_2 - m$.

Three solutions derived from one

Starting with one solution (m, m_1, m_2), we derive three *new* solutions :

$$(m', m_1, m_2), \quad (m, m_1', m_2), \quad (m, m_1, m_2').$$

If the solution we start with is $(1, 1, 1)$, we produce only one new solution, $(2, 1, 1)$ (up to permutation).

If we start from $(2, 1, 1)$, we produce only two *new* solutions, $(1, 1, 1)$ and $(5, 2, 1)$ (up to permutation).

A *new* solution means *distinct from the one we start with.*

New solutions

We shall see that any solution different from $(1, 1, 1)$ and from $(2, 1, 1)$ yields three new different solutions – and we shall see also that in each other solution the three numbers m, m_1 and m_2 are pairwise distinct.

Two solutions are *neighbors* if they share two components.

Markoff's tree

Assume we start with (m, m_1, m_2) satisfying $m > m_1 > m_2$. We shall check

$$m_2' > m_1' > m > m'.$$

We order the solution according to the largest coordinate. Then two of the neighbors of (m, m_1, m_2) are larger than the initial solution, the third one is smaller.

Hence if we start from $(1, 1, 1)$, we produce infinitely many solutions, which we organize in a tree : this is *Markoff's tree.*

This algorithm yields all the solutions

Conversely, starting from any solution other than $(1, 1, 1)$, the algorithm produces a *smaller* solution.

Hence by induction we get a sequence of smaller and smaller solutions, until we reach $(1, 1, 1)$.

Therefore the solution we started from was in Markoff's tree.

First branches of Markoff's tree

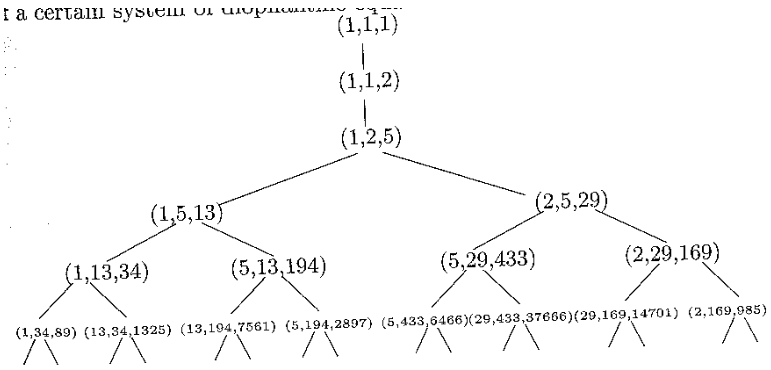
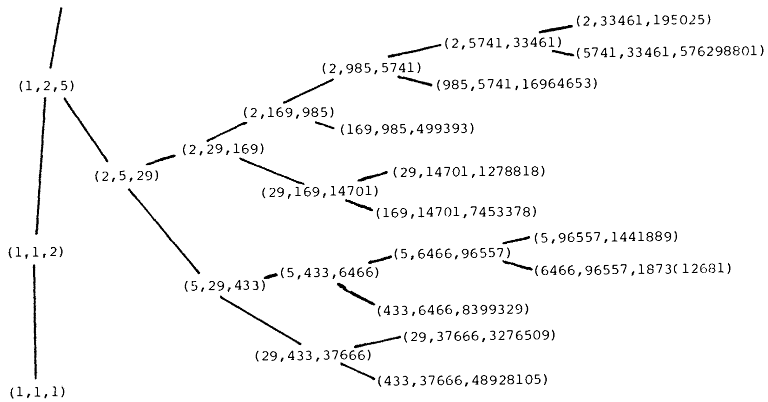


Figure 10. The Tree of Markoff Solutions.

Markoff's tree starting from $(2, 5, 29)$



Markoff's tree up to 100 000

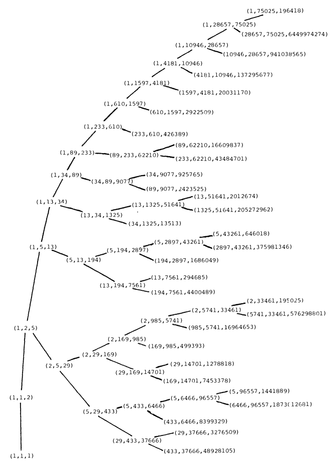
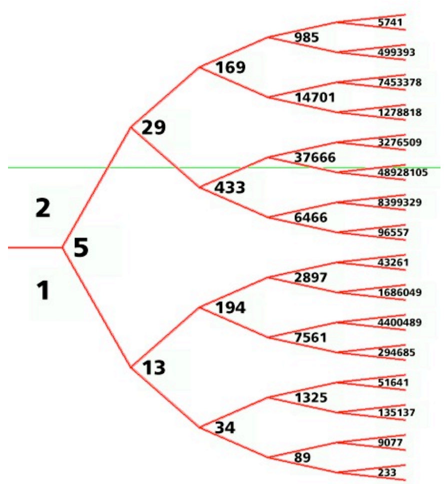


FIGURE 2
Markoff triples (p, q, r) with $\max(p, q) \leq 100000$

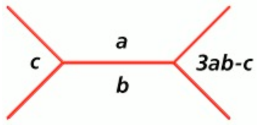
Don Zagier,
On the number of Markoff numbers below a given bound.
Mathematics of Computation, **39** 160 (1982), 709–723.

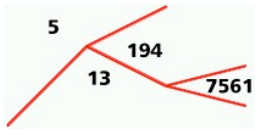
Markoff's tree

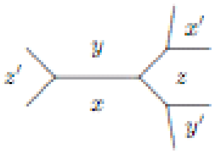


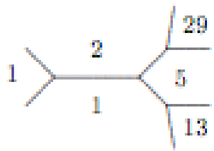
$a^2 + b^2 + c^2 = 3abc$

$$X^2 - 3abX + a^2 + b^2 = (X - c)(X - 3ab + c)$$









The Fibonacci sequence and the Markoff equation

The smallest Markoff number is 1. When we impose $z = 1$ in the Markoff equation $x^2 + y^2 + z^2 = 3xyz$, we obtain the equation

$$x^2 + y^2 + 1 = 3xy.$$

Going along the Markoff's tree starting from $(1, 1, 1)$, we obtain the subsequence of Markoff numbers

$$1, 2, 5, 13, 34, 89, 233, 610, 1597, 4181, 10946, 28657, \dots$$

which is the sequence of Fibonacci numbers with odd indices

$$F_1 = 1, F_3 = 2, F_5 = 5, F_7 = 13, F_9 = 34, F_{11} = 89, \dots$$

Leonardo Pisano (Fibonacci)

The Fibonacci sequence $(F_n)_{n\ge 0}$:

$$0, 1, 1, 2, 3, 5, 8, 13, 21,$$

$$34, 55, 89, 144, 233 \dots$$

is defined by

$$F_0 = 0, F_1 = 1,$$

$$F_n = F_{n-1} + F_{n-2} \quad (n \ge 2).$$

Leonardo Pisano (Fibonacci)
(1170–1250)

Encyclopedia of integer sequences (again)

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, 514229, 832040, 1346269, 2178309, 3524578, 5702887, 9227465, …

The Fibonacci sequence is available online
The On-Line Encyclopedia of Integer Sequences

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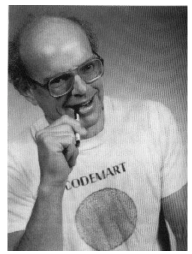


http ://www.research.att.com/~njas/sequences/A000045

Fibonacci numbers with odd indices

Fibonacci numbers with odd indices are Markoff's numbers :

$$F_{m+3}F_{m-1} - F_{m+1}^2 = (-1)^m \quad \text{for} \quad m \geq 1$$

and

$$F_{m+3} + F_{m-1} = 3F_{m+1} \quad \text{for} \quad m \geq 1.$$

Set $y = F_{m+1}$, $x = F_{m-1}$, $x' = F_{m+3}$, so that, for even m,

$$x + x' = 3y, \quad xx' = y^2 + 1$$

and

$$X^2 - 3yX + y^2 + 1 = (X - x)(X - x').$$

Order of the *new* solutions

Let (m, m_1, m_2) be a solution of Markoff's equation

$$m^2 + m_1^2 + m_2^2 = 3mm_1m_2.$$

Denote by m' the other root of the quadratic polynomial

$$X^2 - 3m_1m_2X + m_1^2 + m_2^2.$$

Hence

$$X^2 - 3m_1m_2X + m_1^2 + m_2^2 = (X - m)(X - m')$$

and

$$m + m' = 3m_1m_2, \quad mm' = m_1^2 + m_2^2.$$

$m_1 \neq m_2$

Let us check that *if* $m_1 = m_2$, *then* $m_1 = m_2 = 1$ *:* this holds only for the two exceptional solutions $(1, 1, 1)$, $(2, 1, 1)$.

Assume $m_1 = m_2$. We have

$$m^2 + 2m_1^2 = 3mm_1^2 \quad \text{hence} \quad m^2 = (3m - 2)m_1^2.$$

Therefore m_1 divides m. Let $m = km_1$. We have $k^2 = 3km_1 - 2$, hence k divise 2.
For $k = 1$ we get $m = m_1 = 1$.
For $k = 2$ we get $m_1 = 1$, $m = 2$.

Consider now a solution distinct from $(1, 1, 1)$ or $(2, 1, 1)$: hence $m_1 \neq m_2$.

Two larger, one smaller

Assume $m_1 > m_2$.
Question : *Do we have* $m' > m_1$ *or else* $m' < m_1$ *?*

Consider the number $a = (m_1 - m)(m_1 - m')$.

Since $m + m' = 3m_1m_2$, and $mm' = m_1^2 + m_2^2$, we have

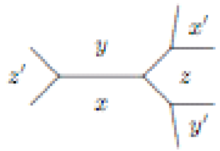
$$\begin{aligned} a &= m_1^2 - m_1(m + m') + mm' \\ &= 2m_1^2 + m_2^2 - 3m_1^2m_2 \\ &= (2m_1^2 - 2m_1^2m_2) + (m_2^2 - m_1^2m_2). \end{aligned}$$

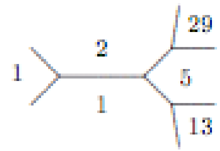
However $2m_1^2 < 2m_1^2m_2$ and $m_2^2 < m_1^2m_2$, hence $a < 0$.

This means that
m_1 is in the interval defined by m and m'.

Order of the solutions

If $m > m_1$, we have $m_1 > m'$ and the new solution (m', m_1, m_2) is smaller than the initial solution (m, m_1, m_2).
If $m < m_1$, we have $m_1 < m'$ and the new solution (m', m_1, m_2) is larger than the initial solution (m, m_1, m_2).





Prime factors

Remark. Let m be a Markoff number with

$$m^2 + m_1^2 + m_2^2 = 3mm_1m_2.$$

The same proof shows that the GCD of m, m_1 and m_2 is 1 : indeed, if p divides m_1, m_2 and m, then p divides the *new* solutions which are produced by the preceding process – going down in the tree shows that p would divide 1.

The odd prime factors of m are all congruent to 1 modulo 4 (since they divide a sum of two relatively prime squares).

If m is even, then the numbers

$$\frac{m}{2}, \quad \frac{3m-2}{4}, \quad \frac{3m+2}{8},$$

are odd integers.

Markoff's Conjecture

The previous algorithm produces the sequence of Markoff numbers. Each Markoff number occurs infinitely often in the tree as one of the components of the solution.

According to the definition, for a Markoff number $m > 2$ there exist a pair (m_1, m_2) of positive integers with $m > m_1 > m_2$ such that $m^2 + m_1^2 + m_2^2 = 3mm_1m_2$.

Question : *Given m, is such a pair (m_1, m_2) unique ?*

The answer is yes, as long as $m \le 10^{105}$.

Frobenius's work

Markoff's Conjecture does not occur in Markoff's 1879 and 1880 papers but in Frobenius's one in 1913.

Ferdinand Georg Frobenius
(1849–1917)

Special cases

The Conjecture has been proved for certain classes of Markoff numbers m like

$$p^n, \quad \frac{p^n \pm 2}{3}$$

for p prime.
A. Baragar (1996),
P. Schmutz (1996),
J.O. Button (1998),
M.L. Lang, S.P. Tan (2005),
Ying Zhang (2007).

Arthur Baragar

http ://www.nevada.edu/ baragar/

Powers of a prime number

Anitha Srinivasan, 2007
A really simple proof of the Markoff conjecture for prime powers

Number Theory Web
Created and maintained by
Keith Matthews, Brisbane, Australia
www.numbertheory.org/pdfs/simpleproof.pdf

The state of the art

10/09/2007, 04/12/2007 : Norbert Riedel
http ://fr.arxiv.org/abs/0709.1499v2
http ://fr.arxiv.org/abs/0709.1499

A triple (a, b, c) of positive integers is called a Markoff triple iff it satisfies the diophantine equation $a^2 + b^2 + c^2 = abc$. Recasting the Markoff tree, whose vertices are Markoff triples, in the framework of integral upper triangular 3×3 matrices, it will be shown that the largest member of such a triple determines the other two uniquely. This answers a question which has been open for almost 100 years.

Flaw in the proof discovered by Serge Perrine.

Why the coefficient 3 ?

Let n be a positive integer.
If the equation $x^2 + y^2 + z^2 = nxyz$ has a solution in positive integers, then either $n = 3$ and x, y, z are relatively prime, or $n = 1$ and the GCD of the numbers x, y, z is 3.

Friedrich Hirzebruch & Don Zagier,
The Atiyah–Singer Theorem and elementary number theory,
Publish or Perish (1974)

Markoff type equations

Bijection between the solutions for $n=1$ and those for $n=3$:

• if $x^2+y^2+z^2=3xyz$, then $(3x,3y,3z)$ is solution of $X^2+Y^2+Z^2=XYZ$, since $(3x)^2+(3y)^2+(3z)^2=(3x)(3y)(3z)$.

• if $X^2+Y^2+Z^2=XYZ$, then X, Y, Z are multiples of 3 and $(X/3)^2+(Y/3)^2+(Z/3)^2=3(X/3)(Y/3)(Z/3)$.

The squares modulo 3 are 0 and 1. If X, Y and Z are not multiples of 3, then $X^2+Y^2+Z^2$ is a multiple of 3.

If one or two (not three) integers among X, Y, Z are multiples of 3, then $X^2+Y^2+Z^2$ is not a multiple of 3.

Equations $x^2+ay^2+bz^2=(1+a+b)xyz$

If we insist that $(1,1,1)$ is a solution, then up to permutations there are only two more Diophantine equations of the type

$$x^2+ay^2+bz^2=(1+a+b)xyz$$

having infinitely many integer solutions, namely those with $(a,b)=(1,2)$ and $(2,3)$:

$$x^2+y^2+2z^2=4xyz \quad \text{and} \quad x^2+2y^2+3z^2=6xyz$$

• $x^2+y^2+z^2$: tessalation of the plane by equilateral triangles
• $x^2+y^2+2z^2=4xyz$: tessalation of the plane by isoceles rectangle triangles
• $x^2+2y^2+3z^2=6xyz$: tessalation ?

Laurent's phenomenon

Connection with Laurent polynomials.
James Propp, *The combinatorics of frieze patterns and Markoff numbers,*
`http ://fr.arxiv.org/abs/math/0511633`
If f, g, h are Laurent polynomials in two variables x and y, i.e., polynomials in x, x^{-1}, y, y^{-1}, in general

$$h\big(f(x,y),g(x,y)\big)$$

is not a Laurent polynomial :

$$f(x)=\frac{x^2+1}{x}=x+\frac{1}{x},$$

$$f\big(f(x)\big)=\frac{\left(x+\frac{1}{x}\right)^2+1}{x+\frac{1}{x}}=\frac{x^4+3x^2+1}{x(x^2+1)}.$$

Hurwitz's equation (1907)

For each $n\ge 2$ the set K_n of positive integers k for which the equation

$$x_1^2+x_2^2+\cdots+x_n^2=kx_1\cdots x_n$$

has a solution in positive integers is finite.
The largest value of k in K_n is n — with the solution

$$(1,1,\ldots,1).$$

Examples :

$K_3=\{1,\ 3\}$,
$K_4=\{1,\ 4\}$,
$K_7=\{1,\ 2,\ 3,\ 5,\ 7\}$.

Hurwitz's equation

$$x_1^2 + x_2^2 + \cdots + x_n^2 = kx_1 \cdots x_n$$

When there is a solution in positive integers, there are infinitely many solutions, which can be organized in finitely many trees.

A. Baragar proved that *there exists such equations which require an arbitrarily large number of trees*
J. Number Theory (1994), **49** No 1, 27-44.

The analog for the rank of elliptic curves over the rational number field is yet a conjecture.

Growth of Markoff's sequence

1978 : order of magnitude of m, m_1 and m_2 for
$m^2 + m_1^2 + m_2^2 = 3mm_1m_2$
with $m_1 < m_2 < m$,

Harvey Cohn

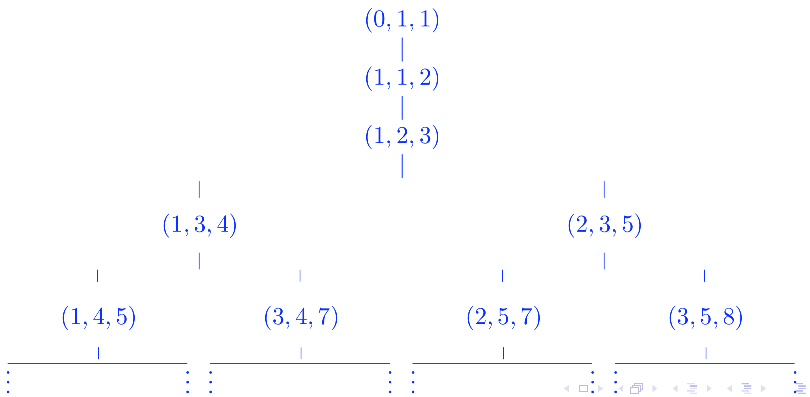
$\log(3m_1) + \log(3m_2) = \log(3m) + o(1)$.

To identify primitive words in a free group with two generators, H. Cohn used *Markoff forms.*

$x \mapsto \log(3x) : (m_1, m_2, m) \mapsto (a, b, c)$ with $a + b \sim c$.

Euclidean tree

Start with $(0, 1, 1)$. From a triple (a, b, c) satisfying $a + b = c$ and $a \le b \le c$, one produces two larger such triples $(a, c, a + c)$ and $(b, c, b + c)$ and a smaller one $(a, b - a, b)$ or $(b - a, a, b)$.



Markoff and Euclidean trees

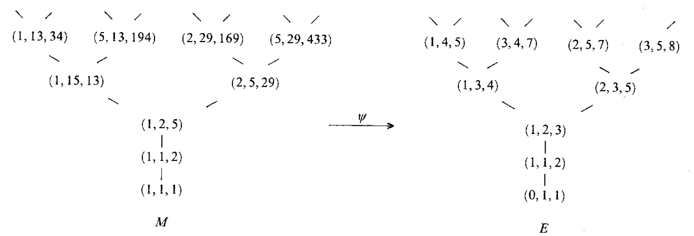


Figure 2. The Markoff tree and the Euclid tree

Tom Cusik & Mary Flahive,
The Markoff and Lagrange spectra,
Math. Surveys and Monographs **30**, AMS (1989).

Growth of Markoff's sequence

Don Zagier (1982) : estimating the number of the Markoff triples bounded by x :

$c(\log x)^2 + O\big(\log x (\log\log x)^2\big)$,

$c = 0,18071704711507\ldots$

Conjecture : the n-th Markoff number m_n is

$$m_n \sim A^{\sqrt{n}} \quad \text{with} \quad A = 10,5101504\cdots$$

Historical origin : rational approximation

Hurwitz's Theorem (1891) : *For any real irrational number x, there exist infinitely many rational numbers p/q such that*

$$\left| x - \frac{p}{q} \right| \le \frac{1}{\sqrt{5}q^2}.$$

Golden ratio
$\Phi = (1+\sqrt{5})/2 = 1,61803398874989484 82\ldots$
Hurwitz's result is optimal.

Adolf Hurwitz
(1859–1919)

The Fibonacci sequence and the Golden ratio

Formula of A. De Moivre (1730), L. Euler (1765), J.P.M. Binet (1843) :

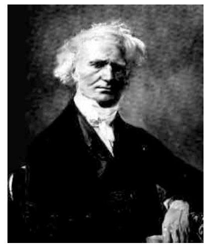
$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n.$$

Formula of De Moivre–Euler–Binet

Abraham de Moivre
(1667–1754)

Leonhard Euler
(1707–1783)

Jacques Philippe Marie Binet
(1786–1856)



F_n is the nearest integer to $\frac{1}{\sqrt{5}}\Phi^n$.

Quadratic relation

One checks by induction

$$F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n \quad \text{for all} \quad n \geq 0.$$

The left hand side is the value at (F_{n+1}, F_n) of the quadratic form

$$X^2 - XY - Y^2 = (X - \Phi Y)(X + \Phi^{-1}Y).$$

The sequence $u_n = F_{n+1}/F_n$, $n \geq 1$ converges to the Golden ratio Φ and

$$F_{n+1}^2 - F_{n+1}F_n - F_n^2 = F_n^2(u_n - \Phi)(u_n + \Phi^{-1}).$$

Quotients of consecutive Fibonacci numbers

One deduces

$$F_n^2|\Phi - u_n| = \frac{1}{\Phi^{-1} + u_n} \to \frac{1}{\Phi^{-1} + \Phi} = \frac{1}{\sqrt{5}}.$$

Hence

$$\lim_{n\to\infty} F_n^2 \left| \Phi - \frac{F_{n+1}}{F_n} \right| = \frac{1}{\sqrt{5}}.$$

Continued fractions

The sequence $u_n = F_{n+1}/F_n$ is also defined by

$$u_1 = 1,\ u_n = 1 + \frac{1}{u_{n-1}}, \quad (n \geq 2).$$

Hence

$$u_n = 1 + \cfrac{1}{1 + \cfrac{1}{u_{n-2}}} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{u_{n-3}}}} = \cdots$$

$$= [1, 1, \dots, 1] \quad n \text{ times}$$

$$= [\overline{1}]$$

Hurwitz's result is optimal

Hurwitz's result

$$\liminf_{q\to\infty} \left(q \min_{p\in\mathbf{Z}} |qx - p| \right) \leq \frac{1}{\sqrt{5}} \quad \text{for all } x \in \mathbf{R} \setminus \mathbf{Q}$$

is optimal : there is equality for $x = \Phi$.
For $|q\Phi - p| \leq 1$, we have

$$1 \leq |q^2 + pq - p^2| = |q\Phi - p| \cdot (q\Phi^{-1} + p)$$

with

$$q\Phi^{-1} + p = q(\Phi + \Phi^{-1}) + p - q\Phi \leq q\sqrt{5} + 1,$$

hence

$$1 \leq |q\Phi - p| \cdot (q\sqrt{5} + 1).$$

Notice that $P(X) = X^2 - X - 1$ has discriminant 5 and $P'(\Phi) = \sqrt{\Delta} = \sqrt{5}$.

Liouville's inequality

Liouville's inequality. *Let α be an algebraic number of degree $d \geq 2$, $P \in \mathbf{Z}[X]$ its minimal polynomial, $c = |P'(\alpha)|$ and $\epsilon > 0$. There exists q_0 such that, for any $p/q \in \mathbf{Q}$ with $q \geq q_0$,*

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{(c+\epsilon)q^d}.$$

Joseph Liouville, 1844

Markoff's constant

For $x \in \mathbf{R} \setminus \mathbf{Q}$ denote by $\lambda(x) \in [\sqrt{5}, +\infty]$ the least upper bound of the numbers $\gamma > 0$ such that there exist infinitely many $p/q \in \mathbf{Q}$ satisfying

$$\left|x - \frac{p}{q}\right| \leq \frac{1}{\gamma q^2}.$$

This means

$$\frac{1}{\lambda(x)} = \liminf_{q \to \infty} \bigl(q \min_{p \in \mathbf{Z}} |qx - p|\bigr).$$

Hurwitz : $\lambda(x) \geq \sqrt{5}$ for any x and $\lambda(\Phi) = \sqrt{5}$.

Markoff's constant

An irrational real number x is *badly approximable* by rational numbers if its Markoff's constant is finite. This means that there exists $\gamma > 0$ such that, for any $p/q \in \mathbf{Q}$,

$$\left|x - \frac{p}{q}\right| \geq \frac{1}{\gamma q^2}.$$

For instance Liouville's numbers have an infinite Markoff's constant.

A real number is badly approximable if and only if the sequence $(a_n)_{n\geq 0}$ of partial quotients in its continued fraction expansion

$$x = [a_0,\ a_1,\ a_2,\ \dots, a_n, \dots]$$

is bounded.

Badly approximable numbers

Any quadratic irrational real number has a finite Markoff's constant (= is badly approximable).

It is not known whether there exist real algebraic numbers of degree ≥ 3 which are badly approximable.

It is not known whether there exist real algebraic numbers of degree ≥ 3 which are *not* badly approximable . . .

One conjectures that *any irrational real number which is not quadratic and which is badly approximable is transcendental.*

Lebesgue measure

The set of badly approximable real numbers has zero measure for Lebesgue's measure.

Henri Léon Lebesgue
(1875–1941)

Properties of the Markoff's constant

We have

$$\lambda(x+1)=\lambda(x): \qquad \left|x+1-\frac{p}{q}\right| = \left|x-\frac{p+q}{q}\right|$$

and

$$\lambda(-x)=\lambda(x): \qquad \left|-x-\frac{p}{q}\right| = \left|x+\frac{p}{q}\right|,$$

Also $\lambda(1/x)=\lambda(x)$:

$$p^2\left|\frac{1}{x}-\frac{q}{p}\right| = q^2\left|\frac{p}{qx}\right|\cdot\left|x-\frac{p}{q}\right|.$$

The modular group

The multiplicative group generated by the three matrices

$$\begin{pmatrix}1&1\\0&1\end{pmatrix}, \quad \begin{pmatrix}1&0\\0&-1\end{pmatrix}, \quad \begin{pmatrix}0&1\\1&0\end{pmatrix}$$

is the group $\mathrm{GL}_2(\mathbf{Z})$ of 2×2 matrices $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ with coefficients in $\mathbf{Z}$ and determinant ± 1.

J-P. Serre – *Cours d'arithmétique*, Coll. SUP, Presses Universitaires de France, Paris, 1970.

$$\begin{pmatrix}a&b\\c&d\end{pmatrix} x = \frac{ax+b}{cx+d}$$

$$\begin{pmatrix}1&1\\0&1\end{pmatrix} x = x+1 \qquad \begin{pmatrix}1&0\\0&-1\end{pmatrix} x = -x \qquad \begin{pmatrix}0&1\\1&0\end{pmatrix} x = \frac{1}{x}$$

$$\lambda(x+1)=\lambda(x) \qquad \lambda(-x)=\lambda(x) \qquad \lambda(1/x)=\lambda(x)$$

Consequence : *Let $x\in\mathbf{R}\setminus\mathbf{Q}$ and let a, b, c, d be rational integers satisfying $ad-bc=\pm 1$. Set*

$$y=\frac{ax+b}{cx+d}.$$

Then $\lambda(x)=\lambda(y)$.

Hurwitz's work (continued)

The inequality $\lambda(x) \geq \sqrt{5}$ *for all real irrational* x is optimal for the Golden ratio and for all the *noble* irrational numbers whose continued fraction expansion ends with an infinite sequence of 1's – these numbers are the roots of the quadratic polynomials having discriminant 5 :

$$\Phi = [1, 1, 1, \dots] = [\overline{1}].$$

Adolf Hurwitz, 1891

The first elements of the spectrum

Hurwitz's inequality $\lambda(x) \geq \sqrt{5}$ is optimal for the Golden ratio Φ and all the numbers related to Φ by a homography of determinant ± 1 :

$$\frac{a\Phi + b}{c\Phi + d} \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}).$$

For all the other real numbers we have $\lambda(x) \geq 2\sqrt{2}$. This is optimal for

$$\sqrt{2} = 1,4142135623730950488016887242096980785\dots$$

whose continued fraction expansion is

$$[1;\ 2,\ 2,\ \dots, 2,\ \dots] = [1; \overline{2}].$$

Minima of quadratic forms

Let $f(X, Y) = aX^2 + bXY + cY^2$ be a quadratic form with real coefficients. Denote by $\Delta(f)$ its discriminant $b^2 - 4ac$.

Consider the minimum $m(f)$ of $|f(x, y)|$ on $\mathbf{Z}^2 \setminus \{(0, 0)\}$. Assume $\Delta(f) \neq 0$ and set

$$C(f) = m(f)/\sqrt{|\Delta(f)|}.$$

Let α and α' be the roots of $f(X, 1)$:

$$f(X, Y) = a(X - \alpha Y)(X - \alpha' Y),$$

$$\{\alpha, \alpha'\} = \left\{ \frac{1}{2a} \left(-b \pm \sqrt{\Delta(f)} \right) \right\}.$$

Example with $\Delta < 0$

The form

$$f(X, Y) = X^2 + XY + Y^2$$

has discriminant $\Delta(f) = -3$ and minimum $m(f) = 1$, hence

$$C(f) = \frac{m(f)}{\sqrt{|\Delta(f)|}} = \frac{1}{\sqrt{3}}.$$

For $\Delta < 0$, the form

$$f(X, Y) = \sqrt{\frac{|\Delta|}{3}} (X^2 + XY + Y^2)$$

has discriminant Δ and minimum $\sqrt{|\Delta|/3}$. Again

$$C(f) = \frac{1}{\sqrt{3}}.$$

Definite quadratic forms ($\Delta < 0$)

If the discriminant is negative, J.L. Lagrange and Ch. Hermite (letter to Jacobi, August 6, 1845) proved $C(f) \leq 1/\sqrt{3}$ with equality for $f(X,Y) = X^2 + XY + Y^2$.
For each $\varrho \in (0, 1/\sqrt{3}]$, there exists such a form f with $C(f) = \varrho$.

Joseph-Louis Lagrange (1736–1813)

Charles Hermite (1822–1901)

Carl Gustav Jacob Jacobi (1804–1851)

Example with $\Delta > 0$

The form

$$f(X,Y) = X^2 - XY - Y^2$$

has discriminant $\Delta(f) = 5$ and minimum $m(f) = 1$, hence

$$C(f) = \frac{m(f)}{\sqrt{\Delta(f)}} = \frac{1}{\sqrt{5}}.$$

For $\Delta > 0$, the form

$$f(X,Y) = \sqrt{\frac{\Delta}{5}}(X^2 - XY - Y^2)$$

has discriminant Δ and minimum $\sqrt{\Delta/5}$. Again

$$C(f) = \frac{1}{\sqrt{5}}.$$

Indefinite quadratic forms ($\Delta > 0$)

Assume $\Delta > 0$

A. Korkine and E.I.. Zolotarev proved in 1873 $C(f) \leq 1/\sqrt{5}$ with equality for $f_0(X,Y) = X^2 - XY - Y^2$.
For all forms which are not equivalent to f_0 under $\mathrm{GL}(2, \mathbf{Z})$, they prove $C(f) \leq 1/\sqrt{8}$.
$1/\sqrt{5} = 0,447\,213\,595\ldots$
$1/\sqrt{8} = 0,353\,553\,391\ldots$

Gap !

Egor Ivanovich Zolotarev (1847–1878)

Indefinite quadratic forms ($\Delta > 0$).

The works by Korkine and Zolotarev inspired Markoff who pursued the study of this question.
He produced infinitely many values $C(f_i)$, $i = 0, 1, \ldots$, between $1/\sqrt{5}$ and $1/3$, with the same property as f_0.
These values form a sequence which converges to $1/3$. He constructed them by means of the tree of solutions of the Markoff equation.

A. Markoff, 1879 and 1880.

Indefinite quadratic forms ($\Delta > 0$)

Assume $f((X,Y) = aX^2 + bXY + cY^2 \in \mathbf{R}[X,Y]$ with $a > 0$ has discriminant $\Delta > 0$.

If $|f(x,y)|$ is small with $y \neq 0$, then x/y is close to a root of $f(X,1)$, say α.

Then
$$|x - \alpha' y| \sim | y| \cdot |\alpha - \alpha'|$$
and $\alpha - \alpha' = \sqrt{\Delta}/a$.

Hence

$$|f(x,y)| = |a(x-\alpha y)(x - \alpha' y)| \sim y^2\sqrt{\Delta}\left|\alpha - \frac{x}{y}\right|.$$

Lagrange spectrum and Markoff spectrum

Markoff spectrum = set of values taken by

$$\frac{1}{C(f)} = \sqrt{\Delta(f)}/m(f)$$

when f runs over the set of quadratic forms $ax^2 + bxy + cy^2$ with real coefficients of discriminant $\Delta(f) = b^2 - 4ac > 0$ and $m(f) = \inf_{(x,y)\in\mathbf{Z}^2\setminus\{0\}} |f(x,y)|$.
Lagrange spectrum = set of values taken by Markoff's constant (!)

$$\lambda(x) = 1/\liminf_{q\to\infty} q(\min_{p\in\mathbf{Z}} |qx - p|)$$

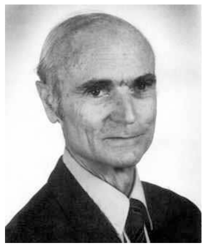
when x runs over the set of real numbers.
The Markoff spectrum contains the Lagrange spectrum.
The intersection with the intervall $[\sqrt{5}, 3[$ is the same for both of them, and is a discrete sequence.

Fuchsian groups and hyperbolic Riemann surfaces

Markoff's tree can be seen as the dual of the triangulation of the hyperbolic upper half plane by the images of the fundamental domain of the modular invariant under the action of the modular group
.

Lazarus Immanuel Fuchs
(1833–1902)

Triangulation of polygons, metric properties of polytopes

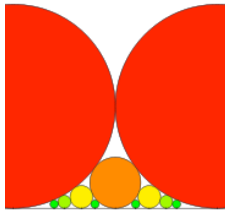
Harold Scott MacDonald Coxeter (1907–2003)	Robert Alexander Rankin (1915–2001)	John Horton Conway
		

Ford circles

The Ford circle associated to the irreducible fraction p/q is tangent to the real axis at the point p/q and has radius $1/2q^2$.

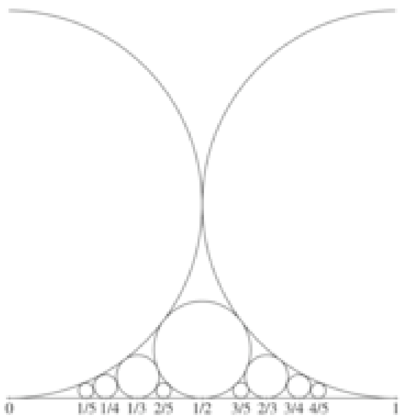
Ford circles associated to two consecutive elements in a Farey sequence are tangent.

Lester Randolph Ford (1886–1967)



Amer. Math. Monthly (1938).

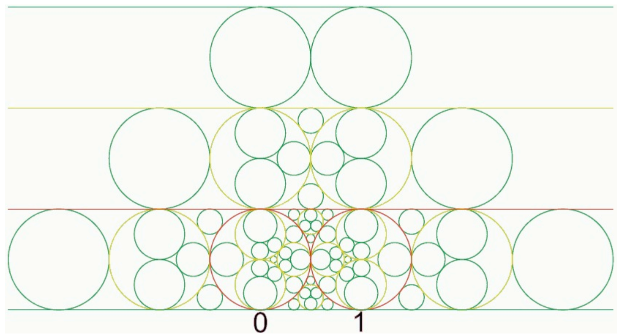
Farey sequence of order 5



$$\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}$$

Complex continued fraction

The third generation of Asmus Schmidt's complex continued fraction method.



http ://www.maa.org/editorial/mathgames/mathgames_03_15_04.html

Continued fractions and hyperbolic geometry

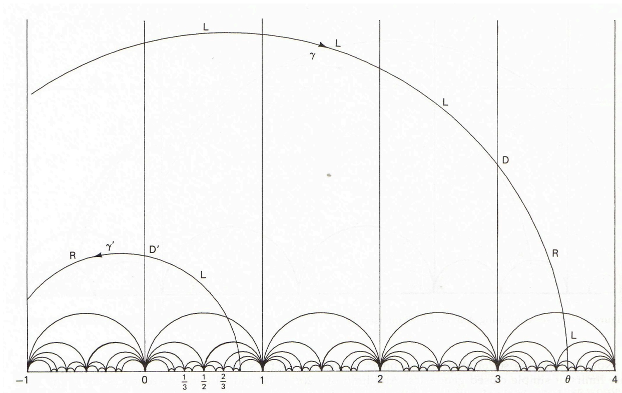


Figure 8. Reading off the continued fraction expansion of θ from 3 : θ = 3 + ...

The Geometry of Markoff Numbers

Caroline Series,
The Geometry of Markoff Numbers,
The Mathematical Intelligencer **7** N.3 (1985), 20–29.

Fricke groups

The subgroup Γ of $\mathrm{SL}_2(\mathbf{Z})$ generated by the two matrices

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

is the free group with two generators.

The Riemann surface quotient of the Poincaré upper half plane by Γ is a *punctured torus*. The minimal lengths of the closed geodesics are related to the $C(f)$, for f indefinite quadratic form.

Free groups.

Fricke proved that if A and B are two generators of Γ, then their traces satisfy

$$(\mathrm{tr}A)^2 + (\mathrm{tr}B)^2 + (\mathrm{tr}AB)^2 = (\mathrm{tr}A)(\mathrm{tr}B)(\mathrm{tr}AB)$$

Harvey Cohn showed that quadratic forms with a Markoff constant $C(f) \in]1/3, 1/\sqrt{5}]$ are equivalent to

$$cx^2 + (d-a)xy - by^2$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a generator of Γ.

Fundamental domain of a punctured disc

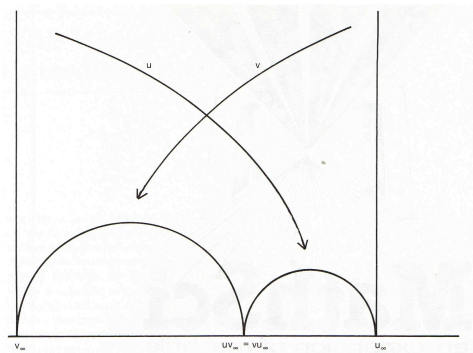


Figure 13. Fundamental region for the punctured torus.

A simple curve on a punctured disc

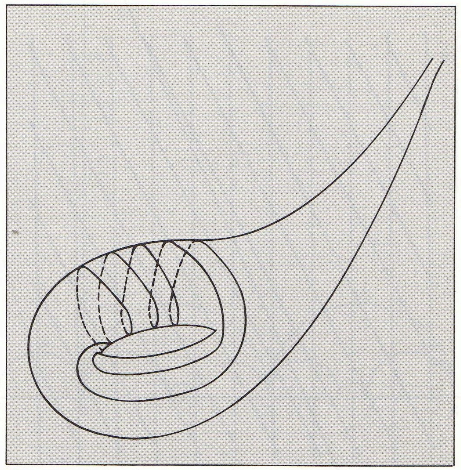


Figure 1. A simple curve on the punctured torus.

Markoff and Diophantine approximation

J.W.S. Cassels,
An introduction to Diophantine approximation,
Cambridge Univ. Press (1957)

John William Scott Cassels

International Conference on Algebra and Related Topics (ICART 2008) *May 29, 2008*

http ://www.math.sc.chula.ac.th/∼icart2008/

On the Markoff Equation

$$x^2 + y^2 + z^2 = 3xyz$$

Michel Waldschmidt

http://www.math.jussieu.fr/∼miw/