Updated: *May 28, 2010*

Diophantine approximation, irrationality and transcendence

Michel Waldschmidt

Course N°12, *May 26, 2010*

These are informal notes of my course given in April – June 2010 at IMPA (*Instituto Nacional de Matematica Pura e Aplicada*), Rio de Janeiro, Brazil.

7 Approximation of functions

We give Lambert's proof of the irrationality of π and e^r for $r \in \mathbf{Q} \setminus \{0\}$, involving continued fractions of analytic functions. Then we give a very short introduction to generalized hypergeometric functions.

7.1 Lambert's proof of the irrationality of π and e^r for $r \in \mathbf{Q} \setminus \{0\}$

The fundamental result of Lambert's paper [3] is:

Theorem 121 (Lambert, 1761)**.** *For any $r \in \mathbf{Q} \setminus \{0\}$, the numbers $\tan r$ and e^r are irrational. In particular the number π is irrational.*

The main tool is continued fractions, and the first goal of Lambert is to develop $\tan x = \sin x/\cos x$ and $(e^x - e^{-x})/(e^x + e^{-x})$ into continued fractions.

Proposition 122. *The functions $\tan x$ and $(e^x - e^{-x})/(e^x + e^{-x})$ can be represented as a continued fraction*

$$\tan x = \frac{x|}{|1} + \frac{-x^2|}{|\ 3} + \frac{-x^2|}{|\ 5} + \cdots + \frac{-x^2\ |}{|2k-1} + \cdots$$

and

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{x|}{|1} + \frac{x^2|}{|\,3} + \frac{x^2|}{|\,5} + \cdots + \frac{x^2\ \ |}{|2k-1} + \cdots$$

Each of these continued fractions converges uniformly to the function in the left hand side on any compact subset of $\mathbf{C}$ on which this function is bounded.

These two formulae are related by

$$\tan t = \frac{1}{i}\cdot\frac{e^{it}-e^{-it}}{e^{it}+e^{-it}}\cdot$$

The next tool is a criterion for irrationality, by means of such irregular continued fractions. Here is Proposition 1, § 4.3.3, of [1].

Proposition 123. *Let $(a_n)_{n\ge1}$ and $(b_n)_{n\ge1}$ be two sequences of rational integers. Assume that the continued fraction*

$$\frac{b_1|}{|a_1}+\frac{b_2|}{|a_2}+\frac{b_3|}{|a_3}+\cdots+\frac{b_n|}{|a_n}+\cdots$$

converges to some real number x. Assume also that there exists a positive integer n_0 such that, for all $n\ge n_0$, we have $0<|b_n|<|a_n|$. Then for each $n\ge1$ the continued fraction

$$\frac{b_n|}{|a_n}+\frac{b_{n+1}|}{|a_{n+1}}+\frac{b_{n+2}|}{|a_{n+2}}+\cdots+\frac{b_{n+m}|}{|a_{n+m}}+\cdots$$

converges to a limit x_n. Further, we have $|x_n|\le1$ for all $n\ge n_0$. Furthermore, if $x_n\ne\pm1$ for all $n\ge n_0$, then x is irrational.

From

$$\frac{b_1|}{|a_1}+\frac{b_2|}{|a_2}+\frac{b_3|}{|a_3}+\cdots+\frac{b_n\qquad|}{|a_n+x_{n+1}},$$

using (51), we deduce

$$x=\frac{A_{n-1}+x_nA_{n-2}}{B_{n-1}+x_nB_{n-2}}\cdot$$

This is an analog of (70) but for generalized continued fractions and with x_n replaced by $1/x_n$. Therefore, x is rational if and only if x_n is rational for at least one $n\ge1$, if and only if x_n is rational for all $n\ge1$.

We assume these two propositions and we complete the proof of the irrationality of $\tan r$ for $r\in\mathbf{Q}$ non–zero.

We shall use several times the following lemma, which means, in short terms

$$a_0+\frac{b_1|}{|a_1}+\frac{b_2|}{|a_2}+\cdots+\frac{b_n|}{|a_n}=a_0+\frac{\lambda_1b_1|}{|\lambda_1a_1}+\frac{\lambda_1\lambda_2b_2|}{|\ \lambda_2a_2}++\cdots+\frac{\lambda_{n-1}\lambda_nb_n|}{|\quad\lambda_na_n}\cdot$$

Lemma 124. *Consider a generalized finite continued fraction and define, as usual (cf. (51))*

$$\begin{pmatrix} A_n & A_{n-1} \\ B_n & B_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ b_1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{n-1} & 1 \\ b_{n-1} & 0 \end{pmatrix} \begin{pmatrix} a_n & 1 \\ b_n & 0 \end{pmatrix}.$$

Let $\lambda_1, \ldots, \lambda_n$ be further variables. Define, for $n \geq 0$, $a'_n = \lambda_n a_n$ and, for $n \geq 1$, $b'_n = \lambda_{n-1}\lambda_n b_n$, with $\lambda_0 = 1$. Then the polynomials A'_n and B'_n defined by

$$\begin{pmatrix} A'_n & A'_{n-1} \\ B'_n & B'_{n-1} \end{pmatrix} = \begin{pmatrix} a'_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a'_1 & 1 \\ b'_1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a'_{n-1} & 1 \\ b'_{n-1} & 0 \end{pmatrix} \begin{pmatrix} a'_n & 1 \\ b'_n & 0 \end{pmatrix}.$$

are

$$A'_n = \lambda_1 \cdots \lambda_n A_n \quad \text{and} \quad B'_n = \lambda_1 \cdots \lambda_n B_n.$$

In particular

$$\frac{A'_n}{B'_n} = \frac{A_n}{B_n}.$$

Proof. This is true for $n = 0$ and $n = 1$, and by induction this follows from the recurrence formulae satisfied by A_n, B_n, A'_n and B'_n:

$$A'_n = a'_n A'_{n-1} + b'_n A'_{n-2}, \qquad B'_n = a'_n B'_{n-1} + b'_n B'_{n-2}.$$

□

Proof of Lambert's irrationality result on $\tan r$ *for* $r \in \mathbf{Q} \setminus \{0\}$. Write $r = p/q$ with $q \geq 1$ and $p \neq 0$ integers. From proposition 122 we deduce

$$\tan p/q = \frac{p/q|}{|\ 1} + \frac{-p^2/q^2|}{|\quad 3} + \frac{-p^2/q^2|}{|\quad 5} + \cdots + \frac{-p^2/q^2|}{|\,2n+1} + \cdots$$

Lemma 124 with $a_0 = 0$, $a_n = 2n - 1$ for $n \geq 1$, $b_1 = p/q$, $b_n = -p^2/q^2$ for $n \geq 2$, $\lambda_n = q$ for $n \geq 1$, yields

$$\tan p/q = \frac{p|}{|q} + \frac{-p^2|}{|\ 3q} + \frac{-p^2|}{|\ 5q} + \cdots + \frac{-p^2\quad|}{|(2n+1)q} + \cdots$$

For $n > \max\{3, p^2/2q\}$, set

$$y_n = \frac{-p^2\quad|}{|(2n+1)q} + \frac{-p^2\quad|}{|(2n+3)q} + \cdots + \frac{-p^2\quad|}{|(2n+m)q} + \cdots$$

so that

$$y_n = -\frac{p^2}{(2n+1)q + y_{n-1}}.$$

One deduces from Proposition 123 that $|y_n| \leq 1$. From the estimate

$$|y_n| = \frac{p^2}{(2n+1)q - |y_{n-1}|} \leq \frac{p^2}{2nq} < 1,$$

it follows that $|y_n| < 1$. Therefore $y_n \neq \pm 1$ for all sufficiently large n, hence again we can apply Proposition 123 and conclude.

$\square$

The proof of Proposition 123 is similar to the proof of Proposition 60, the main difference being that we do not assume the numbers a_n and b_n to be positive - but here we assume the strict inequality $|a_n| > |b_n|$.

Proof of Proposition 123. We start with the following remark. *Let a, b and x be real numbers satisfying $|a| \geq | b| + 1$, $|b| \geq 1$ and $|x| < 1$. Then $a + x$ has the sign of a and*

$$\left| \frac{b}{a+x} \right| < 1.$$

When a and b are rational integers, the hypotheses on a and b hold as soon as $|a| > |b| > 0$.

From this observation and the assumption $0 < |b_n| < |a_n|$, $0 < |b_{n+1}| < |a_{n+1}|$, we deduce that for all $n \geq n_0$,

$$\frac{b_n|}{|a_n} + \frac{b_{n+1}|}{|a_{n+1}} = \frac{b_n}{a_n + \dfrac{b_{n+1}}{a_{n+1}}}$$

has the same sign as b_n/a_n and has modulus < 1. By induction, one finds that, for all $m \geq 0$,

$$\frac{b_n|}{|a_n} + \frac{b_{n+1}|}{|a_{n+1}} + \cdots + \frac{b_{n+m}|}{|a_{n+m}}$$

has the same sign as b_n/a_n and has modulus < 1. Since the continued fraction (of x, hence of x_n) is convergent, it follows that for all $n \geq n_0$, x_n has the same sign as a_{n_0}/b_{n_0} and $|x_n| \leq 1$.

Assume now that $|x_n| < 1$ for all $n \geq n_0$ and that x is rational. By induction, x_n is rational for all $n \geq 1$; write $x_n = u_n/v_n$ with $|u_n| < v_n$ for $n \geq n_0$. From $x_n = b_n/(a_n + x_{n+1})$ it follows that

$$x_{n+1} = -a_n + \frac{b_n}{x_n} = \frac{-a_n u_n + b_n v_n}{u_n}$$

is a rational number of modulus < 1 and denominator $|u_n|$ smaller than the denominator v_n of x_n. By infinite descent we reach a contradiction. $\square$

Remark. *Assume the assumptions of Proposition 123 are satisfied, but $x_n = \pm 1$ for some $n \geq n_0$. Once some x_n is rational, all x_n are rational, therefore $x_n = \pm 1$ for all sufficiently large n. Since the x_n with $n \geq n_0$ have constant sign, we have $x_n = x_{n+1}$, and from $x_n = b_n/(a_n + b_{n+1})$ with $|a_n| > |b_n| > 0$ we deduce $x_n = -1$ and $a_n = b_n - 1 \leq -2$. An example is*

$$1 = \frac{-1\,|}{|-2} + \frac{-1\,|}{|-2} + \cdots + \frac{-1\,|}{|-2} + \cdots = [0, 2, -2, 2, -2, \ldots].$$

It remains to prove Proposition 122.

Proof of Proposition 122. Lambert starts with the power series expansions of sin and cos:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

and

$$\cos x = 1 - x^2 + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

Divide sin by cos and write $\tan x = \sin x / \cos x = x/(1 + A_1)$. The power series A_1 starts with $-x^2/3$. Next write $A_1 = -x^2/(3 + A_2)$, so that

$$\tan x = \frac{x}{1 + A_1} = \frac{x}{1 + \dfrac{-x^2}{3 + A_2}}\cdot$$

The first term of A_2 is $-x^2/5$. For $A_2 = -x^2/(5 + A_3)$ we have

$$\tan x = \frac{x}{1 + \dfrac{-x^2}{3 + \dfrac{-x^2}{5 + A_3}}} = \frac{x|}{|1} + \frac{-x^2|}{|\;3} + \frac{-x^2\;\;|}{|5 + A_3}\cdot$$

The closed formulae for A_1, A_2 and A_3 are given in [1]. Here is the formula for A_k which is computed from

$$\tan x = \frac{x|}{|1} + \frac{-x^2|}{|\;3} + \frac{-x^2|}{|\;5} + \cdots + \frac{-x^2\qquad|}{|2k - 1 + A_k},$$

namely

$$A_k = \frac{\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}x^{2n+2}\frac{(2n+2)(2n+4)\cdots(2n+2k)}{(2n+2k+1)!}}{\displaystyle\sum_{n=0}^{\infty}(-1)^{n}x^{2n}\frac{(2n+2)(2n+4)\cdots(2n+2k-2)}{(2n+2k-1)!}}.$$

One can write also the coefficients respectively

$$\frac{(2n+2)(2n+4)\cdots(2n+2k)}{(2n+2k+1)!} = \frac{2^k(n+k)!}{n!(2n+2k+1)!}$$

and

$$\frac{(2n+2)(2n+4)\cdots(2n+2k-2)}{(2n+2k-1)!} = \frac{2^{k-1}(n+k-1)!}{n!(2n+2k-1)!}.$$

The proof of the convergence of the continued fraction requires to compute the convergents, which is something done by Lambert. He writes

$$\frac{x|}{|1} + \frac{-x^2|}{|\ 3} + \frac{-x^2|}{5} + \cdots + \frac{-x^2\ \ |}{|2n-1} = \frac{P_n}{Q_n}$$

where

$$P_{n+1} = (2n+1)P_n - x^2P_{n-1}, \qquad Q_{n+1} = (2n+1)Q_n - x^2Q_{n-1}$$

for $n \geq 2$, with the initial conditions $P_1 = x$, $Q_1 = 1$, $P_2 = 3x$, $Q_2 = 3 - x^2$. By induction, it follows that the polynomial P_n is odd, of degree n if n is odd and $n-1$ is n is even, while Q_n is an even polynomial, of degree n if n is even and $n-1$ is n is odd. The explicit formulae are

$$P_n = c_np_n, \quad Q_n = c_nq_n, \quad c_n = 1\cdot 3\cdot 5\cdots(2n-1) = \frac{(2n)!}{2^nn!},$$

with

$$p_n = \sum_{1\leq k\leq (n+1)/2}(-1)^{k-1}\frac{x^{2k-1}}{(2k-1)!}\cdot\frac{(2n-2k)(2n-2k-2)\cdots(2n-4k+4)}{(2n-1)(2n-3)\cdots(2n-2k+3)}$$

and

$$q_n = \sum_{0\leq k\leq n/2}(-1)^{k}\frac{x^{2k}}{(2k)!}\cdot\frac{(2n-2k)(2n-2k-2)\cdots(2n-4k+2)}{(2n-1)(2n-3)\cdots(2n-2k+1)}.$$

As n tends to infinity, p_n and q_n converge uniformly on any compact subset of $\mathbf{C}$ to sin and cos: the difference between the sums of the first k terms in the Taylor expansion at the origin of p_n and sin (respectively of q_n and cos) is bounded above by

$$\frac{|x|^{2k+1}}{(2k+1)!} + \frac{|x|^{2k+2}}{(2k+2)!} + \frac{|x|^{2k+3}}{(2k+3)!} + \cdots$$

and therefore p_n/q_n converge to $\tan x$ uniformly on any compact subset of $\mathbf{C}$ where $|\tan x|$ is bounded.

$\square$

Remark. *In the proof of Theorem 121, we may replace the Lambert's irrationality criterion (Proposition 123) for continued fractions by our standard criterion (Proposition 4) involving rational approximations, as follows.*

Writing the function $f(z) = (1/z)\tan z$ as a continued fraction and using (54), we obtain, for $n > 0$,

$$f(z) = \frac{P_n(z)}{Q_n(z)} + \sum_{m>n}\left(\frac{P_m(z)}{Q_m(z)} - \frac{P_{m+1}(z)}{Q_{m+1}(z)}\right) = \frac{P_n(z)}{Q_n(z)} + \sum_{m\geq n}\frac{z^{2m}}{Q_{m-1}Q_m(z)}\cdot$$

The polynomials P_n and Q_n have integral coefficients and degrees $\leq n$; for n tending to infinity, $Q_n(p/q)$ grows like $2^n n!$. One checks that the rational approximation given by $P_n(p/q)/Q_n(p/q)$ is too sharp for $f(p/q)$ to be a rational number.

From Lemma 124, it follows that the continued fraction for $(e^x - e^{-x})/(e^x + e^{-x})$ given in Proposition 122 can be written

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = [0, 1/x, 3/x, 5/x, \ldots, (2k-1)/x, \ldots].$$

For $x = 1/2$ this gives

$$\frac{e+1}{e-1} = [2,\ 6,\ 10,\ 14,\ \ldots,\ 4k+2,\ \ldots]. = [\overline{4k+2}]_{k\geq 0}.$$

Let us deduce Euler's continued fraction expansion for e (see § 1.4)

$$e = [2,\ 1,\ 2,\ 1,\ 1,\ 4,\ 1,\ 1\ldots] = [2, \overline{1, 2k, 1}]_{k\geq 1}.$$

Define p_k/q_k as the k–th convergent of $x = [2, 6, \ldots, 4k+2, \ldots]$ and r_k/s_k as the k–th convergent of $y = [1, 1, 2, 1, 1, 4, \ldots, 1, 2k, 1, \ldots]$. We eliminate the indices which are not congruent to 1 modulo 3 among the 5 relations involving 7 symbols

$$\begin{aligned} r_{3k-3} &= r_{3k-4} + r_{3k-5}, \\ r_{3k-2} &= r_{3k-3} + r_{3k-4}, \\ r_{3k-1} &= 2kr_{3k-2} + r_{3k-3}, \\ r_{3k} &= r_{3k-1} + r_{3k-2}, \\ r_{3k+1} &= r_{3k} + r_{3k-1} \end{aligned}$$

and deduce

$$r_{3k+1} = (4k+2)r_{3k-2} + r_{3k-5}.$$

We do the same for s_k and get

$$\begin{pmatrix} r_{3k+1} & r_{3k-2} \\ s_{3k+1} & s_{3k-2} \end{pmatrix} = \begin{pmatrix} r_{3k-2} & r_{3k-5} \\ s_{3k-2} & s_{3k-5} \end{pmatrix} \begin{pmatrix} 4k+2 & 1 \\ 1 & 0 \end{pmatrix}.$$

These are the same recurrence relations which are satisfied by p_k and q_k. Since

$$p_{-2} = 0, \quad p_{-1} = 1, \quad p_0 = 2, \qquad q_{-2} = 1, \quad q_{-1} = 0, \quad q_0 = 1$$

and

$$r_{-2} = 0 = 2q_{-1}, \quad r_1 = 2 = 2q_0, \qquad s_{-2} = 1 = p_{-1} - q_{-1}, \quad s_1 = 1 = p_0 - q_0,$$

we deduce $r_{3k+1} = 2q_k$ and $s_{3k+1} = p_k - q_k$ for all k. From $y = \lim_{k\to\infty} r_{3k}/s_{3k}$ we deduce $y = 2/(x-1)$. Since $x = (e+1)/(e-1)$, we get $y = e - 1$.

The same argument starting from

$$\frac{e^2+1}{e^2-1} = [\overline{2j+1}]_{j\geq 0} = [1;\ 3,\ 5,\ 7, \ldots],$$

yields Euler's continued fraction expansion for e^2 (see § 1.4)

$$e^2 = [7;\ \overline{3j-1,\ 1,\ 1,\ 3j,\ 12j+6}]_{j\geq 1} = [7;\ 2,\ 1,\ 1,\ 3,\ 18,\ 5,\ 1,\ 1,\ 6,\ 30,\ 8, \ldots],$$

7.2 Hypergeometric functions

A (generalized) *hypergeometric series* is a power series

$$1 + \alpha_1 z + \alpha_2 z^2/2 + \cdots + \alpha_n z^n/n! + \cdots$$

such that there exists a rational fraction $A \in \mathbf{C}(T)$ satisfying, for all $n \geq 0$,

$$\alpha_{n+1} = \alpha_n A(n).$$

Write this rational fraction as

$$A(T) = c\frac{(a_1+T)\cdots(a_p+T)}{(b_1+T)\cdots(b_q+T)}.$$

We assume that A has no pole on $\mathbf{Z}_{\geq 0}$, which means $b_j \notin \mathbf{Z}_{\leq 0}$ for $1 \leq j \leq q$, so that $A(n)$ is defined for all $n \geq 0$. Then

$$\alpha_{n+1} = c\frac{(a_1+n)\cdots(a_p+n)}{(b_1+n)\cdots(b_q+n)}\alpha_n$$

and therefore

$$\alpha_n = c^n\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n},$$

where $(a)_n$ denotes the *Pochhammer symbol*

$$(a)_n = a(a+1)\cdots(a+n-1) \quad \text{for } n \geq 1 \text{ and } (a)_0 = 1.$$

It is also called *raising factorial*: notice that $(1)_n = n!$ and satisfies an number of relations, among which

$$(a)_{k+m} = (a)_k(a+k)_m.$$

For each $n \geq 0$, we have

$$\lim_{a\to\infty}\frac{(a)_n}{a^n} = 1$$

and for each $a \in \mathbf{C} \setminus \mathbf{Z}_{<0}$, we have

$$\lim_{n\to\infty}\frac{(a)_n}{n!} = 1.$$

For p and q non–negative integers, we define

$$ {}_pF_q\left(\begin{matrix} a_1 & a_2 & \cdots & a_p \\ b_1 & b_2 & \cdots & b_q \end{matrix}\middle| z\right) = \sum_{n\geq 0}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\cdot\frac{z^n}{n!}.$$

We shall use also the notation

$$ {}_pF_q\big(a_1, a_2, \cdots, a_p; b_1, b_2, \cdots, b_q; z\big).$$

In the case where some a_i is in $\mathbf{Z}_{\le 0}$, then ${}_pF_q$ is a polynomial. Otherwise, this power series has a radius of convergence which is infinite when $q \ge p$, finite if $q = p-1$, and 0 if $q < p-1$.

For $a_p = b_q = c$ we have

$$ {}_pF_q\left(\begin{matrix} a_1 & a_2 & \cdots & a_{p-1} & c \\ b_1 & b_2 & \cdots & b_{q-1} & c \end{matrix}\middle| z\right) = {}_{p-1}F_{q-1}\left(\begin{matrix} a_1 & a_2 & \cdots & a_{p-1} \\ b_1 & b_2 & \cdots & b_{q-1} \end{matrix}\middle| z\right) $$

Examples. The basic example is ${}_0F_0(z) = e^z$. Other examples are

$$ {}_1F_0\bigl(a;z\bigr) = \sum_{n\ge 0} \frac{a(a+1)\cdots(a+n-1)}{n!} \cdot z^n = (1-z)^{-a} $$

and

$$ {}_2F_1\bigl(1,1;2;z\bigr) = \sum_{n\ge 0} \frac{z^n}{n+1} = -\frac{1}{z}\log(1-z). $$

We consider the special case $p=0$, $q=1$ of Gauss hypergeometric series:

$$ {}_0F_1(c;z) = \sum_{n\ge 0} \frac{z^n}{(c)_n n!}. $$

We denote this function by $f(c;z)$.

Since

$$ \left(\frac{1}{2}\right)_n = \left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right)\cdots\left(\frac{1}{2}+n-1\right) = \frac{(2n)!}{2^{2n}n!} $$

and

$$ \left(\frac{3}{2}\right)_n = \left(\frac{3}{2}\right)\left(\frac{3}{2}+1\right)\cdots\left(\frac{3}{2}+n-1\right) = \frac{(2n+1)!}{2^{2n}n!}, $$

special cases are

$$ f(1/2;z^2) = \sum_{n\ge 0} \frac{(2z)^{2n}}{(2n)!} = \cosh(2z) $$

and

$$ f(3/2;z^2) = \sum_{n\ge 0} \frac{(2z)^{2n}}{(2n+1)!} = \frac{1}{2z}\sinh(2z). $$

From

$$ \frac{1}{(c)_n} = \frac{c+n}{(c)_{n+1}} = \frac{1}{(c+1)_n} + \frac{n}{(c)_{n+1}} $$

one deduces

$$f(c;z) = \sum_{n\geq 0} \frac{z^n}{(c+1)_n n!} + \sum_{n\geq 1} \frac{nz^n}{(c)_{n+1} n!}.$$

The first series is $f(c+1;z)$, the second is

$$\sum_{n\geq 0} \frac{z^{n+1}}{(c)_{n+2} n!} = \frac{z}{c(c+1)} \sum_{n\geq 0} \frac{z^n}{(c+2)_n n!} = \frac{z}{c(c+1)} f(c+2;z).$$

This is the functional equation relating $f(c;z)$, $f(c+1;z)$ and $f(c+2;z)$:

$$f(c;z) = f(c+1;z) + \frac{z}{c(c+1)} f(c+2;z).$$

Hence the function $g(c;z) = f(c;z)/f(c+1;z)$ satisfies

$$g(c,z) = 1 + \frac{z}{c(c+1)} \cdot \frac{1}{g(c+1;z)}.$$

Next define $h(c;z) = (c/z)g(c;z^2)$: we get

$$h(c;z) = \frac{c}{z} + \frac{1}{h(c+1;z)}.$$

Therefore, for $k \geq 1$,

$$h(c;z) = \left[\frac{c}{z}, \frac{c+1}{z}, \ldots, \frac{c+k-1}{z}, h(c+k;z)\right].$$

Replacing h by its value in terms of f yields

$$\frac{c}{z} \cdot \frac{f(c;z^2)}{f(c+1;z^2)} = \left[\frac{c}{z}, \frac{c+1}{z}, \ldots, \frac{c+k-1}{z}, \frac{(c+k)}{z} \cdot \frac{f(c+k;z^2)}{f(c+k+1;z^2)}\right].$$

We now take the limit on k:

Lemma 125. *For c and z positive real numbers, the infinite continued fraction converges and we have*

$$\frac{c}{z} \cdot \frac{f(c;z^2)}{f(c+1;z^2)} = \left[\frac{c}{z}, \frac{c+1}{z}, \ldots, \frac{c+k}{z}, \cdots\right].$$

Proof. We first check the following auxiliary result:

Let $(a_n)_{n\geq 0}$ be a sequence of real numbers, all ≥ 1. Let x be a real number. Assume that for all $n \geq 1$, there exists a real number $x_n \geq 1$ such that

$$x = [a_0, a_1, \ldots, a_{n-1}, x_n].$$

Then the infinite continued fraction $[a_0, a_1, \ldots, a_n, \ldots]$ converges to x.

We already proved this result when the a_n are integers, the proof in the general case is the same: we write

$$[a_0, a_1, \ldots, a_n] = \frac{A_n}{B_n}$$

with $A_n = a_n A_{n-1} + A_{n-2}$, $B_n = a_n B_{n-1} + B_{n-2}$, so that

$$x = \frac{x_{n+1}A_n + A_{n-1}}{x_{n+1}B_n + B_{n-1}},$$

we note that $B_n \geq B_{n-1} + B_{n-2}$, which implies that B_n tends to infinity, and we conclude with the estimate

$$\left| x - \frac{A_n}{B_n} \right| = \frac{1}{B_n(x_{n+1}B_n + B_{n-1})} \leq \frac{1}{B_n^2}.$$

To complete the proof of Lemma 125, we notice that for c and z positive, we have

$$f(c+k+1; z^2) < f(c+k; z^2) \quad \text{and} \quad \frac{c+k}{z} \geq 1$$

for sufficiently large k.

□

In the special cases $c = 1/2$, this provides another proof of the continued fraction expansion from Proposition 122:

$$\frac{e^z - e^{-z}}{e^z + e^{-z}} = [0, 1/z, 3/z, \ldots, (2k-1)/z, \ldots] = \frac{z|}{|1} + \frac{z^2|}{|\ 3} + \frac{z^2|}{|\ 5} + \cdots + \frac{z^2 \quad |}{|2k-1} + \cdots$$

References

[1] P. Eymard and J.-P. Lafon, *The number π*, American Mathematical Society, Providence, RI, 2004. Translated from the 1999 French original by Stephen S. Wilson.

[2] C. Hermite, *Sur la fonction exponentielle*, C. R. Acad. Sci. Paris, 77 (1873), pp. 18–24, 74–79, 226–233, 285–293. Œuvres de Charles Hermite, Paris: Gauthier-Villars, (1905), III, 150–181. See also *Oeuvres* III, 127–130, 146–149, and *Correspondance Hermite-Stieltjes*, II, lettre 363, 291–295. University of Michigan Historical Math Collection
`http://name.umdl.umich.edu/AAS7821.0001.001`.

[3] H. Lambert, *Mémoire sur quelques propriétés remarquables des quantités transcendantes circulaires et logarithmiques.* Mémoires de l'Académie des Sciences de Berlin, 17 (1761), 1768, p. 265–322; lu en 1767; Math. Werke, t. II., 1767.

[4] S. Shirali, *Continued fraction for e.* Resonance, vol. **5** N°1, Jan. 2000, 14–28.
`http://www.ias.ac.in/resonance/`.