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Diophantine approximation, irrationality and transcendence

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9.2 Integer valued entire functions

We have seen in § 9.1 that there is no hope to prove a general transcendence theorem on the values of entire functions. One needs to be less ambitious, and the most natural thing to do is to put restrictions on the functions. For instance the functions produced in § 9.1 with large exceptional sets do not satisfy differential equations (more precisely, as we have seen, it is possible to produce such functions which do not satisfy differential equations – it is another challenge to prove that none of them satisfies a differential equation!). We shall see, with the Schneider–Lang Theorem, that general transcendence results can be proved for functions satisfying some differential equations.

However, one of the earliest progresses in the theory came from adding restrictions not on the functions, but on the numbers. We were considering in § 9.1 algebraic values of transcendental functions at algebraic points. A much more restricted question is to investigate integer values at integral points. This is the story that we are telling now. We even start with a more specific topic by looking at zero values. Next we consider Pólya's pioneer work on integer valued entire functions, we pursue with Gel'fond's extension to Gaussian integers, and then with his proof of the transcendence of e^{π}.

When f is a complex function which is bounded on a disc $|z| \le r$, we set

$$|f|_r = \sup_{|z|=r} |f(z)|.$$

9.2.1 Weierstraß canonical products

Recall that if f is an analytic function on a simply connected open subset D of $\mathbf{C}$ without zero in D, then there exists a analytic function g in D such that $f = e^g$. If f has only finitely many zeros, then $f(z) = A(z)e^{g(z)}$, where A is a polynomial (having the same zeros as f) and g is analytic in D. We are interested in having a similar decomposition when f has infinitely many zeroes - recall that if f is not the zero function, then the zeroes are isolated.

We assume $D = \mathbf{C}$ (hence f is an *entire* function). Its zeros form a discrete set, one can order them by non–decreasing modulus: let $(\alpha_0, \alpha_1, \ldots)$ be this sequence of zeros of f, counting multiplicities. It will be convenient to assume $f(0) \neq 0$, hence $|\alpha_0| > 0$.

We further assume that f has finite order of growth ϱ, namely (cf [10] Chap. X § 3):

$$\varrho = \limsup_{r\to\infty} \frac{\log\log|f|_r}{\log r}.$$

Recall the Taylor expansion at the origin of $\log(1-z)$:

$$\log(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \cdots \frac{z^m}{m} - \cdots$$

For $m \geq 0$, one defines the *Weierstraß factor* ([10] Chap. X § 2) as:

$$E(z, m) = (1-z)e^{z+z^2/2+z^3/3+\cdots+z^m/m};$$

in particular $E(z, 0) = 1 - z$. This function is very close to 1 (especially when m is large) for $|z|$ not too large: according to [10] Chap. X § 2 Lemma 2.2, for $|z| \leq 1/2$ one has $|\log E(z, m)| \leq 2|z|^{m+1}$.

A classical result (see [10] Chap. X § 3 Th. 3.5) is that there exist an integer $m \leq \varrho$ and a polynomial P of degree $\leq \varrho$ such that

$$f(z) = e^{P(z)} \prod_{n\geq 0} E(z/\alpha_n, m).$$

The integer m is the integral part of ϱ if ϱ is not an integer, it is ϱ or $\varrho - 1$ if ϱ is an integer.

Conversely, given a discrete sequence of non–zero complex numbers $(\alpha_n)_{n\geq 0}$, ordered with non–decreasing modulus, there exists a sequence of non–negative numbers $(m_n)_{n\geq 0}$ such that the product

$$\prod_{n\geq 0} E(z/\alpha_n, m_n)$$

is normally convergent over any compact subset of $\mathbf{C}$ (see [16] Chap. VII § 7.6 and [10] Chap. X § 2 Th. 2.3). When this property is true for a constant sequence $m_n = m$, $(n \geq 0)$, and when m is the smallest integer such that the product

$$\prod_{n\geq 0} E(z/\alpha_n, m)$$

is convergent, then this product is called the *canonical product of Weierstraß associated with the sequence* $(\alpha_n)_{n\geq 0}$.

Examples.

• (See [16] Chap. XII and [10] Chap. XII § 2).

The canonical product of Weierstraß associated with the non–negative integers $\mathbf{Z}_{\geq 0}$ is

$$z\prod_{n\geq 1}\left(1-\frac{z}{n}\right)e^{z/n} = -\frac{e^{\gamma z}}{\Gamma(-z)}.$$

• (see [16] Chap. XII § 12.4 and [10] Chap. X § 2).

The canonical product of Weierstraß associated with the rational integers $\mathbf{Z}$, is

$$z\prod_{n\in\mathbf{Z}\setminus\{0\}}\left(1-\frac{z}{n}\right)e^{z/n} = \pi^{-1}\sin(\pi z) = \frac{-z}{\Gamma(z)\Gamma(1-z)}.$$

• (see [16] Chap. XX and [10] Chap. XI § 4). and [1, 9, 14]). Let $\Omega = \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$ be a lattice in $\mathbf{C}$. The *Weierstraß canonical product* attached to Ω is the *Weierstraß sigma function* σ_Ω defined by

$$\sigma_\Omega(z) = z\prod_{\omega\in\Omega\setminus\{0\}}\left(1-\frac{z}{\omega}\right)e^{\frac{z}{\omega}+\frac{z^2}{2\omega^2}}.$$

Exercise 11. *Show that the function*

$$g(z) = \sum_{n\geq 0}(-1)^n\frac{\pi^{2n}}{2^{2n}(2n)!}z^n$$

has the infinite product expansion

$$g(z) = \prod_{n\in\mathbf{Z}}\left(1-\frac{z}{(2n+1)^2}\right).$$

`Hint:` *Check* $g(t^2) = \cos(\pi t/2)$.

An entire function f is said to be of finite exponential type if the number

$$\alpha = \limsup_{r\to\infty}\frac{\log|f|_r}{r}$$

is finite. In this case f is said to be of exponential type α. Notice that a function of finite exponential type has order ≤ 1; if the order is < 1, then the type α is zero.

Lemma 151. *A function of exponential type < 1 which vanishes for all $n = 0, 1, \ldots$ is the zero function.*

The proof relies on the following auxiliary result:

Lemma 152 (Jensen's Formula)**.** *If g is an analytic function in an open set containing the closed disk disc $|z| \leq r$ with zeros $(a_j)_{1\leq j\leq k}$ in this disc and if $g(0) \neq 0$, then*

$$\log |g(0)| + \sum_{j=1}^{k} \log \frac{r}{|a_j|} = \frac{1}{2\pi} \int_0^{2\pi} \log |g(re^{i\theta})| d\theta.$$

Sketch of proof of Jensen's Formula 152. Assume first that g has no zero in the closed disk disc $|z| \leq r$. Then there is an open disk containing this closed disk, where g has no zero, and therefore there is an analytic function h in a neighborhood of the disc $|z| \leq r$ such that $g = e^h$. Since $|g| = e^{\Re e h}$, the formula follows by taking the real part of

$$h(0) = \frac{1}{2i\pi} \int_{|z|=r} h(z) \frac{dz}{z} = \frac{1}{2\pi} \int_0^{2\pi} h(re^{i\theta}) d\theta.$$

In the general case, one can write $g(z) = (z - a_1) \cdots (z - a_k)e^{h(z)}$, where h is analytic. By multiplicativity of both sides of the conclusion of Lemma 152, the formula reduces to the following one: *for any complex number α,*

$$\int_0^1 \log |e^{2i\pi t} - \alpha| dt = \log \max\{1, |\alpha|\}.$$

(See for instance [11], pp. 5–6, or [10] Chap. IX Th. 1.3).

□

Proof of Lemma 151. Assume f is not the zero function and vanishes at all the non–negative integers $n = 0, 1, \ldots$ Since the zeroes of f are isolated, there exists $z_0 \in (0, 1)$ such that $f(z_0) \neq 0$. Use Jensen's formula 152 for the function $g(z) = f(z_0 + z)$ with $r = N - z_0$, where N is a large integer. The set of zeroes of g in the disc $|z| \leq r$ contains the elements $n - z_0$, $1 \leq n \leq N - z_0$. For $1 \leq n \leq N - z_0$ we have $(N - z_0)/(n - z_0) \geq N/n$. For the other zeros we use the trivial estimate $\log(r/|a_j|) \geq 0$. Also $|g|_r \leq |f|_N$.

We deduce an upper bound of the right hand side of Jensen's Formula by using the assumption: there exists $c > 0$ and $\lambda < 1$ such that $|f|_N \le ce^{\lambda N}$:

$$\frac{1}{2\pi}\int_0^{2\pi} \log|g(re^{i\theta})|d\theta \le \log|g|_r \le \log|f|_N \le \lambda N + \log c.$$

Hence

$$\log|f(z_0)| \le \lambda N - \sum_{n=1}^{N} \log(N/n) + \log c = \lambda N - N\log N + \log N! + \log c.$$

Since $\lambda < 1$, it follows from Stirling's formula:

$$N! \simeq N^N e^{-N}\sqrt{2\pi N} \tag{153}$$

that $\lambda N - N\log N + \log N!$ tends to $-\infty$ as N tends to infinity, which contradicts $f(z_0) \ne 0$. □

Remark on Jensen's Formula. In many situations, one can replace Jensen's formula (Lemma 152) by Schwarz's Lemma (see § 10.4), which gives an upper bound for $|f|_r$ when f has N zeroes (counting multiplicities) in $|z| \le r$: for $R > r$ one has

$$|f|_r \le \left(\frac{R^2 + r^2}{2rR}\right)^{-N} |f|_R. \tag{154}$$

However, here, it would give a weaker result: in order to reach the conclusion of Lemma 151, using (154), one needs to assume that f has exponential type $\le \gamma$ where γ satisfies

$$\gamma < \sup_{\lambda > 1} \frac{1}{\lambda} \log\left(\frac{\lambda^2 + 1}{2\lambda}\right) < \frac{1}{5}.$$

Remark on Stirling's Formula (153). We needed only a weak form of Stirling's formula. Asymptotic expansions (see the definition in Chap. VIII of [16]) for the logarithm of the Gamma function are known:

$$\log\Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) - \int_0^{+\infty} \frac{P_1(t)}{z+t}dt$$

for

$$-\pi + \delta < \arg z < \pi + \delta \quad \text{with} \quad 0 < \delta < \pi,$$

where $P_1(t) = t - \lfloor t \rfloor - 1/2$. Denote by $(B_n)_{n\ge 0}$ the sequence of Bernoulli numbers, which are defined by ([16] § 7.1)

$$\frac{x}{e^x - 1} = \sum_{n\ge 0} B_n \frac{x^n}{n!}\cdot$$

The first non–zero values are

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},\ B_8 = -\frac{1}{30},\ B_{10} = \frac{5}{66}\cdot$$

For z with argument $\le (\pi/2) - \delta$ with $\delta > 0$, we have (see Chap. XII § 12.33 of [16]):

$$\log\Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) + \frac{B_2}{1\cdot 2\cdot z} + \frac{B_4}{3\cdot 4\cdot z^2} + \frac{B_6}{5\cdot 6\cdot z^3} + \cdots$$

9.2.2 Pólya and 2^z

Satz I in [12] is the following result.

Theorem 155 (Pólya)**.** *If an entire function f satisfies $f(n) \in \mathbf{Z}$ for all $n = 0, 1, \dots,$ and*

$$\lim_{r\to\infty} \frac{r^{1/2}|f|_r}{2^r} = 0,$$

then f is a polynomial.

A consequence of Pólya's Theorem 155 is that an entire function of exponential type $< \log 2$ is a polynomial. In loose terms, it means that the function 2^z is the transcendental function mapping $\mathbf{Z}_{\ge 0}$ to $\mathbf{Z}$ which grows the least rapidly.

In his 1929 paper [12], Pólya also considered entire functions mapping $\mathbf{Z}$ to $\mathbf{Z}$: he proved that the smallest such transcendental function is

$$\frac{1}{\sqrt{5}}\left(\left(\frac{3+\sqrt{5}}{2}\right)^z - \left(\frac{3-\sqrt{5}}{2}\right)^z\right).$$

After Pólya's work, a number of papers have been written on the subject. In particular Ch. Pisot used the Laplace–Borel transform to prove that an entire function mapping $\mathbf{Z}_{\ge 0}$ to $\mathbf{Z}$ of exponential type $\le \log 2 = 0.69314718\ldots$ is of the form $A(z) + B(z)2^z$, where A and B are polynomials. See [6, 7].

Pólya's proof involves the calculus of finite differences [4] which we now introduce.

9.2.3 Calculus of finite differences

Given a function f and points $x_0, x_1, \ldots, x_m$ where f is analytic, one defines inductively analytic functions $f_1, f_2, \ldots$ as follows:

$$f_1(z) = \frac{f(z) - f(z_0)}{z - z_0}, \quad f_2(z) = \frac{f_1(z) - f_1(z_1)}{z - z_1}, \quad f_3(z) = \frac{f_2(z) - f_2(z_2)}{z - z_2}, \ldots$$

so that

$$\begin{aligned} f(z) &= f(z_0) + (z - z_0) f_1(z), \\ f_1(z) &= f_1(z_1) + (z - z_1) f_2(z), \\ f_2(z) &= f_2(z_2) + (z - z_2) f_3(z), \ldots \end{aligned}$$

This gives the expansion

$$\begin{aligned} f(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)(z - z_1) + c_3(z - z_0)(z - z_1)(z - z_2) + \cdots \\ + c_m(z - z_0)(z - z_1) \cdots (z - z_{m-1}) + (z - z_0)(z - z_1) \cdots (z - z_m) f_{m+1}(z) \end{aligned}$$

with $c_0 = f(z_0)$, $c_1 = f_1(z_1)$, $\ldots$, $c_m = f_m(z_m)$.

Here is a first set of formulae for the coefficients $c_0, c_1, \ldots c_m$. For simplicity we assume that the points $x_0, x_1, \ldots, x_m$ are pairwise distinct. Define first

$$[x_0] = f(x_0), \ [x_1] = f(x_1), \ \ldots, [x_m] = f(x_m),$$

and next set

$$[x_0, x_1] = \frac{[x_0] - [x_1]}{x_0 - x_1}, \ [x_1, x_2] = \frac{[x_1] - [x_2]}{x_1 - x_2}, \ldots, [x_{m-1}, x_m] = \frac{[x_{m-1}] - [x_m]}{x_{m-1} - x_m},$$

$$[x_0, x_1, x_2] = \frac{[x_0, x_1] - [x_1, x_2]}{x_0 - x_2}, \ [x_1, x_2, x_3] = \frac{[x_1, x_2] - [x_2, x_3]}{x_1 - x_3}, \ldots,$$

and so on, up to

$$[x_0, x_1, \ldots, x_m] = \frac{[x_0, x_1, \ldots, x_{m-1}] - [x_1, x_2, \ldots, x_m]}{x_0 - x_m}.$$

Then

$$c_0 = [x_0], \quad c_1 = [x_0, x_1], \quad \ldots, \quad c_m = [x_0, \ldots, x_m].$$

We now explain another way of getting such an expansion, by means of an identity due to Ch. Hermite (see [13]):

$$\frac{1}{x - z} = \frac{1}{x - x_0} + \frac{z - x_0}{x - x_0} \cdot \frac{1}{x - z}.$$

We replace the last factor $1/(x-z)$ by repeating the same formula with x_0 replaced by x_1:

$$\frac{1}{x-z} = \frac{1}{x-x_0} + \frac{z-x_0}{x-x_0} \cdot \left(\frac{1}{x-x_1} + \frac{z-x_1}{x-x_1} \cdot \frac{1}{x-z} \right) \cdot$$

Inductively we deduce

$$\frac{1}{x-z} = \sum_{j=0}^{m} \frac{(z-x_0)(z-x_1)\cdots(z-x_{j-1})}{(x-x_0)(x-x_1)\cdots(x-x_j)} \\ + \frac{(z-x_0)(z-x_1)\cdots(z-x_m)}{(x-x_0)(x-x_1)\cdots(x-x_m)} \cdot \frac{1}{x-z} \cdot$$

Now we multiply by $(1/2i\pi)f(x)$ and integrate along a simple contour $\mathcal{C}$ which contains all the x_i as well as z: this produces *Newton interpolation expansion*

$$f(z) = \sum_{j=0}^{m} c_j (z-x_0)\cdots(z-x_{j-1}) + R_m(z)$$

with

$$c_j = \frac{1}{2i\pi} \int_{\mathcal{C}} \frac{f(x)dx}{(x-x_0)(x-x_1)\cdots(x-x_j)} \quad (0 \le j \le m-1)$$

and

$$R_m(z) = (z-x_0)(z-x_1)\cdots(z-x_m) \cdot \frac{1}{2i\pi} \int_{\mathcal{C}} \frac{f(x)dx}{(x-x_0)(x-x_1)\cdots(x-x_m)(x-z)} \cdot$$

Similar formulae exist when the points x_i are not distinct: when one repeats m times the same x_i, one considers the values $f^{(s)}(x_i)$ of the successive derivatives of f at x_i, for $s = 0, \ldots, m-1$. See § 9.2.8 and [10] Chap. IX § 2.

9.2.4 Proof of Pólya's Theorem

Proof. The Newton's interpolation series introduced in § 9.2.3 associated with the function f and the points $x_j = j$ for $j \ge 0$ is the formal series

$$F(z) = \sum_{n \ge 0} c_n z(z-1)\cdots(z-n+1),$$

where, for $n \geq 0$,

$$c_n = \sum_{k=0}^{n} \frac{f(k)}{\prod\limits_{\substack{0 \leq j \leq n \\ j \neq k}} (k-j)} \cdot$$

Since

$$\prod_{\substack{0 \leq j \leq n \\ j \neq k}} (k-j) = k(k-1) \cdots 2 \cdot 1 \cdot (-1)(-2) \cdots (k-n) = (-1)^{n-k} k!(n-k)!,$$

we deduce

$$c_n = \frac{1}{n!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(k).$$

Hence c_n is a rational number and more precisely $n!c_n$ is a rational integer. We are going to prove that c_n vanishes for sufficiently large n. In order to do so, we produce an upper bound for $|c_n|$ by using the hypothesis of Theorem 155, namely

$$|f|_r = \epsilon(r) r^{-1/2} 2^r$$

where $\epsilon(r) \to 0$ as $r \to \infty$. From the integral formula

$$c_n = \frac{1}{2i\pi} \int_{|z|=r_n} \frac{f(z)dz}{z(z-1)\cdots(z-n)}$$

which is valid for any $r_n > n$, we deduce

$$|c_n| \leq \epsilon(r_n) r_n^{-1/2} 2^{r_n} \frac{1}{(r_n-1)(r_n-2)\cdots(r_n-n)} \cdot$$

The best choice [12] is $r_n = 2n$. Using Stirling's formula (153) we obtain

$$\begin{aligned} n!|c_n| &\leq \frac{\epsilon(2n)}{\sqrt{2n}} 2^{2n} \frac{n!(n-1)!}{(2n-1)!} \\ &= \frac{\epsilon(2n)}{\sqrt{2n}} 2^{2n+1} \frac{n!^2}{(2n)!} \\ &\sim \frac{\epsilon(2n)}{\sqrt{2n}} 2^{2n+1} \frac{(n^n e^{-n} \sqrt{2\pi n})^2}{(2n)^{2n} e^{-2n} \sqrt{4\pi n}} = \epsilon(2n)\sqrt{2\pi}. \end{aligned}$$

Hence $|c_n| < 1/n!$ for sufficiently large n, and therefore $c_n = 0$ for sufficiently large n, which means that the interpolation series F is a polynomial. Since $f - F$ vanishes for all $n = 0, 1, \ldots$ (by the construction of the interpolation series) and has exponential type $< \log 2 < 1$, it follows from Lemma 151 that $f = F$, hence f is a polynomial. $\square$

Remark. In [12], Pólya explains his choice of $r_n = 2n$ by letting $r_n = n/\xi$ with $0 < \xi < 1$, and by performing all the details of the computation with ξ. He shows that the optimal value for ξ is obtained when the function $\xi^\xi(1-\xi)^{1-\xi}$ assumes its minimal value, which is at $\xi = 1/2$, and then he completes the proof with this choice.

9.2.5 Integer valued entire functions on Gaussian integers

In 1926, S. Fukasawa extended Pólya's result to the Gaussian integers: he proved that if f is an entire function mapping $\mathbf{Z}[i]$ to $\mathbf{Z}[i]$ and if, for any $\epsilon > 0$, there exists $\theta_\epsilon > 0$ such that

$$|f|_r \le e^{\theta_\epsilon r^{\sigma-\epsilon}} \quad \text{with} \quad \sigma = \frac{1440}{919 + 27\sqrt{5}} = 1.470\ldots,$$

then f is a polynomial. In 1929, A.O. Gel'fond [3] refined the result and obtained the right exponent 2 in place of $\sigma - \epsilon$: he proved that an entire function f mapping $\mathbf{Z}[i]$ to $\mathbf{Z}[i]$ and satisfying

$$|f|_r \le e^{\gamma r^2} \quad \text{with} \quad \gamma < \frac{\pi}{2(1 + e^{164/\pi})^2} \simeq 0.7 \cdot 10^{-45}$$

is a polynomial.

The proofs by Fukasawa and Gel'fond rely on Newton's interpolation series at the points in $\mathbf{Z}[i]$.

That the exponent 2 cannot be improved is shown by the Weierstraß sigma function associated to $\mathbf{Z}[i]$. Gel'fond wrote that his estimate for the constant γ is not the right limit for the problem. In 1980, D.W. Masser showed that the result cannot hold with γ replaced by a constant larger than $\pi/(2e)$. In 1981, F. Gramain [5] proved that the result holds with $\pi/(2e)$, which is therefore best possible:

> *If f is an entire function which is not a polynomial and maps $\mathbf{Z}[i]$ to $\mathbf{Z}[i]$, then*
>
> $$\limsup_{r\to\infty} \frac{1}{r^2} \log |f|_r \ge \frac{\pi}{2e}\cdot$$

9.2.6 The constant of Gramain–Weber

The work by Masser and Gramain on entire functions mapping $\mathbf{Z}[i]$ to $\mathbf{Z}[i]$ gave rise to the following problem, which is still unsolved. For each integer

$k \geq 2$, let r_k be the minimal radius of a closed disk in $\mathbf{R}^2$ containing at least k points of $\mathbf{Z}^2$, and for $n \geq 2$ define

$$\delta_n = -\log n + \sum_{k=2}^{n} \frac{1}{\pi r_k^2}.$$

The limit $\delta = \lim_{n\to\infty} \delta_n$ exists (it is an analogue in dimension 2 of the Euler constant), and the best known estimates for it are [8]

$$1.811 \cdots < \delta < 1.897 \ldots$$

(see also [2]). F. Gramain conjectures that

$$\delta = 1 + \frac{4}{\pi}\bigl(\gamma L(1) + L'(1)\bigr),$$

where γ is Euler's constant and

$$L(s) = \sum_{n\geq 0} (-1)^n (2n+1)^{-s}$$

is the L function of the quadratic field $\mathbf{Q}(i)$ (Dirichlet beta function). Since $L(1) = \pi/4$ and

$$L'(1) = \sum_{n\geq 0} (-1)^{n+1} \cdot \frac{\log(2n+1)}{2n+1} = \frac{\pi}{4}\bigl(3\log\pi + 2\log 2 + \gamma - 4\log\Gamma(1/4)\bigr),$$

Gramain's conjecture is equivalent to

$$\delta = 1 + 3\log\pi + 2\log 2 + 2\gamma - 4\log\Gamma(1/4) = 1.822825\ldots$$

Other problems related to the lattice $\mathbf{Z}[i]$ are described in the section "*On the borders of geometry and arithmetic*" of [15].

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