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Diophantine approximation, irrationality and transcendence

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The work by Fukasawa on integer valued entire functions at the points of $\mathbf{Z}[i]$ requires estimates on the number of points of $\mathbf{Z}[i]$ into a disc. More generally, Fukasawa showed that if A is a domain bounded by finitely many curves of finite length, if we set

$$A = \int\int_{(D)} dxdy, \qquad B = \int\int_{(D)} \log\sqrt{x^2+y^2}dxdy,$$

then the number of points in $Dt \cap \mathbf{Z}[i]$ satisfies

$$At^2\log t + Bt^2 + O(t\log t) \quad \text{as} \quad t\to\infty.$$

For the unit disc $D = \{z \in \mathbf{C}\ ;\ |z| \le 1\}$, one has $A = \pi$ and $B = -\pi/2$. One deduces

$$\log \prod_{\substack{0\neq\omega\in\mathbf{Z}[i]\\ |\omega|\le t}} |\omega| = \sum_{\substack{0\neq\omega\in\mathbf{Z}[i]\\ |\omega|\le t}} \log|\omega| = \pi r^2\log r - \frac{\pi}{2}r^2 + o(r^2).$$

This yields

Lemma 156. *An entire function f satisfying $f(\mathbf{Z}[i]) = \{0\}$ and, for all sufficiently large r,*

$$|f|_r \le e^{\kappa r^2}$$

with $\kappa < \pi/2$, is a polynomial.

Proof. Like in the proof of Lemma 151, this follows from Jensen's formula, but here one replaces Stirling's formula by the estimates

$$\sum_{|\omega|\le r} 1 = \pi r^2 + o(r^2)$$

and

$$\sum_{\substack{0\neq\omega\in\mathbf{Z}[i]\\|\omega|\leq t}} \log(|\omega|/r) = \pi r^2 \log r - \frac{\pi}{2} r^2 - \pi r^2 \log r + o(r^2) = -\frac{\pi}{2} r^2 + o(r^2).$$

□

9.2.7 Transcendence of e^π

In [2], just after his paper [1] on integer valued entire functions on $\mathbf{Z}[i]$, A.O. Gel'fond extended his proof and obtained the following outstanding result:

Theorem 157 (Ge'lfond)**.** *The number*

$$e^\pi = 23,140\,692\,632\,779\,269\,005\,729\,086\,367\ \dots$$

is transcendental.

This was the first step towards a solution of the seventh of the 23 problems raised by D. Hilbert at the International Congress of Mathematicians in Paris in 1900: *for algebraic α and β with $\alpha \neq 0$, $\alpha \neq 1$ and β irrational, the number α^β is transcendental.*

The number α^β is defined as $\alpha^\beta = \exp(\beta \log \alpha)$, where $\log \alpha$ is any logarithm of α. The condition $\alpha \neq 1$ may be replaced by $\log \alpha \neq 0$, both statements are equivalent.

Taking $\alpha = -1$, $\log \alpha = i\pi$, $\beta = -i$ gives $\alpha^\beta = e^\pi$.

Proof of Theorem 157. . Gel'fond starts by ordering $\mathbf{Z}[i]$ by non–decreasing modulus, and for those of the same modulus by increasing arguments in $[0, 2\pi)$:

$$\mathbf{Z}[i] = \{x_0, x_1, x_2, \dots, x_n, \dots\}$$

with $x_0 = 0$. Hence

$$\{x_0, x_1, x_2, \dots\} = \{0,\ 1,\ i,\ -1,\ -i,\ 1+i,\ -1+i,\ -1-i,\ 2,\ 2i, \dots\}.$$

If the disc $|z| \leq r_n$ contains the points x_i for $0 \leq i \leq n$, then the number $n+1$ of these points is

$$n + 1 = \pi r_n^2 + \alpha r_n + o(r_n)$$

with $\alpha < 2\sqrt{2}\pi$, hence $|x_n| = \sqrt{n/\pi} + o(\sqrt{n})$.

For $n \geq 1$, define $P_n(z) = z(z - x_1) \cdots (z - x_{n-1})$. Gel'fond expands the function $e^{\pi z}$ into a series of P_n:

$$e^{\pi z} = \sum_{k=0}^{n} A_k P_k(z) + R_n(z),$$

where, following 9.2.3,

$$A_k = \frac{1}{2i\pi} \int_{|\zeta|=n} \frac{e^{\pi\zeta} d\zeta}{P_{k+1}(\zeta)} \quad \text{and} \quad R_n(z) = \frac{P_{n+1}(z)}{2i\pi} \int_{|\zeta|=n} \frac{e^{\pi\zeta}}{P_{k+1}(\zeta)} \cdot \frac{d\zeta}{\zeta - z}.$$

Since the zeroes of P_{k+1} are simple, the residue formula gives, for $n \geq 0$,

$$A_n = \sum_{k=0}^{n} \frac{e^{\pi x_k}}{\omega_{n,k}}, \quad \text{with} \quad \omega_{n,k} = \prod_{\substack{0 \leq j \leq n \\ j \neq k}} (x_k - x_j).$$

The number $e^{\pi x_k}$ is $\pm e^{\pi \Re e(x_k)}$ and $\Re e(x_k)$ is a rational integer of absolute value $\leq \sqrt{n/\pi} + o(\sqrt{n})$. Hence A_n is a polynomial in e^{π} and $e^{-\pi}$ of degree $\leq \sqrt{n/\pi} + o(\sqrt{n})$ and coefficients in $\mathbf{Q}(i)$. The integral over the circle $|\zeta| = n$ yields the upper bound

$$|A_n| \leq \frac{e^{\pi n}}{\prod_{0 \leq j \leq n} (n - |x_j|)} \leq e^{-n \log n + \pi n + O(\sqrt{n})}.$$

In his previous work [1], Gel'fond proved that the least common multiple Ω_n of the numbers $\omega_{n,k}$ for $0 \leq k \leq n$ (which is also the least common denominator of the numbers $1/\omega_{n,k}$ for $0 \leq k \leq n$) satisfies

$$\Omega_n \leq e^{\frac{1}{2} n \log n + 163n + o(n)}.$$

The product $\Omega_n A_n$ is in $\mathbf{Z}[i][e^{\pi}, e^{-\pi}]$:

$$\Omega_n A_n = \sum_{k=0}^{n} B_{kn} e^{\pi x_k} \quad \text{with} \quad B_{kn} = \Omega_n / \omega_{n,k} \in \mathbf{Z}[i]$$

and

$$\max_{0 \leq k \leq n} |B_{kn}| \leq e^{\frac{1}{2} n \log n + 163n - \frac{1}{2} n \log n + 3\pi n + o(n)} \leq e^{173n + o(n)}.$$

Assuming e^{π} is algebraic, Liouville's inequality (Lemma 26) implies $A_n = 0$ for all sufficiently large n, and therefore the interpolation series

$$F(z) = \sum_{n \geq 0} A_n P_n(z)$$

is a polynomial. This polynomial F, by construction, takes the value $e^{\pi x_k}$ at $z = x_k$, which means that the entire function $e^{\pi z} - F(z)$ vanishes on $\mathbf{Z}[i]$. But this function has exponential type π, hence order 1, and Lemma 156 implies that this function is the zero function. This is a contradiction with the fact that $e^{\pi z}$ is a transcendental function.

□

9.2.8 Interpolation formulae

In the easiest case where there are no multiplicities, the interpolation problem is to find a function f taking given values at distinct points. When x_i and y_i are m given points $(0 \le i \le m-1)$, with x_i pairwise distinct, there is a unique polynomial P of degree $< m$ satisfying $P(x_i) = y_i$ for $0 \le i \le m-1$. This polynomial is

$$f(z) = \sum_{j=0}^{m-1} y_j f_j(z),$$

where f_j is the solution of the same problem for the special case where $y_i = \delta_{ij}$ (Kronecker symbol, which is 1 for $i = j$ and 0 otherwise). Explicitly,

$$f_j(z) = \prod_{\substack{0 \le i \le m-1 \\ i \ne j}} \frac{z - x_i}{x_j - x_i} \cdot$$

Similar formulae exist when the x_i may be repeated. As a simple example, if $x_i = x_0$ for $0 \le i \le m$, then the condition on f becomes $f^{(j)}(x_0) = y_j$ $(0 \le j < m)$, and the solution is given by the Taylor's expansion

$$f(z) = \sum_{j=0}^{m-1} y_j f_j(z) \quad \text{with} \quad f_j(z) = \frac{1}{j!}(z - x_0)^j \cdot$$

In the very general case, one way to produce such formulae is to introduce integral formulae.

Let $Q(z)$ be a monic polynomial with roots $z_1, \ldots, z_n$, and for $1 \le i \le n$ let $m_i \ge 1$ be the multiplicity of z_i as a root of Q:

$$Q(z) = \prod_{i=1}^{n} (z - z_i)^{m_i}.$$

Let R be a real number with $R > \max_{1 \le i \le n} |z_i|$, so that the disc $|z| < R$ contains all points z_i. We denote by Γ the circle $|z| = R$. Further, for

$1 \leq i \leq n$, let r_i be a real number in the range

$$0 < r_i < \min_{\substack{1 \leq k \leq n \\ k \neq i}} |z_i - z_k|.$$

We denote by Γ_i the circle $|z_i| \leq r_i$: it contains z_i, but no z_k for $k \neq i$. The following formula is due to Hermite: *for f analytic in an open domain containing the disc $|z| \leq R$ and for z in the open disc $|z| < R$ distinct from all z_i,*

$$\frac{f(z)}{Q(z)} = \frac{1}{2i\pi} \int_\Gamma \frac{f(\zeta)}{Q(\zeta)} \cdot \frac{d\zeta}{\zeta - z} - \frac{1}{2i\pi} \sum_{i=1}^{n} \sum_{j=0}^{m_i-1} \frac{f^{(j)}(z_i)}{j!} \int_{\Gamma_i} \frac{(\zeta - z_i)^j}{Q(\zeta)} \cdot \frac{d\zeta}{\zeta - z}.$$

The proof is a simple application of the residue formula (see for instance [3] Chap. IX § 2): the first integral divided by $2i\pi$ is the sum of the residues of the function

$$\varphi(\zeta) = \frac{f(\zeta)}{Q(\zeta)} \cdot \frac{1}{\zeta - z}$$

at the poles in $|z| < R$. The pole $\zeta = z$ is simple, and the residue is $f(z)/Q(z)$, which gives the left hand side. Also, each sum

$$\sum_{j=0}^{m_i-1} \frac{f^{(j)}(z_i)}{j!} \int_{\Gamma_j} \frac{(\zeta - z_i)^j}{Q(\zeta)} \cdot \frac{d\zeta}{\zeta - z}$$

in the right hand side is $2i\pi$ times the residue at $\zeta = z_i$ of $\varphi(\zeta)$. Hence the formula drops out.

If f is a polynomial of degree $< M$ where $M = m_1 + \cdots + m_n$, then the first integral vanishes.

For $1 \leq i_0 \leq n$ and $0 \leq j_0 < m_i$, define the function $f_{i_0,j_0}(z)$ on the open set $|z - z_{i_0}| > r_{i_0}$ by

$$f_{i_0,j_0}(z) = -\frac{1}{j!} \cdot \frac{1}{2i\pi} Q(z) \int_{|\zeta - z_{i_0}| = r_{i_0}} \frac{(\zeta - z_{i_0})^{j_0}}{Q(\zeta)} \cdot \frac{d\zeta}{\zeta - z}.$$

Here, r_{i_0} is any number satisfying $0 < r_{i_0} < \min_{i \neq i_0} |z_i - z_{i_0}|$. Computing the integral by means of the residue Theorem shows that the integral extends to a meromorphic function in $\mathbf{C}$ with a single pole at $z = z_{i_0}$ of order $\leq m_i$. Also, letting $|z|$ tend to infinity shows that $f_{i_0,j_0}(z)$ is a polynomial of degree $< M$. Hence f_{i_0,j_0} is the unique polynomial of degree $< M$ satisfying

$$f^{(j)}_{i_0,j_0}(z_i) = \delta_{(i_0,j_0),(i,j)} \quad \text{where} \quad \delta_{(i_0,j_0),(i,j)} = \begin{cases} 1 & \text{if } i = i_0 \text{ and } j = j_0, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that, given distinct points $z_1, \ldots, z_n$, positive integers $m_1, \ldots, m_n$ and complex numbers y_{ij} $(1 \le i \le n,\ 0 \le j \le m_i - 1)$, there is a unique polynomial of degree $< M$, where $M = m_1 + \cdots + m_n$, satisfying the M conditions $f^{(j)}(z_i) = y_{ij}$ for $1 \le i \le n$ and $0 \le j \le m_i - 1$. This polynomial is given by

$$\sum_{i=1}^{n} \sum_{j=0}^{m_i-1} y_{ij} f_{ij}.$$

9.2.9 Rational interpolation

We just mention another kind of interpolation formula, which was introduced by René Lagrange in 1935, and used more recently by Tanguy Rivoal [4] for producing Diophantine results, including a new proof of Apéry's theorem on the irrationality of $\zeta(3)$.

One starts with the formula

$$\frac{1}{x-z} = \frac{\alpha-\beta}{(x-\alpha)(x-\beta)} + \frac{x-\beta}{x-\alpha} \cdot \frac{z-\alpha}{z-\beta} \cdot \frac{1}{x-z}.$$

Iterating and integrating yields

$$f(z) = \sum_{n=0}^{N-1} B_n \frac{(z-\alpha_1)\cdots(z-\alpha_n)}{(z-\beta_1)\cdots(z-\beta_n)} + \tilde{R}_N(z).$$

This is an expansion of f into rational fractions, with given zeroes and poles.

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10 The Schneider–Lang Theorem

The Theorem of Schneider-Lang is a general statement dealing with values of meromorphic functions of one or several complex variables, satisfying differential equations.

The first general result dealing with analytic or meromorphic functions of one variable and containing the solution to Hilbert's seventh problem appears in [4]. In fact one can deduce the transcendence of α^β (Gel'fond-Schneider Theorem 1.4) from this theorem, either by using the two functions z and α^z without derivatives (Schneider's method), or else e^z and $e^{\beta z}$ with derivatives (Gel'fond's method). The statement is rather complicated, and Th. Schneider made successful attempts to simplify it [5]. Schneider's criteria in [5], Chap. II, § 3, Th.12 and 13 deal only with Gel'fond's method, i.e. involve derivatives. Further simplifications have been introduced by S. Lang later: either for Schneider's method (see [1], Chap. III, § 1, Th.1), or else for Gel'fond's method and functions satisfying differential equations (see [1], Chap. III, § 1, Th.1 and [3], Appendix 1). This last result is known as the *Theorem of Schneider-Lang*.

10.1 Statement and first corollaries

`Content of the course:` Theorem of Schneider–Lang, corollaries: theorem of Hermite–Lindemann, Theorem of Gel'fond–Schneider.
Outline of the proof.
`References:` [6] (Chap. 3, § 3.7) and [7] (§ 2.2).
See also [5] (Chap. II, § 3, Th.12 and 13); [1] (Chap. III, § 1, Th.1); [3] (Appendix 1).
There is also a proof in [2] (Chap. IX § 3) for the special case where the number field is **Q**: this allows to avoid any use of algebraic number theory.

References

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