Updated: *June 22, 2010*

# Diophantine approximation, irrationality and transcendence

*Michel Waldschmidt*

Course N°21, *June 28, 2010*

These are informal notes of my course given in April – June 2010 at IMPA (*Instituto Nacional de Matematica Pura e Aplicada*), Rio de Janeiro, Brazil.

The text below is taken from [4] § 5.2.

## 11.2 Modular functions and Ramanujan functions

S. Ramanujan introduced the following functions

$$P(q)=1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n},\quad Q(q)=1+240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n},\quad R(q)=1-504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}\cdot$$

They are special cases of Fourier expansions of Eisenstein series. Recall the Bernoulli numbers $B_k$ defined by:

$$\frac{z}{e^z-1}=1-\frac{z}{2}+\sum_{k=1}^{\infty}(-1)^{k+1}B_k\frac{z^{2k}}{(2k)!},$$

$$B_1=1/6,\quad B_2=1/30,\quad B_3=1/42.$$

For $k\geq 1$ the normalized Eisenstein series of weight $k$ is

$$E_{2k}(q)=1+(-1)^k\frac{4k}{B_k}\sum_{n=1}^{\infty}\frac{n^{2k-1}q^n}{1-q^n}\cdot$$

The connection with (161) is

$$E_{2k}(q)=\frac{1}{2\zeta(2k)}\cdot G_k(\tau),$$

for $k \geq 2$, where $q = e^{2\pi i\tau}$. In particular

$$G_2(\tau) = \frac{\pi^4}{3^2 \cdot 5} \cdot E_4(q), \qquad G_3(\tau) = \frac{2\pi^6}{3^3 \cdot 5 \cdot 7} \cdot E_6(q).$$

With Ramanujan's notation we have

$$P(q) = E_2(q), \quad Q(q) = E_4(q), \quad R(q) = E_6(q).$$

The discriminant $\Delta$ and the modular invariant $J$ are related with these functions by Jacobi's product formula

$$\Delta = \frac{(2\pi)^{12}}{12^3} \cdot (Q^3 - R^2) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1-q^n)^{24} \quad \text{and} \quad J = \frac{(2\pi)^{12} Q^3}{\Delta} = \frac{(2^4 3^2 5 G_2)^3}{\Delta} \cdot$$

Let $q$ be a complex number, $0 < |q| < 1$. There exists $\tau$ in the upper half plane $\mathcal{H}$ such that $q = e^{2\pi i\tau}$. Select any twelfth root $\omega$ of $\Delta(q)$ . The invariants $g_2$ and $g_3$ of the Weierstraß $\wp$ function attached to the lattice $(\mathbf{Z} + \mathbf{Z}\tau)\omega$ satisfy $g_2^3 - 27g_3^2 = 1$ and

$$P(q) = 3\frac{\omega}{\pi} \cdot \frac{\eta}{\pi}, \quad Q(q) = \frac{3}{4}\left(\frac{\omega}{\pi}\right)^4 g_2, \quad R(q) = \frac{27}{8}\left(\frac{\omega}{\pi}\right)^6 g_3.$$

According to formulae (162) and (163), here are a few special values

- For $\tau = i$, $q = e^{-2\pi}$,

$$P(e^{-2\pi}) = \frac{3}{\pi}, \quad Q(e^{-2\pi}) = 3\left(\frac{\omega_1}{\pi}\right)^4, \qquad (183)$$
$$R(e^{-2\pi}) = 0 \quad \text{and} \quad \Delta(e^{-2\pi}) = 2^6 \omega_1^{12},$$

  with

$$\omega_1 = \frac{\Gamma(1/4)^2}{\sqrt{8\pi}} = 2.6220575542\ldots$$

- For $\tau = \varrho$, $q = -e^{-\pi\sqrt{3}}$,

$$P(-e^{-\pi\sqrt{3}}) = \frac{2\sqrt{3}}{\pi}, \; Q(-e^{-\pi\sqrt{3}}) = 0, \qquad (184)$$
$$R(-e^{-\pi\sqrt{3}}) = \frac{27}{2}\left(\frac{\omega_1}{\pi}\right)^6, \; \Delta(-e^{-\pi\sqrt{3}}) = -2^4 3^3 \omega_1^{12},$$

  with

$$\omega_1 = \frac{\Gamma(1/3)^3}{2^{4/3}\pi} = 2.428650648\ldots$$

## 11.3 Nesterenko's result

In 1976, D. Bertrand pointed out that Schneider's Theorem 173 on the transcendence of $\omega/\pi$ implies:
*For any $q \in \mathbf{C}$ with $0 < |q| < 1$, at least one of the two numbers $Q(q)$, $R(q)$ is transcendental.*

He also proved the $p$-adic analog by means of a new version of the Schneider–Lang criterion for meromorphic functions (he allows one essential singularity) which he applied to Jacobi–Tate elliptic functions. Two years later he noticed that G.V. Chudnovsky's Theorem 178 yields:
*For any $q \in \mathbf{C}$ with $0 < |q| < 1$, at least two of the numbers $P(q)$, $Q(q)$, $R(q)$ are algebraically independent.*

The following result of Yu.V. Nesterenko goes one step further:

**Theorem 185** (Nesterenko, 1996)**.** *For any $q \in \mathbf{C}$ with $0 < |q| < 1$, three of the four numbers $q$, $P(q)$, $Q(q)$, $R(q)$ are algebraically independent.*

Among the tools used by Nesterenko in his proof is the following result due to K. Mahler:

*The functions $P$, $Q$, $R$ are algebraically independent over $\mathbf{C}(q)$.*

Also he uses the fact that they satisfy a system of differential equations for $D = q\, d/dq$ discovered by S. Ramanujan in 1916:

$$12\frac{DP}{P} = P - \frac{Q}{P}, \quad 3\frac{DQ}{Q} = P - \frac{R}{Q}, \quad 2\frac{DR}{R} = P - \frac{Q^2}{R}\cdot$$

One of the main steps in his original proof is his following zero estimate:

**Theorem 186** (Nesterenko's zero estimate)**.** *Let $L_0$ and $L$ be positive integers, $A \in \mathbf{C}[q, X_1, X_2, X_3]$ a non-zero polynomial in four variables of degree $\le L_0$ in $q$ and $\le L$ in each of the three other variables $X_1, X_2, X_3$. Then the multiplicity at the origin of the analytic function $A\big(q, P(q), Q(q), R(q)\big)$ is at most $2 \cdot 10^{45} L_0 L^3$.*

In the special case where $J(q)$ is algebraic, P. Philippon produced an alternative proof for Nesterenko's result where this zero estimate 186 is not used; instead of it, he used Philibert's measure of algebraic independence for $\omega/\pi$ and $\eta/\pi$. However Philibert's proof requires a zero estimate for algebraic groups.

Using (183) one deduces from Theorem 185

**Corollary 187.** *The three numbers $\pi$, $e^\pi$, $\Gamma(1/4)$ are algebraically independent.*

while using (184) one deduces

**Corollary 188.** *The three numbers* $\pi$, $e^{\pi\sqrt{3}}$, $\Gamma(1/3)$ *are algebraically independent.*

Consequences of Corollary 187 are the transcendence of the numbers

$$\sigma_{\mathbf{Z}[i]}(1/2) = 2^{5/4}\pi^{1/2}e^{\pi/8}\Gamma(1/4)^{-2}$$

and (P. Bundschuh)

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1}{2} + \frac{\pi}{2} \cdot \frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}}.$$

D. Duverney, K. and K. Nishioka and I. Shiokawa as well as D. Bertrand derived from Nesterenko's Theorem 185 a number of interesting corollaries, including the following ones

**Corollary 189.** *Rogers-Ramanujan continued fraction:*

$$RR(\alpha) = 1 + \cfrac{\alpha}{1 + \cfrac{\alpha^2}{1 + \cfrac{\alpha^3}{1 + \ddots}}}$$

*is transcendental for any algebraic* $\alpha$ *with* $0 < |\alpha| < 1$.

**Corollary 190.** *Let* $(F_n)_{n\geq 0}$ *be the Fibonacci sequence:* $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$. *Then the number*

$$\sum_{n=1}^{\infty} \frac{1}{F_n^2}$$

*is transcendental.*

Jacobi Theta Series are defined by

$$\theta_2(q) = 2q^{1/4}\sum_{n\geq 0} q^{n(n+1)} = 2q^{1/4}\prod_{n=1}^{\infty}(1-q^{4n})(1+q^{2n}),$$

$$\theta_3(q) = \sum_{n\in\mathbf{Z}} q^{n^2} = \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1})^2,$$

$$\theta_4(q) = \theta_3(-q) = \sum_{n\in\mathbf{Z}} (-1)^n q^{n^2} = \prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n-1})^2.$$

**Corollary 191.** . *Let $i$, $j$ and $k \in \{2,3,4\}$ with $i \neq j$. Let $q \in \mathbf{C}$ satisfy $0 < |q| < 1$. Then each of the two fields*

$$\mathbf{Q}\big(q, \theta_i(q), \theta_j(q), D\theta_k(q)\big) \quad \textit{and} \quad \mathbf{Q}\big(q, \theta_k(q), D\theta_k(q), D^2\theta_k(q)\big)$$

*has transcendence degree $\geq 3$ over $\mathbf{Q}$.*

As an example, *for an algebraic number $q \in \mathbf{C}$ with $0 < |q| < 1$, the three numbers*

$$\sum_{n\geq 0} q^{n^2}, \quad \sum_{n\geq 1} n^2 q^{n^2}, \quad \sum_{n\geq 1} n^4 q^{n^2}$$

*are algebraically independent.* In particular *the number*

$$\theta_3(q) = \sum_{n\in\mathbf{Z}} q^{n^2}$$

*is transcendental.* The number $\theta_3(q)$ was explicitly considered by Liouville as far back as 1851.

The proof of Yu.V. Nesterenko is effective and yields quantitative refinements (measures of algebraic independence).

# References


[1] *Introduction to algebraic independence theory*, vol. 1752 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2001. With contributions from F. Amoroso, D. Bertrand, W. D. Brownawell, G. Diaz, M. Laurent, Yuri V. Nesterenko, K. Nishioka, Patrice Philippon, G. Rémond, D. Roy and M. Waldschmidt, Edited by Nesterenko and Philippon.

[2] M. Waldschmidt, *Sur la nature arithmétique des valeurs de fonctions modulaires*, Astérisque, (1997), pp. Exp. No. 824, 3, 105–140. Séminaire Bourbaki, Vol. 1996/97.

[3] ——, *Transcendance et indépendance algébrique de valeurs de fonctions modulaires*, in Number theory (Ottawa, ON, 1996), vol. 19 of CRM Proc. Lecture Notes, Amer. Math. Soc., Providence, RI, 1999, pp. 353–375.

[4] ——, *Elliptic functions and transcendence*, in Surveys in number theory, vol. 17 of Dev. Math., Springer, New York, 2008, pp. 143–188.