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On transcendental entire functions with infinitely many derivatives taking integer values at two points

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Abstract

Given a subset $S = \{s_0, s_1\}$ of the complex plane with two points and an infinite subset $\mathscr{S}$ of $S \times \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of nonnegative integers, we ask for a lower bound for the order of growth of a transcendental entire function f such that $f^{(n)}(s) \in \mathbb{Z}$ for all $(s, n) \in \mathscr{S}$.

We first take $\mathscr{S} = \{s_0, s_1\} \times 2\mathbb{N}$, where $2\mathbb{N} = \{0, 2, 4, \dots\}$ is the set of nonnegative even integers. We prove that an entire function f of sufficiently small exponential type such that $f^{(2n)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$ for all sufficiently large n must be a polynomial. The estimate we reach is optimal, as we show by constructing a uncountable set of examples. The main tool, both for the proof of the estimate and for the construction of examples, is Lidstone polynomials.

The same proof works for $\mathscr{S} = \{s_0, s_1\} \times (2\mathbb{N} + 1)$ and yields a lower bound for the order of a transcendental entire function satisfying $f^{(2n+1)}(s_0) \in \mathbb{Z}$ and $f^{(2n+1)}(s_1) \in \mathbb{Z}$ for all sufficiently large n.

Our next example is $(\{s_0\} \times (2\mathbb{N} + 1)) \cup (\{s_1\} \times 2\mathbb{N})$ (odd derivatives at s_0 and even derivatives at s_1). We use analogs of Lidstone polynomials which have been introduced by J.M. Whittaker and studied by I.J. Schoenberg.

Finally, using results of W. Gontcharoff, A. J. Macintyre and J.M. Whittaker, we prove lower bounds for the exponential type of a transcendental entire function f such that, for each sufficiently large n, one at least of the two numbers $f^{(n)}(s_0)$, $f^{(n)}(s_1)$ is in $\mathbb{Z}$.

Keywords

Integer valued entire functions, Hurwitz functions, Lidstone polynomials, exponential type, Pólya's Theorem.

1 Introduction

The order of an entire function f is

$$\varrho(f) = \limsup_{r\to\infty} \frac{\log\log|f|_r}{\log r} \quad \text{where} \quad |f|_r = \sup_{|z|=r} |f(z)|.$$

The exponential type of an entire function is

$$\tau(f) = \limsup_{r\to\infty} \frac{\log|f|_r}{r}\cdot$$

If the exponential type is finite, then f has order ≤ 1. If f has order < 1, then the exponential type is 0.

An alternative definition is the following: f is of exponential type $\tau(f)$ if and only if, for all $z_0 \in \mathbb{C}$,

$$\limsup_{n\to\infty} |f^{(n)}(z_0)|^{1/n} = \tau(f), \tag{1}$$

where $f^{(n)}$ denotes the n–th derivative $(\mathrm{d}^n/\mathrm{d}z^n)f$ of f. The equivalence between the two definitions follows from Cauchy's inequalities (10) and Stirling's Formula (11). If (1) is true for one $z_0 \in \mathbb{C}$, then it is true for all $z_0 \in \mathbb{C}$.

Given a finite set of points S in the complex plane and an infinite subset $\mathscr{S}$ of $S \times \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of nonnegative integers, we ask for a lower bound for the order of growth of a transcendental entire function f such that $f^{(n)}(s) \in \mathbb{Z}$ for all $(s, n) \in \mathscr{S}$. This question has been studied by a number of authors in the special case where $\mathscr{S} = S \times \mathbb{N}$. When $S = \{0\}$, a function satisfying these conditions, namely $f^{(n)}(0) \in \mathbb{Z}$ for all $n \geq 0$, is called a *Hurwitz function*. The order of a transcendental Hurwitz function is ≥ 1 [Pólya 1921] – see Proposition 2.1 below. Assume now $S = \{0, 1, \dots, k-1\}$ with $k \geq 2$. According to [Straus 1950, Th. 1], the order of a transcendental function satisfying $f^{(n)}(\ell) \in \mathbb{Z}$ for all $\ell = 0, 1, \dots, k-1$ and $n \geq 0$ is at least k. The example of the function $\exp\bigl(z(z-1)\cdots(z-k+1)\bigr)$ shows that the bound for the order is sharp. For $k = 2$, refined estimates are obtained in [Sato and Straus 1964, §3] and [Sato and Straus 1965, §4]. See also [Sato 1971, §7 and §8] and the survey [Sato 1985] with 59 references.

If we replace the assumption $f^{(n)}(s) \in \mathbb{Z}$ with $f^{(n)}(s) = 0$ for all $(s, n) \in \mathscr{S}$, we come across a question which has been the object of extensive works. It is the main topic of [Whittaker 1935, Chap. III] and [Gel'fond 1952, Chap. 3]. It is related with the interpolation problem of the existence and unicity of an entire function f for which the values $f^{(n)}(s)$ for $(s, n) \in \mathscr{S}$ are given. For $S = \{0\}$, the Taylor expansion solves the interpolation problem. The next most often studied case is $S = \{0, 1\}$ and $\mathscr{S} = S \times 2\mathbb{N}$, where $2\mathbb{N} = \{0, 2, 4, \dots\}$ is the set of nonnegative even integers; the basic tool is given by Lidstone polynomials.

In the present paper, we consider a set $S = \{s_0, s_1\}$ of only two complex numbers (with only a short excursion to the case where S may have more than two points in Theorem 1.9. We investigate more general sets in [Waldschmidt 2020]). Using an argument going back to Pólya, we reduce the study of entire functions, the derivatives of even order of which take integer values at two points, to the study of those functions where the same derivatives vanish at these two points. Our main assumption on the growth of our functions f is

$$\limsup_{r\to\infty} \mathrm{e}^{-r}\sqrt{r}|f|_r < \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\max\{|s_0|,|s_1|\}}. \tag{2}$$

If a function f satisfies

$$\limsup_{r\to\infty} \mathrm{e}^{-r}\sqrt{r}|f|_r < \gamma$$

for some constant $\gamma > 0$, then for $z_0 \in \mathbb{C}$ the function $\tilde{f}(z) = f(z + z_0)$ satisfies

$$\limsup_{r\to\infty} \mathrm{e}^{-r}\sqrt{r}|\tilde{f}|_r < \gamma \mathrm{e}^{|z_0|},$$

while the derivative f' of f satisfies

$$\limsup_{r\to\infty} \mathrm{e}^{-r}\sqrt{r}|f'|_r < \gamma \mathrm{e}.$$

The exponential type of such a function is ≤ 1; in the other direction, a function of exponential type < 1 satisfies

$$\limsup_{r\to\infty} \mathrm{e}^{-r}\sqrt{r}|f|_r = 0.$$

We will prove in Section 2, Proposition 2.2, that, for an entire function f satisfying the growth condition (2) and for $|z_0| \leq \max\{|s_0|, |s_1|\}$, the set of $n \geq 0$ for which $f^{(n)}(z_0) \in \mathbb{Z} \setminus \{0\}$ is finite.

In Section 3, we introduce the so–called *Lidstone polynomials* and we prove several estimates for their growth.

In Section 4, we give a lower bound for the growth of transcendental entire functions satisfying $f^{(2n)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$ for all sufficiently large n. On the other hand, we will also show that there are transcendental entire functions f of order 0 for which $f^{(2n)}(s_0) = 0$ for all $n \geq 0$ and $f^{(2n)}(s_1) = 0$ for infinitely many n.

In Section 5 (resp. Section 7), we consider a variant by studying the set of entire functions which satisfy the conditions $f^{(2n+1)}(s_0) \in \mathbb{Z}$ and $f^{(2n+1)}(s_1) \in \mathbb{Z}$ (resp. $f^{(2n+1)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$) for all sufficiently large n. The proof in Section 5 involves Lidstone polynomials, while the proofs of the results from Section 7 rest on analogs of Lidstone polynomials which have been introduced by J.M. Whittaker in 1933 and studied by I.J. Schoenberg in 1936 (Section 6).

In Section 8, we give a lower bound for the growth of transcendental entire functions satisfying the property that for each sufficiently large n, one at least of the two numbers $f^{(n)}(s_0)$, $f^{(n)}(s_1)$ is in $\mathbb{Z}$. In the periodic case we use results of W. Gontcharoff (1930) and A.J. Macintyre (1954), in the general case we use results of W. Gontcharoff (1930) and J.M. Whittaker (1933).

1.1 Derivatives of even order at two points

Our first result is a lower bound for the growth of a transcendental entire function whose derivatives of even order at two points s_0 and s_1 belong to $\mathbb{Z}$.

Theorem 1.1. *Let s_0, s_1 be two distinct complex numbers and f an entire function of exponential type $\tau(f)$ satisfying $f^{(2n)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$ for all sufficiently large n. Assume f satisfies the growth condition (2). Then there exist a polynomial $P \in \mathbb{C}[z]$ and complex numbers $c_1, c_2, \dots, c_L$ with*

$$L\pi \leq |s_1 - s_0|\tau(f)$$

such that

$$f(z) = P(z) + \sum_{\ell=1}^{L} c_\ell \sin\left(\ell\pi \frac{z - s_0}{s_1 - s_0}\right).$$

Recall that assumption (2) implies $\tau(f) \leq 1$. It follows from Theorem 1.1 that, if $|s_1 - s_0| \leq \pi$, then any transcendental entire function f satisfying $f^{(2n)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$ for all sufficiently large n has exponential type ≥ 1. Here are examples of such functions of exponential type 1. Let $a_0 \in \mathbb{Z}$ and $a_1 \in \mathbb{Z}$ with $(a_0, a_1) \neq (0, 0)$. Define

$$f_{a_0,a_1}(z) = a_0 \frac{\sinh(z - s_1)}{\sinh(s_0 - s_1)} + a_1 \frac{\sinh(z - s_0)}{\sinh(s_1 - s_0)}.$$

Then $f_{a_0,a_1}(s_0) = a_0$, $f_{a_0,a_1}(s_1) = a_1$ and $f''_{a_0,a_1} = f_{a_0,a_1}$, hence $f^{(2n)}_{a_0,a_1}(s_0) = a_0$ and $f^{(2n)}_{a_0,a_1}(s_1) = a_1$ for all $n \geq 0$. This function does not satisfy (2).

In the case $|s_1 - s_0| \geq \pi$, we deduce from Theorem 1.1 that any transcendental entire function f satisfying $f^{(2n)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$ for all sufficiently large n has exponential type $\geq \pi/|s_1 - s_0|$. For $\ell \geq 1$, the function

$$f_\ell(z) = \sin\left(\ell\pi \frac{z - s_0}{s_1 - s_0}\right)$$

has exponential type $\ell\pi/|s_1 - s_0|$ and satisfies $f_\ell^{(2n)}(s_0) = f^{(2n)}(s_1) = 0$ for all $n \geq 0$.

Corollary 1.2. *Let f be an entire function satisfying (2) for which $f^{(2n)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$ for all sufficiently large n. Then the set of $n \geq 0$ such that $f^{(2n)}(s_0) \neq 0$ is finite, and also the set of $n \geq 0$ such that $f^{(2n)}(s_1) \neq 0$ is finite. If the exponential type of f satisfies $\tau(f) < \frac{\pi}{|s_1 - s_0|}$, then f is a polynomial.*

We now show that the assumption (2) on the growth of f in Corollary 1.2 is essentially best possible.

Notation 1.3. We denote by ν the unique positive real number satisfying

$$\mathrm{e}^{\nu} - \mathrm{e}^{-\nu} = 4\nu.$$

The numerical value is $\nu = 2.177\,3\ldots$ Both ν and e^{ν} are transcendental.

Theorem 1.4. *Let s_0, s_1 be two distinct complex numbers such that*

$$|s_1 - s_0| < \nu. \tag{3}$$

Then there exist a constant γ and an uncountable set of transcendental entire functions f satisfying $f^{(2n)}(s_0) = 0$ and $f^{(2n)}(s_1) \in \{-1, 0, 1\}$ for all $n \geq 0$, for which the set $\{n \geq 0 \mid f^{(2n)}(s_1) \neq 0\}$ is infinite, and such that

$$\limsup_{r \to \infty} \mathrm{e}^{-r} \sqrt{r} |f|_r \leq \gamma. \tag{4}$$

1.2 Derivatives of odd order at two points

The following variant of Theorem 1.1 deals with the set $S \times (2\mathbb{N} + 1)$ (odd order of the derivatives). We cannot simply use Theorem 1.1 for the first derivative f' of the given function f, since (2) may not be satisfied for f'.

Theorem 1.5. *Let s_0, s_1 be two distinct complex numbers and f an entire function of exponential type $\tau(f)$ satisfying $f^{(2n+1)}(s_0) \in \mathbb{Z}$ and $f^{(2n+1)}(s_1) \in \mathbb{Z}$ for all sufficiently large n. Assume f satisfies the growth condition (2). Then there exist a polynomial $P \in \mathbb{C}[z]$ and complex numbers $c_1, c_2, \ldots, c_L$ with*

$$L\pi \leq |s_1 - s_0| \tau(f)$$

such that

$$f(z) = P(z) + \sum_{\ell=1}^{L} c_\ell \cos\left(\ell\pi \frac{z - s_0}{s_1 - s_0}\right).$$

Under the assumptions of Theorem 1.5, the set of $n \geq 0$ such that $f^{(2n+1)}(s_0) \neq 0$ is finite, and also the set of $n \geq 0$ such that $f^{(2n+1)}(s_1) \neq 0$ is finite. Further, if the exponential type of f satisfies $\tau(f) < \frac{\pi}{|s_1 - s_0|}$, then f is a polynomial.

A polynomial is determined only up to an additive constant by its derivatives of odd order at two points. An expansion of a polynomial in terms of these derivatives, analogous to (12) below, is obtained by taking primitives of the Lidstone polynomials (defined up to an additive constant – notice that Λ'_{n+1} is a primitive of Λ_n). Such expansions have been studied in [Costabile et al., 2018, §3] under the name *Even Lidstone–type sequences*.

1.3 Derivatives of odd order at one point and even order at the other

The next result deals with $\mathscr{S} = (\{s_0\} \times (2\mathbb{N} + 1)) \cup (\{s_1\} \times 2\mathbb{N})$.

Theorem 1.6. *Let s_0, s_1 be two distinct complex numbers. Let f be an entire function of exponential type $\tau(f)$ satisfying $f^{(2n+1)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$ for all sufficiently large n. Assume f satisfies (2). Then there exist a polynomial $P \in \mathbb{C}[z]$ and complex numbers $c_0, c_1, \ldots, c_L$ with*

$$(2L + 1)\frac{\pi}{2} \leq |s_1 - s_0| \tau(f)$$

such that

$$f(z) = P(z) + \sum_{\ell=0}^{L} c_\ell \cos\left(\frac{(2\ell+1)\pi}{2} \cdot \frac{z-s_0}{s_1-s_0}\right).$$

In the case $|s_1-s_0| \le \pi/2$, any transcendental entire function f satisfying $f^{(2n+1)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$ for all sufficiently large n has exponential type ≥ 1. Here are examples of such functions of exponential type 1. Let $a_0 \in \mathbb{Z}$ and $a_1 \in \mathbb{Z}$ with $(a_0, a_1) \ne (0,0)$. Define

$$f_{a_0,a_1}(z) = a_0 \frac{\sinh(z-s_1)}{\cosh(s_0-s_1)} + a_1 \frac{\cosh(z-s_0)}{\cosh(s_1-s_0)}.$$

Then $f'_{a_0,a_1}(s_0) = a_0$, $f_{a_0,a_1}(s_1) = a_1$ and $f''_{a_0,a_1} = f_{a_0,a_1}$, hence $f^{(2n+1)}_{a_0,a_1}(s_0) = a_0$ and $f^{(2n)}_{a_0,a_1}(s_1) = a_1$ for all $n \ge 0$.

In the case $|s_1-s_0| \ge \pi/2$, any transcendental entire function f satisfying $f^{(2n+1)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$ for all sufficiently large n has exponential type $\ge \pi/(2|s_1-s_0|)$. For $\ell \ge 0$, the function

$$f_\ell(z) = \cos\left(\frac{(2\ell+1)\pi}{2} \cdot \frac{z-s_0}{s_1-s_0}\right)$$

has exponential type $\frac{(2\ell+1)\pi}{2|s_1-s_0|}$ and satisfies $f_\ell^{(2n+1)}(s_0) = f_\ell^{(2n)}(s_1) = 0$ for all $n \ge 0$.

Corollary 1.7. *Let f be an entire function satisfying (2) for which $f^{(2n+1)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$ for all sufficiently large n. Then the two sets*

$$\{n \ge 0 \mid f^{(2n+1)}(s_0) \ne 0\} \quad \textit{and} \quad \{n \ge 0 \mid f^{(2n)}(s_1) \ne 0\}$$

are finite. If the exponential type of f satisfies $\tau(f) < \frac{\pi}{2|s_1-s_0|}$, then f is a polynomial.

The assumption (2) in Corollary 1.7 is essentially optimal:

Theorem 1.8. *Let s_0, s_1 be two distinct complex numbers satisfying*

$$|s_1 - s_0| < \log(2+\sqrt{3}) = 1.316\,957\,8\cdots. \tag{5}$$

There exist a constant γ' and an uncountable set of transcendental entire functions f satisfying $f^{(2n+1)}(s_0) = 0$ and $f^{(2n)}(s_1) \in \{-1, 0, 1\}$ for all $n \ge 0$, such that the set of $n \ge 0$ with $f^{(2n)}(s_1) \ne 0$ is infinite and such that

$$\limsup_{r\to\infty} \mathrm{e}^{-r}\sqrt{r}|f|_r \le \gamma'. \tag{6}$$

1.4 Sequence of derivatives

We propose some generalisations of Corollary 1.7, where we assume that for each sufficiently large integer n, one at least of the two numbers $f^{(n)}(s_0)$, $f^{(n)}(s_1)$ is in $\mathbb{Z}$.

We start with the case of a periodic sequence. Let $m \ge 2$ be a positive integer. Let $\sigma_0, \sigma_1, \ldots, \sigma_{m-1}$ be complex numbers, not necessarily distinct: we will be interested in the case where they all belong to a set with two elements, but the next result is not restricted to two points. Set $\zeta = \mathrm{e}^{2i\pi/m}$ and denote by τ the smallest modulus of a zero of the function $\Delta(t)$, where $\Delta(t)$ is the determinant of the $m \times m$ matrix

$$\left(\zeta^{k\ell}\mathrm{e}^{\zeta^k t\sigma_\ell}\right)_{0\le k,\ell\le m-1} = \begin{pmatrix} \mathrm{e}^{t\sigma_0} & \mathrm{e}^{t\sigma_1} & \mathrm{e}^{t\sigma_2} & \cdots & \mathrm{e}^{t\sigma_{m-1}} \\ \mathrm{e}^{\zeta t\sigma_0} & \zeta\mathrm{e}^{\zeta t\sigma_1} & \zeta^2\mathrm{e}^{\zeta t\sigma_2} & \cdots & \zeta^{m-1}\mathrm{e}^{\zeta t\sigma_{m-1}} \\ \mathrm{e}^{\zeta^2 t\sigma_0} & \zeta^2\mathrm{e}^{\zeta^2 t\sigma_1} & \zeta^4\mathrm{e}^{\zeta^2 t\sigma_2} & \cdots & \zeta^{2(m-1)}\mathrm{e}^{\zeta^2 t\sigma_{m-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathrm{e}^{\zeta^{m-1} t\sigma_0} & \zeta^{m-1}\mathrm{e}^{\zeta^{m-1} t\sigma_1} & \zeta^{2(m-1)}\mathrm{e}^{\zeta^{m-1} t\sigma_2} & \cdots & \zeta^{(m-1)^2}\mathrm{e}^{\zeta^{m-1} t\sigma_{m-1}} \end{pmatrix}.$$

Theorem 1.9. *Let m and τ as before. Let f be a transcendental entire function of exponential type $< \tau$ satisfying*

$$\limsup_{r\to\infty} \mathrm{e}^{-r}\sqrt{r}|f|_r < \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\max\{|\sigma_0|,|\sigma_1|,\ldots,|\sigma_{m-1}|\}}. \tag{7}$$

Assume that for each sufficiently large n, we have

$$f^{(mn+j)}(\sigma_j) \in \mathbb{Z} \text{ for } j = 0, 1, \ldots, m-1.$$

Then f is a polynomial.

This result is optimal:

Proposition 1.10. (a) *Let α be a zero of $\Delta(t)$. There exist $c_0, c_1, \ldots, c_{m-1}$ in $\mathbb{C}$, not all zero, such that the function*

$$f(z) = c_0\mathrm{e}^{\alpha z} + c_1\mathrm{e}^{\zeta\alpha z} + \cdots + c_{m-1}\mathrm{e}^{\zeta^{m-1}\alpha z}$$

satisfies

$$f^{(mn+j)}(\sigma_j) = 0 \text{ for } j = 0, 1, \ldots, m-1 \text{ and } n \ge 0.$$

(b) *Assume $\tau > 1$. Given $a_0, a_1, \ldots, a_{m-1}$ in $\mathbb{C}$, there exists a unique entire function of exponential type ≤ 1 satisfying*

$$f^{(mn+j)}(\sigma_j) = a_j \text{ for } j = 0, 1, \ldots, m-1 \text{ and } n \ge 0.$$

The function given by (a) is a transcendental entire function of exponential type $|\alpha|$. If $(a_0, a_1, \ldots, a_{m-1}) \ne (0, 0, \ldots, 0)$, we will prove that the function f given by (b) is a transcendental entire function of exponential type 1. Notice that f does not satisfy the assumption (7) of Theorem 1.9.

Here is a corollary of Theorem 1.9. We fix again an integer $m \ge 2$ and we denote by τ_m the smallest modulus of a zero of the function

$$1 + \frac{t^m}{m!} + \frac{t^{2m}}{(2m)!} + \cdots + \frac{t^{nm}}{(nm)!} + \cdots$$

When $\sigma_0 = s_1$ and $\sigma_i = s_0$ for $i = 1, ..., m-1$, the smallest modulus of a zero of the determinant $\Delta(t)$ is $\tau_m/|s_1 - s_0|$ (see §8.1).

Since $\tau_2 = \pi/2$, Corollary 1.7 is the case $m = 2$ of the next result.

Corollary 1.11. *Let s_0 and s_1 be two distinct complex numbers. Let f be a transcendental entire functions satisfying (2). Assume that the exponential type $\tau(f)$ of f satisfies*

$$\tau(f) < \frac{\tau_m}{|s_1 - s_0|}.$$

Assume further that for each sufficiently large n, we have

$$f^{(n)}(s_0) \in \mathbb{Z} \text{ for } n \not\equiv 0 \bmod m \quad \text{and} \quad f^{(n)}(s_1) \in \mathbb{Z} \text{ for } n \equiv 0 \bmod m.$$

Then f is a polynomial.

We can extend this result to the case $s_0 = s_1 = 0$ in view of Proposition 2.1 below due to [Pólya 1921].

Corollary 1.11 is sharp: from part (a) of Proposition 1.10 it follows that there exists a transcendental entire function f of type $\tau_m/|s_1 - s_0|$ satisfying

$$f^{(n)}(s_0) = 0 \text{ for } n \equiv 0 \bmod m \quad \text{and} \quad f^{(n)}(s_1) = 0 \text{ for } n \not\equiv 0 \bmod m.$$

Also, from part (b) of Proposition 1.10 it follows that if $\tau_m > |s_1 - s_0|$, given $a_0, a_1, \dots, a_{m-1}$ in $\mathbb{C}$, not all of which are zero, there exists a unique entire function f of exponential type ≤ 1 satisfying, for all $n \geq 0$,

$$f^{(n)}(s_0) = a_j \text{ for } n \equiv j \bmod m \text{ and } 1 \leq j \leq m-1 \text{ and } \; f^{(n)}(s_1) = a_0 \text{ for } n \equiv 0 \bmod m.$$

This function is transcendental of exponential type 1. When $a_0 = a_1 = \cdots = a_{m-1} = 0$, it is 0.

The next and last result deals with a situation more general than the case of two points in Theorem 1.9, since no periodicity is assumed, and we assume only that one at least of the three numbers $f^{(n)}(s_0)$, $f^{(n)}(s_1)$, $f^{(n)}(s_0)f^{(n)}(s_1)$ is in $\mathbb{Z}$. The assumption on the type in Theorem 1.12 may not be optimal.

Theorem 1.12. *Let s_0, s_1 be two distinct complex numbers. Let f is an entire function of exponential type $\tau(f)$ satisfying (2). Assume*

$$\tau(f) < \frac{1}{|s_1 - s_0|}.$$

Assume that, for all sufficiently large n, one at least of the three numbers

$$f^{(n)}(s_0), f^{(n)}(s_1), f^{(n)}(s_0)f^{(n)}(s_1)$$

is in $\mathbb{Z}$. Then f is a polynomial.

2 On a result of Pólya

Recall that a Hurwitz function is an entire function satisfying $f^{(n)}(0) \in \mathbb{Z}$ for all $n \geq 0$. Here is one of the earliest results on Hurwitz functions [Pólya 1921].

Proposition 2.1. *A transcendental Hurwitz function f satisfies*

$$\limsup_{r\to\infty} \mathrm{e}^{-r}\sqrt{r}|f|_r \geq \frac{1}{\sqrt{2\pi}}\cdot$$

The uncountable set of entire functions

$$f(z) = \sum_{n=0}^{\infty} e_n \frac{z^{2^n}}{2^n!} \quad \text{for which} \quad \limsup_{r\to\infty} \mathrm{e}^{-r}\sqrt{r}|f|_r = \frac{1}{\sqrt{2\pi}}, \tag{8}$$

where $e_n \in \{-1, 1\}$, shows that Proposition 2.1 is optimal. This does not mean that it is the final word. On the one hand, [Sato and Straus 1964, §2 Corollary 1] and [Sato and Straus 1965, §3 Corollary] have proved more precise results, including the following :

For every $\epsilon > 0$, there exists a transcendental Hurwitz function with

$$\limsup_{r\to\infty} \sqrt{2\pi r}\,\mathrm{e}^{-r}\left(1 + \frac{1+\epsilon}{24r}\right)^{-1} |f|_r < 1,$$

while every Hurwitz function for which

$$\limsup_{r\to\infty} \sqrt{2\pi r}\,\mathrm{e}^{-r}\left(1 + \frac{1-\epsilon}{24r}\right)^{-1} |f|_r \leq 1$$

is a polynomial.

On the other hand, our Corollary 2.4 below extends the range of validity of Proposition 2.1.

Proposition 2.2. *Let f be an entire function and let $A \geq 0$. Assume*

$$\limsup_{r\to\infty} \mathrm{e}^{-r}\sqrt{r}|f|_r < \frac{\mathrm{e}^{-A}}{\sqrt{2\pi}}\cdot \tag{9}$$

Then there exists $n_0 > 0$ such that, for $n \geq n_0$ and for all $z \in \mathbb{C}$ in the disc $|z| \leq A$, we have

$$|f^{(n)}(z)| < 1.$$

Remark 2.3. When $A = 0$, Pólya's example (8) shows that the upper bound in the assumption (9) of Proposition 2.2 is optimal.

For the proof of Proposition 2.2, we will use Cauchy's inequalities for an entire function f:

$$\frac{|f^{(n)}(z_0)|}{n!} r^n \leq |f|_{r+|z_0|}, \tag{10}$$

which are valid for all $z_0 \in \mathbb{C}$, $n \geq 0$ and $r > 0$. We will also use Stirling's Formula:

$$N^N \mathrm{e}^{-N}\sqrt{2\pi N} < N! < N^N \mathrm{e}^{-N}\sqrt{2\pi N}\mathrm{e}^{1/(12N)}, \tag{11}$$

which is valid for all $N \geq 1$.

Proof of Proposition 2.2. By assumption, there exists $\eta > 0$ such that, for n sufficiently large, we have

$$|f|_n < (1-\eta)\frac{\mathrm{e}^{n-A}}{\sqrt{2\pi n}}.$$

We use Cauchy's inequalities (10) with $r = n - A$: for $|z| \le A$, we have

$$|f^{(n)}(z)| \le \frac{n!}{(n-A)^n}|f|_n.$$

Hence (11) yields

$$|f^{(n)}(z)| \le (1-\eta)\mathrm{e}^{-A+1/(12n)}\left(1-\frac{A}{n}\right)^{-n}.$$

For n sufficiently large the right hand side is < 1. $\square$

We deduce the following refinement of Proposition 2.1:

Corollary 2.4. *Let f be a transcendental entire function. Let $A \ge 0$. Assume (9). Then the set*

$$\left\{(n, z_0) \in \mathbb{N} \times \mathbb{C} \ \mid\ |z_0| \le A,\ f^{(n)}(z_0) \in \mathbb{Z} \setminus \{0\}\right\}$$

is finite.

3 Lidstone polynomials

The theory of Lidstone polynomials and series has a long and rich history. We recall the definition and the basic results which we will need.

3.1 Definition and properties

We denote by δ_{ij} the Kronecker symbol:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \ne j. \end{cases}$$

By induction on n, one defines a sequence of polynomials $(\Lambda_n)_{n\ge 0}$ in $\mathbb{Q}[z]$ by the conditions $\Lambda_0(z) = z$ and

$$\Lambda_n'' = \Lambda_{n-1}, \qquad \Lambda_n(0) = \Lambda_n(1) = 0 \quad \text{for all } n \ge 1.$$

For $n \ge 0$, the polynomial Λ_n, has degree $2n+1$ and leading term $\frac{1}{(2n+1)!}z^{2n+1}$. From the definition one deduces

$$\Lambda_n^{(2k)}(0) = 0 \text{ and } \Lambda_n^{(2k)}(1) = \delta_{k,n} \text{ for all } n \ge 0 \text{ and } k \ge 0.$$

This definition goes back to [Lidstone 1930]. See also [Poritsky 1932], [Whittaker 1934], [Whittaker 1935, §9], [Schoenberg 1936], [Buck 1948, §9], [Buck 1955], [Boas and Buck 1964, Chap. I §4], [Costabile and Serpe, 2007], [Costabile et al., 2018, §1]. A consequence of the definition is that any polynomial $f \in \mathbb{C}[z]$ has a finite expansion

$$f(z) = \sum_{n=0}^{\infty} \Big(f^{(2n)}(0)\Lambda_n(1-z) + f^{(2n)}(1)\Lambda_n(z) \Big) \tag{12}$$

with only finitely many nonzero terms in the series.

Applying (12) to the polynomial z^{2n+1} yields the following recurrence formula [Costabile and Serpe, 2007, Th. 2]: for $n \geq 0$,

$$\Lambda_n(z) = \frac{1}{(2n+1)!} z^{2n+1} - \sum_{h=0}^{n-1} \frac{1}{(2n-2h+1)!} \Lambda_h(z). \tag{13}$$

For instance,

$$\Lambda_0(z) = z, \quad \Lambda_1(z) = \frac{1}{6}(z^3 - z)$$

and [Lidstone 1930, §6 p. 18]

$$\Lambda_2(z) = \frac{1}{120} z^5 - \frac{1}{36} z^3 + \frac{7}{360} z = \frac{1}{360} z(z^2-1)(3z^2-7).$$

It follows from (12) that for $n \geq 0$, a basis of the $\mathbb{Q}$–space of polynomials in $\mathbb{Q}[z]$ of degree $\leq 2n+1$ is given by the $2n+2$ polynomials

$$\Lambda_0(z), \Lambda_1(z), \dots, \Lambda_n(z), \quad \Lambda_0(1-z), \Lambda_1(1-z), \dots, \Lambda_n(1-z).$$

Another consequence of (12) is

$$\frac{z^{2n}}{(2n)!} = \Lambda_n(1-z) + \sum_{h=0}^{n} \frac{1}{(2n-2h)!} \Lambda_h(z)$$

for $n \geq 0$.

Lidstone expansion formula (12) for polynomials extends to entire functions of finite exponential type — see [Poritsky 1932, Th. 1], [Whittaker 1934, Th. 2], [Schoenberg 1936, Th. 1], [Buck 1955, p. 795], [Boas and Buck 1964, § 4]. If f has exponential type $< \pi$, then (12) holds for f, the series being uniformly convergent on any compact of $\mathbb{C}$. Therefore, if an entire function f has exponential type $< \pi$ and satisfies $f^{(2n)}(0) = f^{(2n)}(1) = 0$ for all sufficiently large n, then f is a polynomial. The following result ([Buck 1955, Theorem p. 795], [Boas and Buck 1964, Th. 4.6]) deals with entire functions f of any finite exponential type.

Proposition 3.1. *Let f be an entire function of finite exponential type $\tau(f)$ satisfying $f^{(2n)}(0) = f^{(2n)}(1) = 0$ for all $n \geq 0$. Then there exist complex numbers $c_1, \dots, c_L$ with $L \leq \tau(f)/\pi$ such that*

$$f(z) = \sum_{\ell=1}^{L} c_\ell \sin(\ell \pi z).$$

Let $t \in \mathbb{C}$, $t \notin i\pi\mathbb{Z}$. The entire function

$$f(z) = \frac{\sinh(zt)}{\sinh(t)} = \frac{\mathrm{e}^{zt} - \mathrm{e}^{-zt}}{\mathrm{e}^{t} - \mathrm{e}^{-t}}$$

satisfies

$$f'' = t^2 f, \quad f(0) = 0, \quad f(1) = 1,$$

hence $f^{(2n)}(0) = 0$ and $f^{(2n)}(1) = t^{2n}$ for all $n \geq 0$. Applying the remark before Proposition 3.1 yields the following expansion, valid for $0 < |t| < \pi$ and $z \in \mathbb{C}$:

$$\frac{\sinh(zt)}{\sinh(t)} = \sum_{n=0}^{\infty} t^{2n} \Lambda_n(z). \tag{14}$$

Using Cauchy's residue Theorem with (14), we deduce the integral formula [Whittaker 1934, p. 454–455]:

$$\Lambda_n(z) = (-1)^n \frac{2}{\pi^{2n+1}} \sum_{s=1}^{S} \frac{(-1)^{s+1}}{s^{2n+1}} \sin\bigl(s\pi z\bigr) + \frac{1}{2\pi i} \int_{|t|=(2S+1)\pi/2} t^{-2n-1} \frac{\sinh(zt)}{\sinh(t)} \mathrm{d}t$$

for $S = 1, 2, \dots$ and $z \in \mathbb{C}$. In particular, with $S = 1$ we have

$$\Lambda_n(z) = (-1)^n \frac{2}{\pi^{2n+1}} \sin(\pi z) + \frac{1}{2\pi i} \int_{|t|=3\pi/2} t^{-2n-1} \frac{\sinh(zt)}{\sinh(t)} \mathrm{d}t. \tag{15}$$

3.2 Replacing 0 and 1 with s_0 and s_1

Let s_0 and s_1 be two distinct complex numbers. Define, for $n \geq 0$,

$$\widetilde{\Lambda}_n(z) = (s_1 - s_0)^{2n} \Lambda_n\left(\frac{z}{s_1 - s_0}\right).$$

This sequence of polynomials is also defined by induction by

$$\widetilde{\Lambda}_0(z) = \frac{z}{s_1 - s_0}$$

and, for $n \geq 1$,

$$\widetilde{\Lambda}_n'' = \widetilde{\Lambda}_{n-1}, \quad \widetilde{\Lambda}_n(0) = \widetilde{\Lambda}_n(s_1 - s_0) = 0.$$

Hence

$$\widetilde{\Lambda}_n^{(2k)}(0) = 0 \text{ and } \widetilde{\Lambda}_n^{(2k)}(s_1 - s_0) = \delta_{k,n} \text{ for all } n \geq 0 \text{ and } k \geq 0.$$

It follows that any polynomial $f \in \mathbb{C}[z]$ has an expansion

$$f(z) = \sum_{n=0}^{\infty} \Bigl(f^{(2n)}(s_1) \widetilde{\Lambda}_n(z - s_0) - f^{(2n)}(s_0) \widetilde{\Lambda}_n(z - s_1) \Bigr),$$

with only finitely many nonzero terms in the series.

From (13) we deduce

$$\widetilde{\Lambda}_n(z) = \frac{z^{2n+1}}{(s_1 - s_0)(2n+1)!} - \sum_{h=0}^{n-1} \frac{(s_1 - s_0)^{2n-2h}}{(2n-2h+1)!} \widetilde{\Lambda}_h(z). \tag{16}$$

We will use the following elementary auxiliary lemma.

Lemma 3.2. *There exists an absolute constant $r_0 > 0$ such that, for any $r \geq r_0$ and any t in the interval $0 < t \leq r$, we have*

$$\frac{r}{t}(1 + \log t) + \frac{1}{2} \log t < r + \frac{1}{4r}\cdot$$

Proof. Notice first that the result is true for $0 < t \leq 1$ and $\sqrt{r} \leq t \leq r$.

Let r be an arbitrarily large positive real number. Define, for $t > 0$

$$f(t) = \frac{r}{t}(1 + \log t) + \frac{1}{2} \log t.$$

The derivative f' of f is

$$f'(t) = \frac{1}{2t^2}(t - 2r \log t)$$

and $f'(t)$ has two positive zeroes $1 < t_1 < t_2$, where t_1 is close to 1 while t_2 is close to $2r \log r$ when r is large. Since $f(\mathrm{e}^r) < r = f(1) < f(t_1)$, in the interval $0 < t \leq \mathrm{e}^r$, the function f has its maximum at t_1 with $t_1 = 2r \log t_1$,

$$t_1 = 1 + \frac{1}{2r} + \frac{3}{8r^2} + \frac{1}{3r^3} + O(1/r^4)$$

and

$$\log t_1 = \frac{1}{2r} + \frac{1}{4r^2} + \frac{3}{16r^3} + O(1/r^4)$$

for $r \to \infty$. The maximum is

$$f(t_1) = \frac{r}{t_1} + \frac{1}{2} + \frac{t_1}{4r}$$

and we have

$$\frac{r}{t_1} = \frac{1}{2 \log t_1} = r - \frac{1}{2} - \frac{1}{8r} + O(1/r^2),$$

so that

$$f(t_1) = r + \frac{1}{8r} + O(1/r^2) < r + \frac{1}{4r}$$

for sufficiently large r. □

Setting $t = r/N$ and using the left hand side of Stirling's Formula (11), we deduce from Lemma 3.2:

Corollary 3.3. *For sufficiently large r and for all $N \geq 1$, we have*

$$\frac{r^N}{N!} \leq \frac{\mathrm{e}^{r+(1/4r)}}{\sqrt{2\pi r}}.$$

Corollary 3.3 will be used in the proof of part (ii) of the following result.

Lemma 3.4. *Let s_0 and s_1 be two distinct complex numbers. There exist positive numbers γ_1, γ_2 and γ_3, depending only on s_0 and s_1, such that the following holds.*
(i) *For $r \geq 0$ and $n \geq 0$, we have*

$$|\widetilde{\Lambda}_n|_r \leq \gamma_1 \frac{|s_1 - s_0|^{2n}}{(2n+1)!} \max\left\{\frac{r}{|s_1 - s_0|}, 2n+1\right\}^{2n+1}.$$

(ii) *Assume (3). Then, for sufficiently large r, we have, for all $n \geq 0$,*

$$|\widetilde{\Lambda}_n|_r \leq \gamma_2 \frac{\mathrm{e}^{r+1/(4r)}}{\sqrt{2\pi r}}.$$

(iii) *For $r \geq 0$ and $n \geq 0$,*

$$|\widetilde{\Lambda}_n|_r \leq \gamma_3 \left(\frac{|s_1 - s_0|}{\pi}\right)^{2n} \mathrm{e}^{\frac{3\pi r}{2|s_1 - s_0|}}.$$

Proof.
(i) Let $(\kappa_0, \kappa_1, \kappa_2, \dots)$ be a sequence of positive numbers satisfying $\kappa_0 \geq 1$ and, for $n \geq 1$,

$$\kappa_n \geq 1 + \sum_{h=0}^{n-1} \frac{\kappa_h}{(2n-2h+1)!}.$$

By induction we prove the estimate, for $z \in \mathbb{C}$,

$$|\widetilde{\Lambda}_n(z)| \leq \kappa_n \frac{|s_1 - s_0|^{2n}}{(2n+1)!} \max\left\{\frac{|z|}{|s_1 - s_0|}, 2n+1\right\}^{2n+1}. \tag{17}$$

Formula (17) is true for $n = 0$. Assume that, for some $n \geq 1$, (17) is true for n replaced with $h = 0, 1, \dots, n-1$. Then for $0 \leq h \leq n-1$ we have

$$|\widetilde{\Lambda}_h(z)| \leq \kappa_h \frac{|s_1 - s_0|^{2h}}{(2h+1)!} \max\left\{\frac{|z|}{|s_1 - s_0|}, 2n+1\right\}^{2h+1}.$$

We use the upper bound

$$\frac{(2n+1)!}{(2h+1)!} \leq (2n+1)^{2n-2h}.$$

We deduce, for $0 \leq h \leq n-1$,

$$|s_1 - s_0|^{2n-2h} |\widetilde{\Lambda}_h(z)| \leq \kappa_h \frac{|s_1 - s_0|^{2n}}{(2n+1)!} \max\left\{\frac{|z|}{|s_1 - s_0|}, 2n+1\right\}^{2n+1}.$$

Now (16) implies

$$|\widetilde{\Lambda}_n(z)| \leq \left(1 + \sum_{h=0}^{n-1} \frac{\kappa_h}{(2n-2h+1)!}\right) \frac{|s_1 - s_0|^{2n}}{(2n+1)!} \max\left\{\frac{|z|}{|s_1 - s_0|}, 2n+1\right\}^{2n+1},$$

which proves (17).

We deduce part (i) of Lemma 3.4 by taking for the sequence $(\kappa_h)_{h\geq 0}$ a constant sequence $\kappa_h = \gamma_1$ with

$$\gamma_1 = 1 + \gamma_1 \sum_{\ell \geq 1} \frac{1}{(2\ell+1)!}.$$

This proves (i) with the explicit value

$$\gamma_1 = \frac{2}{4 - \mathrm{e} + \mathrm{e}^{-1}} = 1.212\,416\,8\ldots$$

(ii) Let r be an arbitrarily large positive real number. Let $(\tilde{\kappa}_n)_{n\geq 0}$ be another sequence of positive real numbers satisfying $\tilde{\kappa}_0 \geq 1/|s_1 - s_0|$ and, for $n \geq 1$,

$$\tilde{\kappa}_n \geq \frac{1}{|s_1 - s_0|} + \sum_{h=0}^{n-1} \tilde{\kappa}_h \frac{|s_1 - s_0|^{2n-2h}}{(2n-2h+1)!}. \tag{18}$$

We prove the estimate

$$|\widetilde{\Lambda}_n|_r \leq \tilde{\kappa}_n \frac{\mathrm{e}^{r+(1/4r)}}{\sqrt{2\pi r}}. \tag{19}$$

This is true for $n = 0$, since r is sufficiently large. Assume that it is true for all h with $0 \leq h < n$ for some $n \geq 1$. Using the induction hypothesis with (16), we obtain

$$|\widetilde{\Lambda}_n|_r \leq \frac{r^{2n+1}}{|s_1 - s_0|(2n+1)!} + \frac{\mathrm{e}^{r+(1/4r)}}{\sqrt{2\pi r}} \sum_{h=0}^{n-1} \tilde{\kappa}_h \frac{|s_1 - s_0|^{2n-2h}}{(2n-2h+1)!}.$$

Now (19) follows from (18) and Corollary 3.3. We take for the sequence $(\tilde{\kappa}_h)_{h\geq 0}$ a constant sequence $\tilde{\kappa}_h = \gamma_2$ with

$$\gamma_2 = \frac{1}{|s_1 - s_0|} + \gamma_2 \sum_{\ell \geq 1} \frac{|s_1 - s_0|^{2\ell}}{(2\ell+1)!}.$$

Since (3) can be written

$$4|s_1 - s_0| - \mathrm{e}^{|s_1 - s_0|} + \mathrm{e}^{-|s_1 - s_0|} > 0,$$

we deduce part (ii) of Lemma 3.4 with

$$\gamma_2 = \frac{2}{4|s_1 - s_0| - \mathrm{e}^{|s_1 - s_0|} + \mathrm{e}^{-|s_1 - s_0|}}.$$

(iii) From (15) we deduce

$$|\Lambda_n|_r \leq \mathrm{e}^{(3\pi/2)r}\pi^{-2n}\left(\frac{2}{\pi}\mathrm{e}^{-\pi r/2} + \frac{2^{2n+1}}{3^{2n}} \sup_{|t|=3\pi/2} \frac{1}{|\mathrm{e}^t - \mathrm{e}^{-t}|}\right).$$

The proof of Lemma 3.4 is complete. □

From part (iii) of Lemma 3.4 we deduce the following corollary.

Corollary 3.5. *Assume $|s_1 - s_0| < \pi$. There exists a constant $\gamma_4 > 0$ such that, for r sufficiently large,*

$$\sum_{n \geq \gamma_4 r} |\widetilde{\Lambda}_n|_r < 1.$$

The assumption $|s_1 - s_0| < \pi$ cannot be relaxed: indeed, for $z \notin \mathbb{Z}$, the function $t \mapsto \frac{\sinh(zt)}{\sinh(t)}$ has a pole at $t = i\pi$, hence its Taylor series at the origin (14) has radius of convergence π and is not bounded on the closed disc $|t| \leq \pi$.

Proof of Corollary 3.5. Let N be a positive integer. From part (iii) of Lemma 3.4 we deduce

$$\begin{aligned}\sum_{n\geq N} |\widetilde{\Lambda}_n|_r &\leq \gamma_3 \mathrm{e}^{\frac{3\pi r}{2|s_1-s_0|}} \sum_{n\geq N}\left(\frac{|s_1-s_0|}{\pi}\right)^{2n} \\ &= \frac{\gamma_3\pi^2}{\pi^2 - |s_1-s_0|^2}\mathrm{e}^{\frac{3\pi r}{2|s_1-s_0|}}\left(\frac{|s_1-s_0|}{\pi}\right)^{2N}.\end{aligned}$$

The right hand side is < 1 as soon as

$$\frac{3\pi r}{2|s_1-s_0|} + \log\frac{\gamma_3\pi^2}{\pi^2 - |s_1-s_0|^2} < 2N\log\frac{\pi}{|s_1-s_0|},$$

and this is true for r sufficiently large and $N \geq \gamma_4 r$, provided that

$$\gamma_4 > \frac{3\pi}{4|s_1-s_0|(\log\pi - \log|s_1-s_0|)}.$$

□

4 Derivatives of even order at two points

Proof of Theorem 1.1. Let f satisfy the assumptions of Theorem 1.1. Using Corollary 2.4, we deduce from the assumption (2) that the sets

$$\{n \geq 0 \mid f^{(2n)}(s_0) \neq 0\} \text{ and } \{n \geq 0 \mid f^{(2n)}(s_1) \neq 0\}$$

are finite. Hence

$$P(z) = \sum_{n=0}^{\infty}\left(f^{(2n)}(s_1)\widetilde{\Lambda}_n(z-s_0) - f^{(2n)}(s_0)\widetilde{\Lambda}_n(z-s_1)\right)$$

is a polynomial satisfying

$$P^{(2n)}(s_0) = f^{(2n)}(s_0) \text{ and } P^{(2n)}(s_1) = f^{(2n)}(s_1) \text{ for all } n \geq 0.$$

The function $\tilde{f}(z) = f(z) - P(z)$ has the same exponential type as f and satisfies

$$\tilde{f}^{(2n)}(s_0) = \tilde{f}^{(2n)}(s_1) = 0 \text{ for all } n \geq 0.$$

Set

$$\hat{f}(z) = \tilde{f}\big(s_0 + z(s_1 - s_0)\big),$$

so that

$$\hat{f}^{(2n)}(0) = \hat{f}^{(2n)}(1) = 0 \text{ for all } n \geq 0.$$

The exponential types of f and $\hat{f}$ are related by

$$\tau(\hat{f}) = |s_1 - s_0|\tau(f).$$

From Proposition 3.1 we deduce that there exist complex numbers $c_1, c_2, \dots, c_L$ such that

$$\hat{f}(z) = \sum_{\ell=1}^{L} c_\ell \sin(\ell\pi z),$$

and therefore

$$\tilde{f}(z) = \sum_{\ell=1}^{L} c_\ell \sin\left(\ell\pi \frac{z - s_0}{s_1 - s_0}\right).$$

Theorem 1.1 follows. □

Proof of Theorem 1.4. Assume $|s_1 - s_0| < \pi$. From Proposition 3.1, it follows that an entire function f of exponential type ≤ 1 for which $f^{(2n)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$ for all sufficiently large n is of the form

$$f(z) = \sum_{n=0}^{\infty} \left(f^{(2n)}(s_1)\widetilde{\Lambda}_n(z - s_0) - f^{(2n)}(s_0)\widetilde{\Lambda}_n(z - s_1)\right),$$

and also that f is not a polynomial if and only if one at least of the two sets $\{n \geq 0 \mid f^{(2n)}(s_0) \neq 0\}$, $\{n \geq 0 \mid f^{(2n)}(s_1) \neq 0\}$ is infinite. We construct such functions by requiring $f^{(2n)}(s_0) = 0$ for all $n \geq 0$ and $f^{(2n)}(s_1) = 0$ for all $n \geq 0$ outside a lacunary sequence.

Define, for $k \geq 0$, $N_k = \gamma_4^{2^k - 1}$, where γ_4 is the constant in Corollary 3.5, so that $N_0 = 1$ and $N_{k+1} = \gamma_4 N_k^2$. For $n \geq 1$, let $e_n = 0$ if $N_k < n < N_{k+1}$, and $e_{N_k} \in \{+1, -1\}$ for $k \geq 0$, so that there is an uncountable set of such lacunary sequences $(e_n)_{n\geq 0}$. Using Lemma 3.4 we will prove that

$$f(z) := \sum_{n\geq 1} e_n \widetilde{\Lambda}_n(z - s_0)$$

defines an entire function which satisfies (4), hence has order ≤ 1. It will follow that we have $f^{(2n)}(s_0) = 0$, $f^{(2n)}(s_1) = e_n$ for all $n \geq 0$. Since infinitely many e_n are not 0, this function f is transcendental.

It remains to check the upper bound for $|f|_r$. Let $r \geq |s_0|$ be an arbitrary large positive number. Let k be the least positive integer such that $N_k > \sqrt{r + |s_0|}$. From part (i) of Lemma 3.4, using the bounds

$$N_{k-1} \leq \sqrt{r + |s_0|} \leq \sqrt{2r},$$

we deduce, for sufficiently large r,

$$\begin{aligned}\sum_{n<N_k} |e_n|\ |\widetilde{\Lambda}_n|_{r+|s_0|} &\leq \sum_{1\leq n\leq N_{k-1}} |\widetilde{\Lambda}_n|_{r+|s_0|} \\ &< \gamma_1 \frac{N_{k-1}}{|s_1 - s_0|}(2r)^{2N_{k-1}+1} \\ &\leq \gamma_1 r^{3\sqrt{r}} \\ &< \frac{\mathrm{e}^r}{r}\cdot\end{aligned}$$

Assuming (3), we can use part (ii) of Lemma 3.4 and get

$$|\widetilde{\Lambda}_{N_k}|_{r+|s_0|} \leq \gamma_2 \frac{\mathrm{e}^{r+|s_0|+1/(4r)}}{\sqrt{2\pi r}}\cdot$$

Since $\gamma_4(r + |s_0|) \leq \gamma_4 N_k^2 = N_{k+1}$, Corollary 3.5 yields

$$\sum_{n>N_k} |e_n|\ |\widetilde{\Lambda}_n|_{r+|s_0|} \leq \sum_{n\geq N_{k+1}} |\widetilde{\Lambda}_n|_{r+|s_0|} < 1.$$

Combining these three estimates, we conclude

$$\limsup_{r\to\infty} \mathrm{e}^{-r}\sqrt{r}|f|_r \leq \gamma \text{ with } \gamma = \gamma_2 \frac{\mathrm{e}^{|s_0|}}{\sqrt{2\pi}},$$

which is an explicit version of (4):

$$\gamma = \frac{\mathrm{e}^{|s_0|}}{\sqrt{2\pi}} \cdot \frac{2}{4|s_1 - s_0| - \mathrm{e}^{|s_1-s_0|} + \mathrm{e}^{-|s_1-s_0|}}\cdot$$

□

5 Derivatives of odd order at two points

Proof of Theorem 1.5. Let f satisfy the assumptions of Theorem 1.5. Using Corollary 2.4, we deduce from the assumption (2) that the sets

$$\{n \geq 0 \ \mid\ f^{(2n+1)}(s_0) \neq 0\} \text{ and } \{n \geq 0 \ \mid\ f^{(2n+1)}(s_1) \neq 0\}$$

are finite.

Let Q be a primitive of the polynomial

$$\sum_{n=0}^{\infty}\Big(f^{(2n+1)}(s_1)\widetilde{\Lambda}_n(z-s_0)-f^{(2n+1)}(s_0)\widetilde{\Lambda}_n(z-s_1)\Big).$$

We have

$$Q^{(2n+1)}(s_0)=f^{(2n+1)}(s_0)\ \text{ and }\ Q^{(2n+1)}(s_1)=f^{(2n+1)}(s_1)\ \text{ for all }\ n\geq 0,$$

hence the function $\tilde{f}(z)=f(z)-Q(z)$ satisfies

$$\tilde{f}^{(2n+1)}(s_0)=\tilde{f}^{(2n+1)}(s_1)=0\ \text{ for all }\ n\geq 0.$$

Set

$$\hat{f}(z)=\tilde{f}\big(s_0+z(s_1-s_0)\big),$$

so that

$$\hat{f}^{(2n+1)}(0)=\hat{f}^{(2n+1)}(1)=0\ \text{ for all }\ n\geq 0.$$

From Proposition 3.1 we deduce that there exist complex numbers $c'_1,c'_2,\dots,c'_L$ with $L\pi\leq\tau(\hat{f}')$ such that

$$\hat{f}'(z)=\sum_{\ell=1}^{L}c'_\ell\sin(\ell\pi z).$$

The exponential types of f, $\tilde{f}$, $\hat{f}$ and $\hat{f}'$ are related by

$$\tau(f)=\tau(\tilde{f})\quad\text{and}\quad\tau(\hat{f}')=\tau(\hat{f})=|s_1-s_0|\tau(\tilde{f}).$$

Theorem 1.5 follows. □

6 Whittaker polynomials

6.1 Definition and properties

We now consider the set $\mathscr{S}=(\{0\}\times(2\mathbb{N}+1))\cup(\{1\}\times 2\mathbb{N})\subset\{0,1\}\times\mathbb{N}$: we take odd derivatives at 0 and even derivatives at 1. The analogs of Lidstone polynomials have been introduced by [Whittaker 1934, §6 p. 457–458], and studied by [Schoenberg 1936]. See also [Gel'fond 1952, Chap. III §4].

Following [Whittaker 1934], one defines a sequence $(M_n)_{n\geq 0}$ of even polynomials by induction on n with $M_0=1$,

$$M''_n=M_{n-1},\quad M_n(1)=M'_n(0)=0\text{ for all }n\geq 1.$$

For all $n\geq 0$, the polynomial M_n is even of degree $2n$ and leading term $\frac{1}{(2n)!}z^{2n}$. From the definition one deduces

$$M_n^{(2k+1)}(0)=0\text{ and }M_n^{(2k)}(1)=\delta_{k,n}\text{ for all }n\geq 0\text{ and }k\geq 0.$$

As a consequence, any polynomial $f \in \mathbb{C}[z]$ has an expansion

$$f(z) = \sum_{n=0}^{\infty} \left(f^{(2n)}(1) M_n(z) - f^{(2n+1)}(0) M'_{n+1}(1-z) \right), \tag{20}$$

with only finitely many nonzero terms in the series.

Applying (20) to the polynomial z^{2n} yields the following recurrence formula:

$$M_n(z) = \frac{1}{(2n)!} z^{2n} - \sum_{h=0}^{n-1} \frac{1}{(2n-2h)!} M_h(z). \tag{21}$$

For instance

$$M_1(z) = \frac{1}{2}(z^2 - 1), \quad M_2(z) = \frac{1}{24}(z^4 - 6z^2 + 5) = \frac{1}{24}(z^2 - 1)(z^2 - 5),$$

$$M_3(z) = \frac{1}{720}(z^6 - 15z^4 + 75z^2 - 61) = \frac{1}{720}(z^2 - 1)(z^4 - 14z^2 + 61).$$

Whittaker [Whittaker 1934, § 6], proved that the expansion (20) holds for entire functions of exponential type $< \pi/2$. Here is the analog of Proposition 3.1 for Whittaker polynomials [Schoenberg 1936, Th. 2]:

Proposition 6.1. *Let f be an entire function of finite exponential type $\tau(f)$ satisfying $f^{(2n+1)}(0) = f^{(2n)}(1) = 0$ for all $n \geq 0$. Then there exist complex numbers $c_0, \ldots, c_L$ with $(2L+1)\pi/2 \leq \tau(f)$ such that*

$$f(z) = \sum_{\ell=0}^{L} c_\ell \cos\left(\frac{(2\ell+1)\pi}{2} z \right).$$

Therefore, if an entire function f has exponential type $< \pi/2$ and satisfies $f^{(2n+1)}(0) = f^{(2n)}(1) = 0$ for all $n \geq 0$, then $f = 0$.

In (14), we considered, for $t \in \mathbb{C}$, $t \notin i\pi\mathbb{Z}$, the entire function $z \mapsto \frac{\sinh(zt)}{\sinh(t)}$; now we consider, for $t \in \mathbb{C}$, $t \notin i\frac{\pi}{2} + i\pi\mathbb{Z}$, the entire function

$$f(z) = \frac{\cosh(zt)}{\cosh(t)} = \frac{\mathrm{e}^{zt} + \mathrm{e}^{-zt}}{\mathrm{e}^{t} + \mathrm{e}^{-t}},$$

which satisfies

$$f'' = t^2 f, \quad f(1) = 1, \quad f'(0) = 0,$$

hence $f^{(2n)}(1) = t^{2n}$ and $f^{(2n+1)}(0) = 0$ for all $n \geq 0$. From Proposition 6.1 and the result of Whittaker quoted just before that proposition, it follows that the sequence $(M_n)_{n \geq 0}$ is also defined by the expansion

$$\frac{\cosh(zt)}{\cosh(t)} = \sum_{n=0}^{\infty} t^{2n} M_n(z) \tag{22}$$

for $|t| < \pi/2$ and $z \in \mathbb{C}$.

Using Cauchy's residue Theorem, we deduce from (22) the integral formula

$$M_n(z) = (-1)^n \frac{2^{2n+2}}{\pi^{2n+1}} \sum_{s=0}^{S-1} \frac{(-1)^s}{(2s+1)^{2n+1}} \cos\left(\frac{(2s+1)\pi}{2} z\right) + \frac{1}{2\pi i} \int_{|t|=S\pi} t^{-2n-1} \frac{\cosh(zt)}{\cosh(t)} \mathrm{d}t$$

for $S = 1, 2, \ldots$ and $z \in \mathbb{C}$. In particular, with $S = 1$ we obtain

$$M_n(z) = (-1)^n \frac{2^{2n+2}}{\pi^{2n+1}} \cos(\pi z/2) + \frac{1}{2\pi i} \int_{|t|=\pi} t^{-2n-1} \frac{\cosh(zt)}{\cosh(t)} \mathrm{d}t. \tag{23}$$

6.2 Replacing 0 and 1 with s_0 and s_1

Let s_0 and s_1 be two distinct complex numbers. Define, for $n \geq 0$,

$$\widetilde{M}_n(z) = (s_1 - s_0)^{2n} M_n\left(\frac{z}{s_1 - s_0}\right).$$

This sequence of polynomials is also defined by induction by $\widetilde{M}_0(z) = 1$ and, for $n \geq 1$,

$$\widetilde{M}_n'' = \widetilde{M}_{n-1}, \quad \widetilde{M}_n'(0) = \widetilde{M}_n(s_1 - s_0) = 0.$$

Hence

$$\widetilde{M}_n^{(2k+1)}(0) = 0 \text{ and } \widetilde{M}_n^{(2k)}(s_1 - s_0) = \delta_{k,n} \text{ for all } n \geq 0 \text{ and } k \geq 0.$$

It follows that any polynomial $f \in \mathbb{C}[z]$ has an expansion

$$f(z) = \sum_{n=0}^{\infty} \left(f^{(2n)}(s_1)\widetilde{M}_n(z - s_0) + f^{(2n+1)}(s_0)\widetilde{M}_{n+1}'(z - s_1)\right),$$

with only finitely many nonzero terms in the series.

From (20) we deduce

$$\widetilde{M}_n(z) = \frac{z^{2n}}{(2n)!} - \sum_{h=0}^{n-1} \frac{(s_1 - s_0)^{2n-2h}}{(2n-2h)!} \widetilde{M}_h(z). \tag{24}$$

Here is the analog of Lemma 3.4 for the sequence of polynomials $\widetilde{M}_n$:

Lemma 6.2. *Let s_0, s_1 be two distinct complex numbers. There exist positive contants γ_1', γ_2' and γ_3' such that the following holds.*
(i) *For $r \geq 0$ and $n \geq 0$, we have*

$$|\widetilde{M}_n|_r \leq \gamma_1' \frac{|s_1 - s_0|^{2n}}{(2n)!} \max\left\{\frac{r}{|s_1 - s_0|}, 2n\right\}^{2n}.$$

(ii) *Assume* (5). *Then, for sufficiently large r and for all $n \geq 0$,*

$$|\widetilde{M}_n|_r \leq \gamma_2' \frac{\mathrm{e}^{r+1/(4r)}}{\sqrt{2\pi r}}.$$

(iii) *For $r \geq 0$ and $n \geq 0$,*

$$|\widetilde{M}_n|_r \leq \gamma_3' \left(\frac{2|s_1 - s_0|}{\pi} \right)^{2n} \mathrm{e}^{\frac{\pi r}{|s_1 - s_0|}}.$$

Proof.
(i) Let $(\kappa_0', \kappa_1', \kappa_2', \dots)$ be a sequence of positive numbers satisfying $\kappa_0' \geq 1$ and, for $n \geq 1$,

$$\kappa_n' \geq 1 + \sum_{h=0}^{n-1} \frac{\kappa_h'}{(2n - 2h)!}.$$

By induction we prove the estimate, for $z \in \mathbb{C}$,

$$|\widetilde{M}_n(z)| \leq \kappa_n' \frac{|s_1 - s_0|^{2n}}{(2n)!} \max \left\{ \frac{|z|}{|s_1 - s_0|}, 2n \right\}^{2n}. \tag{25}$$

This is true for $n = 0$ (and $z \neq 0$). Assume that, for some $n \geq 1$, (25) is true for n replaced with $h = 0, 1, \dots, n - 1$. Then for $0 \leq h \leq n - 1$ we have

$$|\widetilde{M}_h(z)| \leq \kappa_h' \frac{|s_1 - s_0|^{2h}}{(2h)!} \max \left\{ \frac{|z|}{|s_1 - s_0|}, 2n \right\}^{2h}.$$

We use the upper bound

$$\frac{(2n)!}{(2h)!} \leq (2n)^{2n-2h}.$$

We deduce, for $0 \leq h \leq n - 1$,

$$|s_1 - s_0|^{2n-2h} |\widetilde{M}_h(z)| \leq \kappa_h' \frac{|s_1 - s_0|^{2n}}{(2n)!} \max \left\{ \frac{|z|}{|s_1 - s_0|}, 2n \right\}^{2n}.$$

Now (24) implies

$$|\widetilde{M}_n(z)| \leq \left(1 + \sum_{h=0}^{n-1} \frac{\kappa_h'}{(2n - 2h)!} \right) \frac{|s_1 - s_0|^{2n}}{(2n)!} \max \left\{ \frac{|z|}{|s_1 - s_0|}, 2n \right\}^{2n},$$

which proves (25).

We deduce part (i) of Lemma 6.2 by taking for the sequence $(\kappa_h')_{h \geq 0}$ a constant sequence $\kappa_h' = \gamma_1'$ with

$$\gamma_1' = 1 + \gamma_1' \sum_{\ell \geq 1} \frac{1}{(2\ell)!}.$$

This proves (i) with the explicit value

$$\gamma_1' = \frac{2}{4 - \mathrm{e} - \mathrm{e}^{-1}} = 2.188\,569\,9\dots$$

(ii) Fix r sufficiently large. Let $(\tilde{\kappa}_n')_{n\geq 0}$ be another sequence satisfying $\tilde{\kappa}_0' > 0$ and, for $n \geq 1$,

$$\tilde{\kappa}_n' \geq 1 + \sum_{h=0}^{n-1} \tilde{\kappa}_h' \frac{|s_1 - s_0|^{2n-2h}}{(2n-2h)!}. \tag{26}$$

We prove the estimate

$$|\widetilde{M}_n|_r \leq \tilde{\kappa}_n' \frac{\mathrm{e}^{r+(1/4r)}}{\sqrt{2\pi r}}. \tag{27}$$

This is true for $n = 0$, since r is sufficiently large and $\tilde{\kappa}_0' > 0$. Assume that, for some $n \geq 1$, (27) is true for all h with $0 \leq h < n$. Using the induction hypothesis with (24), we obtain

$$|\widetilde{M}_n|_r \leq \frac{r^{2n}}{(2n)!} + \frac{\mathrm{e}^{r+(1/4r)}}{\sqrt{2\pi r}} \sum_{h=0}^{n-1} \tilde{\kappa}_h' \frac{|s_1 - s_0|^{2n-2h}}{(2n-2h)!}.$$

Now (27) follows from (26) and Corollary 3.3. We take for the sequence $(\tilde{\kappa}_h')_{h\geq 0}$ a constant sequence $\tilde{\kappa}_h' = \gamma_2'$ with

$$\gamma_2' = 1 + \gamma_2' \sum_{\ell\geq 1} \frac{|s_1 - s_0|^{2\ell}}{(2\ell)!}.$$

Since (5) can be written

$$\mathrm{e}^{|s_1-s_0|} + \mathrm{e}^{-|s_1-s_0|} < 4,$$

this implies part (ii) of Lemma 6.2 with

$$\gamma_2' = \frac{2}{4 - \mathrm{e}^{|s_1-s_0|} - \mathrm{e}^{-|s_1-s_0|}},$$

provided that r is sufficiently large.

(iii) From the integral formula (23) one deduces the upper bound:

$$|M_n|_r \leq \left(\frac{2}{\pi}\right)^{2n} \mathrm{e}^{\pi r} \left(\frac{4}{\pi}\mathrm{e}^{-\pi r/2} + 2^{-2n+1} \sup_{|t|=\pi} \frac{1}{|\mathrm{e}^t + \mathrm{e}^{-t}|}\right).$$

The proof of Lemma 6.2 is complete. □

From part (iii) of Lemma 6.2 we deduce the following corollary.

Corollary 6.3. *Assume $|s_1 - s_0| < \pi/2$. There exists a constant $\gamma_4' > 0$ such that, for r sufficiently large,*

$$\sum_{n\geq \gamma_4' r} |\widetilde{M}_n|_r < 1.$$

From (22) it follows that the assumption $|s_1 - s_0| < \pi/2$ cannot be relaxed.

Proof of Corollary 6.3. Let N be a positive integer. From part (iii) of Lemma 6.2 we deduce

$$\begin{aligned}\sum_{n\geq N} |\widetilde{M}_n|_r &\leq \gamma_3' \mathrm{e}^{\frac{\pi r}{|s_1-s_0|}} \sum_{n\geq N} \left(\frac{2|s_1-s_0|}{\pi}\right)^{2n} \\ &= \frac{\gamma_3'\pi^2}{\pi^2 - 4|s_1-s_0|^2} \mathrm{e}^{\frac{\pi r}{|s_1-s_0|}} \left(\frac{2|s_1-s_0|}{\pi}\right)^{2N}.\end{aligned}$$

The right hand side is < 1 as soon as

$$\frac{\pi r}{|s_1-s_0|} + \log \frac{\gamma_3'\pi^2}{\pi^2 - 4|s_1-s_0|^2} < 2N \log \frac{\pi}{2|s_1-s_0|},$$

and this is true for r sufficiently large and $N \geq \gamma_4' r$, provided that

$$\gamma_4' > \frac{\pi}{2|s_1-s_0|(\log \pi - \log(2|s_1-s_0|))}.$$

□

7 Derivatives of odd order at one point and even at the other

7.1 Proof of Theorem 1.6

Proof of Theorem 1.6. Let f satisfy the assumptions of Theorem 1.6. Using the assumption (2), we deduce from Corollary 2.4 that the sets

$$\{n \geq 0 \mid f^{(2n+1)}(s_0) \neq 0\} \text{ and } \{n \geq 0 \mid f^{(2n)}(s_1) \neq 0\}$$

are finite. Hence

$$P(z) = \sum_{n=0}^{\infty} \left(f^{(2n)}(s_1)\widetilde{M}_n(z-s_0) + f^{(2n+1)}(s_0)\widetilde{M}'_{n+1}(z-s_1)\right)$$

is a polynomial satisfying

$$P^{(2n+1)}(s_0) = f^{(2n+1)}(s_0) \text{ and } P^{(2n)}(s_1) = f^{(2n)}(s_1) \text{ for all } n \geq 0.$$

The function $\tilde{f}(z) = f(z) - P(z)$ has the same exponential type as f and satisfies

$$\tilde{f}^{(2n+1)}(s_0) = \tilde{f}^{(2n)}(s_1) = 0 \text{ for all } n \geq 0.$$

Set

$$\hat{f}(z) = \tilde{f}\big(s_0 + z(s_1-s_0)\big),$$

so that

$$\hat{f}^{(2n+1)}(0) = \hat{f}^{(2n)}(1) = 0 \quad \text{for all} \quad n \geq 0.$$

The exponential types of f and $\hat{f}$ are related by

$$\tau(\hat{f}) = |s_1 - s_0|\tau(f).$$

From Proposition 6.1 we deduce that there exists complex numbers $c_0, c_1, \ldots, c_L$ with $(2L+1)\pi/2 \leq \tau(\hat{f})$ such that

$$\hat{f}(z) = \sum_{\ell=0}^{L} c_\ell \cos\left(\frac{(2\ell+1)\pi}{2} z\right),$$

and therefore

$$\tilde{f}(z) = \sum_{\ell=0}^{L} c_\ell \cos\left(\frac{(2\ell+1)\pi}{2} \cdot \frac{z - s_0}{s_1 - s_0}\right).$$

Theorem 1.6 follows. □

7.2 Proof of Theorem 1.8

Proof of Theorem 1.8. Assume (5). Define, for $k \geq 0$, $N_k = (\gamma_4')^{2^k-1}$, where γ_4' is the constant in Corollary 6.3, so that $N_0 = 1$ and $N_{k+1} = \gamma_4' N_k^2$. For $n \geq 1$, let $e_n = 0$ if $N_k < n < N_{k+1}$, and $e_{N_k} \in \{+1, -1\}$ for $k \geq 0$, so that there is an uncountable set of such lacunary sequences $(e_n)_{n \geq 0}$. Define

$$f(z) := \sum_{n \geq 1} e_n \widetilde{M}_n(z - s_0).$$

Let us check the upper bound for $|f|_r$.

Let r be a sufficiently large positive number. Let k be the least positive integer such that $N_k > \sqrt{r + |s_0|}$. From part (i) of Lemma 6.2, using the bound $N_{k-1} \leq \sqrt{r + |s_0|} \leq \sqrt{2r}$, we deduce, for sufficiently large r,

$$\begin{aligned}
\sum_{n < N_k} |e_n|\, |\widetilde{M}_n|_{r+|s_0|} &\leq \sum_{1 \leq n \leq N_{k-1}} |\widetilde{M}_n|_{r+|s_0|} \\
&< \gamma_1' N_{k-1} (2r)^{2N_{k-1}} \\
&\leq \gamma_1' r^{3\sqrt{r}} \\
&< \frac{\mathrm{e}^r}{r}.
\end{aligned}$$

Assuming (5), we can use part (ii) of Lemma 6.2 and get

$$|\widetilde{M}_{N_k}|_{r+|s_0|} \leq \gamma_2' \frac{\mathrm{e}^{r+|s_0|+1/(4r)}}{\sqrt{2\pi r}}.$$

Since $\gamma_4'(r+|s_0|) < \gamma_4' N_k^2 = N_{k+1}$, Corollary 6.3 yields

$$\sum_{n>N_k} |e_n| \, |\widetilde{M}_n|_{r+|s_0|} \leq \sum_{n\geq N_{k+1}} |\widetilde{M}_n|_{r+|s_0|} < 1.$$

Combining these three estimates, we conclude

$$\limsup_{r\to\infty} \mathrm{e}^{-r}\sqrt{r}|f|_r \leq \gamma' \text{ with } \gamma' = \gamma_2' \frac{\mathrm{e}^{|s_0|}}{\sqrt{2\pi}},$$

which is an explicit version of (6):

$$\gamma' = \frac{\mathrm{e}^{|s_0|}}{\sqrt{2\pi}} \cdot \frac{2}{4 - \mathrm{e}^{|s_1-s_0|} - \mathrm{e}^{-|s_1-s_0|}}.$$

We deduce that f has order ≤ 1 and that $f^{(2n+1)}(s_0) = 0$, $f^{(2n)}(s_1) = e_n$ for all $n \geq 0$. $\square$

8 Sequence of derivatives

8.1 Proofs of Theorem 1.9, Proposition 1.10 and Corollary 1.11

The proof of Theorem 1.9 relies on the following result of [Gontcharoff 1930, Chapter IV, §9] and [Macintyre 1954, §4]. See also [Whittaker 1935, Chap. III] and [Gel'fond 1952, Chap. 3].

Proposition 8.1. *Let $\sigma_0, \sigma_1, \dots, \sigma_{m-1}$ be complex numbers and let τ be defined accordingly as in § 1.4. If f is an entire function of exponential type $< \tau$ satisfying*

$$f^{(mn+j)}(\sigma_j) = 0 \text{ for } j = 0, \dots, m-1 \text{ and all sufficiently large } n,$$

then f is a polynomial

Proof of Theorem 1.9. Using (7) and Corollary 2.4, we deduce from the assumptions of Theorem 1.9 that

$$f^{(mn+j)}(\sigma_j) = 0$$

for all sufficiently large n. It follows from Proposition 8.1 and the assumption $\tau(f) < \tau$ that $f(z)$ is a polynomial. $\square$

Proof of Proposition 1.10. (a) Assume $\Delta(\alpha) = 0$: the $m \times m$ matrix

$$\left(\zeta^{k\ell}\mathrm{e}^{\zeta^k\alpha\sigma_\ell}\right)_{0\leq k,\ell\leq m-1}$$

has rank $< m$. There exists $c_0, c_1, \dots, c_{m-1}$ in $\mathbb{C}$, not all zero, such that the function

$$f(z) = c_0\mathrm{e}^{\alpha z} + c_1\mathrm{e}^{\zeta\alpha z} + \cdots + c_{m-1}\mathrm{e}^{\zeta^{m-1}\alpha z}$$

satisfies

$$f^{(j)}(\sigma_j) = 0 \text{ for } j = 0, 1, \dots, m-1.$$

Since $f^{(m)}(z) = \alpha^m f(z)$, one deduces

$$f^{(mn+j)}(\sigma_j) = 0 \text{ for } j = 0, 1, \ldots, m-1 \text{ and } n \geq 0.$$

(b) Assume $\Delta(1) \neq 0$. For $j = 0, 1, \ldots, m-1$, there exists a unique m–tuple of complex numbers $(c_{j0}, c_{j1}, \ldots, c_{j,m-1})$ such that the function

$$\varphi_j(z) = \sum_{k=0}^{m-1} c_{jk} \mathrm{e}^{\zeta^k z}$$

satisfies

$$\varphi_j^{(\ell)}(\sigma_\ell) = \delta_{j\ell} \quad \text{for} \quad 0 \leq \ell \leq m-1.$$

For $j = 0, 1, \ldots, m-1$, the function $\varphi_j(z)$ has exponential type 1 and is a solution of the differential equation $\varphi_j^{(m)} = \varphi_j$. Let $a_0, a_1, \ldots, a_{m-1}$ in $\mathbb{C}$. Define

$$f(z) = a_0\varphi_0(z) + a_1\varphi_1(z) + \cdots + a_{m-1}\varphi_{m-1}(z).$$

We have

$$f^{(mn+j)}(\sigma_j) = a_j \text{ for } j = 0, 1, \ldots, m-1 \text{ and } n \geq 0.$$

Assume now $\tau > 1$: according to Proposition 8.1, for $a_0 = a_1 = \cdots = a_{m-1} = 0$, the unique solution of exponential type $< \tau$ is $f = 0$. The unicity follows. □

Proof of Corollary 1.11. In case $\sigma_0 = 1$, $\sigma_1 = \sigma_2 = \cdots = \sigma_{m-1} = 0$, the determinant $\Delta(t)$ is

$$\det \begin{pmatrix} \mathrm{e}^t & 1 & 1 & \cdots & 1 \\ \mathrm{e}^{\zeta t} & \zeta & \zeta^2 & \cdots & \zeta^{m-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathrm{e}^{\zeta^{m-1} t} & \zeta^{m-1} & \zeta^{2(m-1)} & \cdots & \zeta^{(m-1)^2} \end{pmatrix}.$$

This determinant is invariant under the transformation $t \mapsto \zeta t$; hence $\Delta(t)$ is a nonzero constant times

$$\mathrm{e}^t + \mathrm{e}^{\zeta t} + \cdots + \mathrm{e}^{\zeta^{m-1} t} = m \sum_{n \geq 0} \frac{t^{nm}}{(nm)!}.$$

Now Corollary 1.11 follows from Theorem 1.9 with $\tau = \tau_m / |s_1 - s_0|$. □

As pointed out by [Macintyre 1954, p. 12], a special case of the results of [Gontcharoff 1930] is that an entire function of exponential type $< \tau_m$ satisfying

$$f^{(n)}(0) = 0 \text{ for } n \equiv 0 \bmod m \quad \text{and} \quad f^{(n)}(1) = 0 \text{ for } n \not\equiv 0 \bmod m$$

is a polynomial. A.J. Macintyre remarks that τ_m is approximately m/e when m is large; he suggests an analogy with Taylor's series which may be considered as the limiting case with $m = \infty$. For Corollary 1.11, when m is large, the assumption (2) implies the assumption on $\tau(f)$. Hence Proposition 2.1 can be viewed as the limiting case of Corollary 1.11.

8.2 Proof of Theorem 1.12

The proof of Theorem 1.12 relies on the following result [Whittaker 1934, Corollary of Theorem 7, p. 468]:

Proposition 8.2. *If an entire function f of exponential type $\tau(f) < 1$ satisfies*

$$f^{(n)}(0)f^{(n)}(1) = 0$$

for all sufficiently large n, then f is a polynomial.

As pointed out in a note added in proof of [Whittaker 1934, p. 469], [Gontcharoff 1930] proved this result earlier under the stronger assumption $\tau(f) < 1/\mathrm{e}$.

Proof of Theorem 1.12. Since f satisfies (2), the assumption of Corollary 2.4 is satisfied, hence $|f^{(n)}(s_j)| < 1$ for n sufficiently large and $j = 0, 1$. Let n be sufficiently large. One at least of the three numbers $f^{(n)}(s_0)$, $f^{(n)}(s_1)$, $f^{(n)}(s_0)f^{(n)}(s_1)$ is an integer of absolute value less than 1, hence it vanishes and therefore the product $f^{(n)}(s_0)f^{(n)}(s_1)$ vanishes. We apply Proposition 8.2 to the function

$$\hat{f}(z) = f\big(s_0 + z(s_1 - s_0)\big),$$

the exponential type of which is $|s_1 - s_0|\tau(f) < 1$.

This completes the proof of Theorem 1.12. □

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