

Introduction to Transcendental Number Theory and to two Conjectures due to Nagata.

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We start by stating the so-called Schneider–Lang criterion for transcendence and showing how it contains the Theorem of Hermite–Lindemann on the transcendence of e , π , and more generally of e^α and $\log \alpha$ for α algebraic, as well as the Theorem of Gel’fond–Schneider on the transcendence of $2^{\sqrt{2}}$, e^π , and more generally of α^β and $(\log \alpha_1)/(\log \alpha_2)$ for α, β, α_1 and α_2 algebraic. Other consequences (Schneider’s results) deal with values of elliptic and quasi elliptic functions.

We will give a detailed sketch of proof and introduce the main tools: Thue–Siegel’s Lemma on the existence of solutions to a system of homogeneous linear equations in integers with integer coefficients, Liouville type lower bounds for non-zero algebraic numbers, zero estimates, Schwarz’s Lemma which produces an upper bound for an analytic function having many zeroes. We will say a few words on interpolation determinants of Laurent, which are a substitute to the auxiliary functions arising from Dirichlet’s box principle.

We consider the question of extending Schwarz’s Lemma to several variables. The solution for Cartesian products is elementary and has a number of applications related with abelian functions: we quote some results in this direction without proof. A more elaborate solution involves L_2 estimates for the solution of the $\bar{\partial}$ equation by Hörmander and Bombieri. We will take for granted this generalized Schwarz’s Lemma and explain its consequences to the investigation of some invariant related to the degree of hypersurfaces with given singularities. It is in this part of the course that two conjectures due to Nagata are introduced.

We will pursue the course with a historical survey of transcendental number theory. Among the topics which will be covered are the Lindemann–Weierstrass theorem, results on Siegel’s E and G functions, Baker’s Theorem on linear independence of logarithms of algebraic numbers, algebraic independence results (Gel’fond, Chudnovski, Nesterenko), Schanuel’s Conjecture. The main special case of Schanuel’s Conjecture is the conjecture on the algebraic independence of logarithms of algebraic numbers. We will explain the state of the art: the six exponentials Theorem (and the four exponentials Conjecture) as well as their generalizations in several variables.

We will present the state of the art on the arithmetic properties of special zeta values and multiple zeta values.

We will also introduce results on Diophantine approximation with some famous theorems, including those due to Dirichlet, Liouville, Thue, Roth and Schmidt. We will discuss questions of effectivity involving Diophantine equations and lower bounds for linear combinations of logarithms, together with some applications.

Finally we will discuss the *abc* Conjecture.