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Transcendental Number Theory: the State of the Art

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Early History *(≤ 1934)*

J. Liouville(1844) – *First examples of transcendental numbers.*

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$$\sum_{n\geq 1} 10^{-n!} \quad \textit{is transcendental}$$

Idea of the proof. If α is an algebraic number, root of a polynomial in $\mathbf{Z}[X]$ of degree d, there exists a constant $c = c(\alpha) > 0$ such that, for any rational number $p/q \neq \alpha$,

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c}{q^d}.$$

Proof of Liouville's inequality

Since $\alpha \neq p/q$, α is root of a polynomial $P \in \mathbf{Z}[X]$ of degree $\leq d$ which does not vanish at p/q. The number $q^d P(p/q)$ is a non-zero rational integer, hence

$$|P(p/q)| \geq q^{-d}.$$

On the other hand

$$|P(p/q)| = |P(p/q) - P(\alpha)| \leq c(\alpha) \left| \alpha - \frac{p}{q} \right|.$$

Proof of the transcendence of

$$\theta = \sum_{n\geq 1} 10^{-n!}$$

For N sufficiently large, set $q = 10^{N!}$ and

$$p = \sum_{n=1}^{N} 10^{N!-n!}.$$

Then

$$0 < \theta - \frac{p}{q} = \sum_{n\geq N+1} 10^{-n!} \leq 2 \cdot 10^{-(N+1)!} = \frac{2}{q^{N+1}}.$$

Same argument by Liouville: *For* $\ell \geq 2$ *in* $\mathbf{Z}$, *irrationality of*

$$\theta' = \sum_{n\geq 1} \ell^{-n^2}$$

For N sufficiently large, set $q = \ell^{N^2}$ and $p = \sum_{n=1}^{N} \ell^{N^2-n^2}$. Then

$$0 < \theta' - \frac{p}{q} = \sum_{n\geq N+1} \ell^{-n^2} \leq 2 \cdot \ell^{-(N+1)^2} = \frac{2}{\ell^{2N+1} q}\cdot$$

If θ' were rational, say $\theta' = a/b$, Liouville's inequality would yield

$$\theta' - \frac{p}{q} \geq \frac{c(\theta')}{q} \quad \text{with} \quad c(\theta') = \frac{1}{b}\cdot$$

Transcendence of

$$\theta' = \sum_{n\geq 1} \ell^{-n^2}$$

proved in 1995 by Yu. V. Nesterenko.

The proof involves theta functions (see below).

Early History ($\leq$ *1934*)

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Early History *(≤ 1934)*

J. Liouville(1844) – *First examples of transcendental numbers.*

G. Cantor (1874) – *"Almost all numbers" are transcendental.*

The set of algebraic numbers is countable, not the set of real or complex numbers.

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Ch. Hermite (1873) – e *is transcendental.*

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Theorem *(Hermite Lindemann) –*

If α *is a non zero algebraic number and* $\log \alpha \neq 0$, *then the number* $\log \alpha$ *is transcendental.*

If β *is a non zero algebraic number, then the number* e^β *is transcendental.*

K. Weierstrass (1885) – *Algebraic independence of*

$$e, \quad e^{\sqrt{2}}, \quad e^{\sqrt{3}}, \ldots$$

Theorem *(Lindemann-Weierstrass) – If* $\beta_1, \ldots, \beta_n$ *are* $\mathbf{Q}$-*linearly independent algebraic numbers, then*

$$e^{\beta_1}, \ldots, e^{\beta_n}$$

are algebraically independent.

Means: $P(e^{\beta_1}, \ldots, e^{\beta_n}) \neq 0$ for $P \in \mathbf{Z}[X_1, \ldots, X_n] \setminus \{0\}$.

Euler (1748) – *Introductio in Analysin Infinitorum*. Suggests the transcendence of $\log \alpha_1 / \log \alpha_2$ when this number is irrational (for algebraic α_1 and α_2).

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Hilbert's seventh Problem (1900) – *Transcendence of* α^β, *for* α *algebraic,* $\alpha\neq 0$, $\alpha\neq 1$ *and* β *algebraic irrational .*

Euler (1748) – *Introductio in Analysin Infinitorum*. Suggests the transcendence of $\log\alpha_1/\log\alpha_2$ when this number is irrational (for algebraic α_1 and α_2).

Hilbert's seventh Problem (1900) – *Transcendence of α^β, for α algebraic, $\alpha \neq 0$, $\alpha \neq 1$ and β algebraic irrational .*

Equivalent: *If α_1, α_2 are two non-zero algebraic numbers and $\log\alpha_1$, $\log\alpha_2$ are $\mathbf{Q}$-linearly independent logarithms of α_1, α_2, then $\log\alpha_1/\log\alpha_2$ is transcendental.*

Equivalence: Recall

$$\alpha^\beta = \exp(\beta\log\alpha).$$

Set $\qquad \alpha_1 = \alpha,\ \alpha_2 = \alpha^\beta,\ \beta = (\log\alpha_2)/\log\alpha_1).$

Euler (1748) – *Introductio in Analysin Infinitorum.* Suggests the transcendence of $\log \alpha_1 / \log \alpha_2$ when this number is irrational (for algebraic α_1 and α_2).

Hilbert's seventh Problem (1900) – *Transcendence of α^β, for α algebraic, $\alpha \neq 0$, $\alpha \neq 1$ and β algebraic irrational .*

Examples: $2^{\sqrt{2}}$ and e^π.

Remark: $e^{i\pi} = -1$.

A. Thue (1908) – Refinement of Liouville's inequality. Application to solving diophantine equations. Use Dirichlet's box principle.

G. Pólya (1914) – *The function* 2^z *is the "smallest" entire transcendental function whose values at non-negative integers are integers.*

C.L. Siegel (1929) – Transcendence and algebraic independence of values of Bessel's functions, and more generally E-functions. Elementary tool: Dirichlet's box principle (Thue-Siegel Lemma).

A.O. Gel'fond (1929) – Extension of Pólya's result to $\mathbf{Z}[i]$.

A.O. Gel'fond (1929) – Transcendence of e^{π}.

Method: Interpolation series for $e^{\pi z}$ at the points of $\mathbf{Z}[i]$.

A.O. Gel'fond and Th. Schneider (1934) – Solution of Hilbert's seventh problem.

Theorem *(Gel'fond–Schneider). If α_1, α_2 are two non-zero algebraic numbers and $\log \alpha_1$, $\log \alpha_2$ are $\mathbf{Q}$-linearly independent logarithms of α_1, α_2, then $\log \alpha_1$, $log \alpha_2$ are linearly independent over the field $\overline{\mathbf{Q}}$ of algebraic numbers.*

Theorem *(Baker, 1966). If $\alpha_1, \ldots, \alpha_n$ are non-zero algebraic numbers and $\log\alpha_1, \ldots, \log\alpha_n$ are $\mathbf{Q}$-linearly independent logarithms of these numbers, then $\log\alpha_1, \ldots, \log\alpha_n$ are linearly independent over $\overline{\mathbf{Q}}$.*

Conjecture. *If $\alpha_1, \ldots, \alpha_n$ are non-zero algebraic numbers and $\log\alpha_1, \ldots, \log\alpha_n$ are $\mathbf{Q}$-linearly independent logarithms of them, then $\log\alpha_1, \ldots, \log\alpha_n$ are algebraically independent.*

Schanuel's Conjecture. *Let $x_1, \ldots, x_n$ be $\mathbf{Q}$-linearly independent complex numbers. Then at least n of the $2n$ numbers*

$$x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}$$

are algebraically independent.

Consequences of Schanuel's Conjecture

Algebraic independence of e *and* π*.*

Transcendence of

$$e^{\pi^2}, \quad 2^{\log 3}, \quad (\log 2)^{\log 3}, \quad \pi^\pi, \quad e^e, \quad \pi^e, \quad \log\log 2, \ldots$$

Algebraic independence of $\mathbf{Q}$*-linearly independent logarithms of algebraic numbers.*

Periods

M. Kontevich and D. Zagier (2000) – *Periods.*

A period is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients over domains of $\mathbf{R}^n$ given by polynomials (in)equalities with rational coefficients.

Examples:

$$\sqrt{2} = \int_{2x^2 \le 1} dx,$$

$$\pi = \int_{x^2+y^2 \le 1} dxdy,$$

$$\log 2 = \int_{1<x<2} \frac{dx}{x},$$

$$\zeta(2) = \int_{1>t_1>t_2>0} \frac{dt_1}{t_1} \cdot \frac{dt_2}{1-t_2} = \frac{\pi^2}{6}.$$

Relations between periods

1 Additivity

$$\int_a^b \big(f(x)+g(x)\big)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

and

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

2 Change of variables

$$\int_{\varphi(a)}^{\varphi(b)} f(t)dt = \int_a^b f\big(\varphi(u)\big)\varphi'(u)du.$$

3 Newton–Leibniz–Stokes

$$\int_a^b f'(t)dt = f(b) - f(a).$$

Conjecture *(Kontsevich–Zagier). If a period has two representations, then one can pass from one formula to another using only rules* 1, 2 *and* 3 *in which all functions and domains of integrations are algebraic with algebraic coefficients.*

$$
\begin{aligned}
\pi &= \int_{x^2+y^2\le 1} dxdy \\
&= 2\int_{-1}^{1} \sqrt{1-x^2}\, dx \\
&= \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} \\
&= \int_{-\infty}^{\infty} \frac{dx}{1-x^2}.
\end{aligned}
$$

Example *(using Baker's Theorem):*
The number

$$\int_0^1 \frac{dx}{1+x^3} = \frac{1}{3}\left(\log 2 + \frac{\pi}{\sqrt{3}}\right)$$

is transcendental.

A.J. van der Poorten (1970) – *On the arithmetic nature of definite integrals of rational functions.*

Let P *and* Q *be polynomials with algebraic coefficients satisfying* $\deg P < \deg Q$ *and let* γ *is either a closed loop or else a path with endpoints which are either algebraic or infinite. If the integral*

$$\int_\gamma \frac{P(z)}{Q(z)} dz$$

exists, then it is either 0 *or transcendental.*

Elliptic and Abelian Integrals

Th. Schneider
(1934) – *Transzendenzuntersuchungen periodischer Funktionen.*
(1937) – *Arithmetische Untersuchungen elliptischer Integrale.*
Transcendence of elliptic integrals of first and second kind.
Th. Schneider (1940) – *Zur Theorie des Abelschen Funktionen und Integrale.*
For any rational numbers a *and* b *which are not integers and such that* $a+b$ *is not an integer, the number*

$$\mathrm{B}(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}=\int_0^1 x^{a-1}(1-x)^{b-1}dx$$

is transcendental.

Remark. For any $p/q \in \mathbf{Q}$ with $p > 0$ and $q > 0$, $\Gamma(p/q)^q$ is a period.

Th. Schneider (1940), S. Lang (1960's), D.W. Masser (1980's).

G. Wüstholz (1989) – *Algebraische Punkte auf analytischen Untergruppen algebraischer Gruppen.*

Extension of Baker's Theorem to commutative algebraic groups.

Transcendence and linear independence over the field of algebraic numbers of abelian integrals of first, second and third kind.

G.V. Chudnovskij (1976) – *Algebraic independence of constants connected with exponential and elliptical functions*

Theorem *(G.V. Chudnovskij). The two numbers*

$$\Gamma(1/4) \quad \textit{and} \quad \pi$$

are algebraically independent and the two numbers

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are algebraically independent.

Yu.V. Nesterenko (1996) – *Modular functions and transcendence questions*

$$\begin{aligned} P(q) &= E_2(q) = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}, \\ Q(q) &= E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n}, \\ R(q) &= E_6(q) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n}. \end{aligned}$$

Theorem *(Yu.V. Nesterenko.) Let $q \in \mathbf{C}$ satisfy $0 < |q| < 1$. Then three at least of the four numbers*

$$q, \; P(q), \; Q(q), \; R(q)$$

are algebraically independent.

Connexion with elliptic functions

For a Weierstrass elliptic function $\wp$ with algebraic invariants g_2 and g_3 and fundamental periods ω_2, ω_1 set $\tau = \omega_1/\omega_2$, $q = e^{2i\pi\tau}$,

$$J(q) = j(\tau) = 1728 \frac{g_2^3}{g_2^3 - 27g_3^2}.$$

Remark.

$$\Delta(q) = \frac{1}{1728}\bigl(Q(q)^3 - R(q)^2\bigr)$$

satisfies

$$J(q) = \frac{Q(q)^3}{\Delta(q)} \quad \text{and} \quad \Delta = q \prod_{n\geq 1}(1-q^n)^{24}.$$

LEMNISCATE $y^2 = 4x^3 - 4x$

$$g_2 = 4, \quad g_3 = 0, \quad j = 1728, \quad \tau = i, \quad q = e^{-2\pi}$$

$$\omega_1 = \frac{\Gamma(1/4)^2}{\sqrt{8\pi}} = 2.6220575542\ldots$$

$$P(e^{-2\pi}) = \frac{3}{\pi}, \qquad Q(e^{-2\pi}) = 3\left(\frac{\omega_1}{\pi}\right)^4,$$

$$R(e^{-2\pi}) = 0, \qquad \Delta(e^{-2\pi}) = \frac{1}{2^6}\left(\frac{\omega_1}{\pi}\right)^{12}.$$

ANHARMONIQUE $\quad y^2 = 4x^3 - 4$

$$g_2 = 0, \quad g_3 = 4, \quad j = 0, \quad \tau = \varrho, \quad q = -e^{-\pi\sqrt{3}}$$

$$\omega_1 = \frac{\Gamma(1/3)^3}{2^{4/3}\pi} = 2.428650648\ldots$$

$$P(-e^{-\pi\sqrt{3}}) = \frac{2\sqrt{3}}{\pi}, \qquad Q(-e^{-\pi\sqrt{3}}) = 0,$$

$$R(-e^{-\pi\sqrt{3}}) = \frac{27}{2}\left(\frac{\omega_1}{\pi}\right)^6, \qquad \Delta(-e^{-\pi\sqrt{3}}) = -\frac{27}{256}\left(\frac{\omega_1}{\pi}\right)^{12}.$$

Corollary. *The three numbers*

$$\Gamma(1/4), \quad \pi \quad \textit{and} \quad e^{\pi}$$

are algebraically independent and the three numbers

$$\Gamma(1/3), \quad \pi \quad \textit{and} \quad e^{\pi\sqrt{3}}$$

are algebraically independent

Conjecture *(Nesterenko)*. *Let* $\tau \in \mathbf{C}$ *have positive imaginary part. Assume that* τ *is not quadratic. Set* $q = e^{2i\pi\tau}$. *Then* 4 *at least of the* 5 *numbers*

$$\tau,\ q,\ P(q),\ Q(q),\ R(q)$$

are algebraically independent.

Remark. Lindemann: $\Gamma(1/2) = \sqrt{\pi}$ *is transcendental.*

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Open problem: *Is* $\Gamma(1/5)$ *transcendental?*

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Remark. The Fermat curve of exponent 5, viz. $x^5 + y^5 = 1$, has genus 2. Its Jacobian is an abelian surface.

Conjecture. *Three of the four numbers*

$$\Gamma(1/5), \quad \Gamma(2/5), \quad \pi \quad \textit{and} \quad e^{\pi\sqrt{5}}$$

are algebraically independent.

Standard Relations

$$\Gamma(a+1) = a\Gamma(a)$$

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)},$$

$$\prod_{k=0}^{n-1} \Gamma\left(a + \frac{k}{n}\right) = (2\pi)^{(n-1)/2} n^{-na+(1/2)} \Gamma(na).$$

Define

$$G(z) = \frac{1}{\sqrt{2\pi}}\Gamma(z).$$

According to the multiplication theorem of Gauss and Legendre, for each positive integer N and each complex number z such that $Nz \not\equiv 0 \pmod{\mathbf{Z}}$,

$$\prod_{i=0}^{N-1} G\left(z + \frac{i}{N}\right) = N^{(1/2)-Nz} G(Nz).$$

The gamma function has no zero and defines a map from $\mathbf{C} \setminus \mathbf{Z}$ to $\mathbf{C}^\times$. Restrict to $\mathbf{Q} \setminus \mathbf{Z}$ and compose with the canonical map $\mathbf{C}^\times \rightarrow \mathbf{C}^\times / \overline{\mathbf{Q}}^\times$. The composite map has period 1, and the resulting mapping

$$\overline{G} : \frac{\mathbf{Q}}{\mathbf{Z}} \setminus \{0\} \rightarrow \frac{\mathbf{C}^\times}{\overline{\mathbf{Q}}^\times}$$

is an odd *distribution* on $(\mathbf{Q}/\mathbf{Z}) \setminus \{0\}$:

$$\prod_{i=0}^{N-1} \overline{G}\left(a + \frac{i}{N}\right) = \overline{G}(Na) \quad \text{for} \quad a \in \frac{\mathbf{Q}}{\mathbf{Z}} \setminus \{0\}$$

and

$$\overline{G}(-a) = \overline{G}(a)^{-1}.$$

Conjecture *(Rohrlich)*. $\overline{G}$ is the universal odd distribution with values in groups where multiplication by 2 is invertible.

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Means: Any multiplicative relation

$$\pi^{b/2}\prod_{a\in\mathbf{Q}}\Gamma(a)^{m_a}\in\overline{\mathbf{Q}}$$

with b and m_a in $\mathbf{Z}$ can be derived for the standard relations.

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Example: (P. Das)

$$\frac{\Gamma(1/3)\Gamma(2/15)}{\Gamma(4/15)\Gamma(1/5)}\in\overline{\mathbf{Q}}.$$

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with b and m_a in $\mathbf{Z}$ can be derived for the standard relations.

This leads to the question whether the distribution relations, the oddness relation and the functional equations of the gamma function generate the ideal over $\overline{\mathbf{Q}}$ of all algebraic relations among the values of $G(a)$ for $a\in\mathbf{Q}$.

Euler's constant

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right) = 0.5772157\ldots,$$

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Open problem: Is γ an irrational number?

Euler's constant

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right) = 0.5772157\ldots,$$

Open problem: Is γ an irrational number?

Conjecture: γ is a transcendental number.

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Conjecture: γ is a transcendental number.

Conjecture: γ is not a period.

J. Sondow (2003) – *Criteria for Irrationality of Euler's Constant.*

A very sharp conjectured lower bound for infinitely many elements in a specific sequence

$$\left| e^{b_0} a_1^{b_1} \cdots a_m^{b_m} - 1 \right|$$

with b_0 arbitrary, and where all the exponents b_i have the same sign, would yield the irrationality of Euler's constant.

Analog of Euler's constant in finite characteristic

L. Carlitz (1935) – *On certain functions connected with polynomials in a Galois field.*

V.G. Drinfel'd (1974) – *Elliptic modules.*

I.I. Wade (1941), J.M. Geijsel (1978), P. Bundschuh (1978), Yu Jing (1980's), G.W. Anderson and D. Thakur (1990). . .

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}, \qquad \gamma = \lim_{s \to 1} \left(\zeta(s) - \frac{1}{s-1} \right).$$

In finite characteristic p runs over the set of monic irreducible polynomials and $\zeta(1)$ converges.

An analog in dimension 2 of Euler constant

For $k \geq 2$, let A_k be the minimal area of a closed disk in $\mathbf{R}^2$ containing at least k points of $\mathbf{Z}^2$. For $n \geq 2$ define

$$\delta_n = -\log n + \sum_{k=2}^{n} \frac{1}{A_k} \quad \text{and} \quad \delta = \lim_{n\to\infty} \delta_n.$$

Gramain conjectures:

$$\delta = 1 + \frac{4}{\pi}\bigl(\gamma L'(1) + L(1)\bigr),$$

where

$$L(s) = \sum_{n\geq 0} (-1)^n (2n+1)^{-s}.$$

Since $L(1) = \pi/4$ and

$$\begin{aligned} L'(1) &= \sum_{n\geq 0}(-1)^{n+1}\cdot\frac{\log(2n+1)}{2n+1} \\ &= \frac{\pi}{4}\bigl(3\log\pi + 2\log 2 + \gamma - 4\log\Gamma(1/4)\bigr), \end{aligned}$$

Gramain's Conjecture can be stated

$$\delta = 1 + 3\log\pi + 2\log 2 + 2\gamma - 4\log\Gamma(1/4) = 1.82282524\ldots$$

Best known estimates for δ (F. Gramain and M. Weber, 1985):

$$1.811\ldots < \delta < 1.897\ldots$$

Transcendence of sums of series

S.D. Adhikari, N. Saradha, T.N. Shorey, R. Tijdeman (2001) – *Transcendental infinite sums.*

N. Saradha, R. Tijdeman (2003) – *On the transcendence of infinite sums of values of rational functions.*

Question: What is the arithmetic nature of

$$\sum_{\substack{n\geq 0\\ Q(n)\neq 0}} \frac{P(n)}{Q(n)} \text{ ?}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1,$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{4n+1} - \frac{3}{4n+2} + \frac{1}{4n+3} + \frac{1}{4n+4} \right) = 0$$

are rational numbers.

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are rational numbers. Telescoping series:

$$\sum_{n=0}^{\infty} \frac{1}{(an+b)(an+a+b)} = \frac{1}{ab},$$

$$\sum_{n=0}^{\infty} \frac{1}{(an+b)(an+a+b)(an+2a+b)} = \frac{1}{2ab(a+b)}.$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)} = \log 2,$$

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(2n+1)(4n+1)} = \frac{\pi}{3},$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1}{2} + \frac{\pi}{2} \cdot \frac{e^{\pi} + e^{-\pi}}{e^{\pi} - e^{-\pi}}$$

are transcendental numbers

Also

$$\sum_{n=0}^{\infty} \frac{1}{(3n+1)(3n+2)(3n+3)}$$

is transcendental.

Question: *Arithmetic nature of*

$$\sum_{n\geq 1} \frac{1}{n^s}$$

for $s \geq 2$ *?*

Zeta Values – Euler Numbers

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} \qquad \text{for } s \geq 2.$$

These are special values of the Riemann Zeta Function: $s \in \mathbf{C}$.

Euler: $\pi^{-2k}\zeta(2k) \in \mathbf{Q}$ *for* $k \geq 1$. (Bernoulli numbers).

Diophantine Question: *What are the algebraic relations among the numbers*

$$\zeta(2), \quad \zeta(3), \quad \zeta(5), \quad \zeta(7) \ldots ?$$

Conjecture. *There is no algebraic relation at all: these numbers*

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Known:

• Hermite-Lindemann: π *is transcendental, hence* $\zeta(2k)$ *also for* $k \geq 1$.

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- Apéry (1978) – $\zeta(3)$ *is irrational.*

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• Hermite-Lindemann: π *is transcendental, hence* $\zeta(2k)$ *also for* $k \geq 1$.

• Apéry (1978) – $\zeta(3)$ *is irrational.*

• Rivoal (2000) + Ball, Zudilin. . . *Infinitely many* $\zeta(2k+1)$ *are irrational + lower bound for the dimension of the* $\mathbf{Q}$*-space they span.*

Let $\epsilon > 0$. For a be a sufficiently large odd integer the dimension of the $\mathbf{Q}$-space spanned by 1, $\zeta(3)$, $\zeta(5)$, $\cdots, \zeta(a)$ is at least

$$\frac{1-\epsilon}{1+\log 2}\log a.$$

W. Zudilin.

• *One at least of the four numbers*

$$\zeta(5), \quad \zeta(7), \quad \zeta(9), \quad \zeta(11)$$

is irrational.

• *There is an odd integer* j *in the range* $[5, 69]$ *such that the three numbers* 1, $\zeta(3)$, $\zeta(j)$ *are linearly independent over* $\mathbf{Q}$.

Linearization of the problem *(Euler).* The product of two zeta values is not quite a zeta value, but something similar.

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From

$$\sum_{n_1\geq 1} n_1^{-s_1} \sum_{n_2\geq 1} n_2{}^{-s_2} = \sum_{n_1>n_2\geq 1} n_1^{-s_1} n_2{}^{-s_2} + \sum_{n_2>n_1\geq 1} n_2^{-s_2} n_1{}^{-s_1} + \sum_{n\geq 1} n^{-s_1-s_2}$$

one deduces, for $s_1 \geq 2$ and $s_2 \geq 2$,

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2)$$

with

$$\zeta(s_1, s_2) = \sum_{n_1>n_2\geq 1} n_1^{-s_1} n_2{}^{-s_2}.$$

For k, $s_1,\ldots,s_k$ positive integers with $s_1 \geq 2$, define $\underline{s} = (s_1,\ldots,s_k)$ and

$$\zeta(\underline{s}) = \sum_{n_1>n_2>\cdots>n_k\geq 1} \frac{1}{n_1^{s_1}\cdots n_k^{s_k}}.$$

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The product of two Multiple Zeta Values is a linear combination, with integer coefficients, of Multiple Zeta Values.

These numbers satisfy a quantity of linear relations with rational coefficients.

A complete description of these relations would in principle settle the problem of the algebraic independence of

$$\pi, \quad \zeta(3), \quad \zeta(5), \ldots, \quad \zeta(2k+1).$$

Goal: *Describe all linear relations among Multiple Zeta Values.*

Example of linear relation.

Euler:

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Euler's result follows from $(t_1,t_2,t_3)\mapsto(1-t_3,1-t_2,1-t_1)$.

Denote by $\mathfrak{Z}_p$ the $\mathbf{Q}$-vector subspace of $\mathbf{R}$ spanned by the real numbers $\zeta(\underline{s})$ with $\underline{s}$ of weight $s_1 + \cdots + s_k = p$, with $\mathfrak{Z}_0 = \mathbf{Q}$ and $\mathfrak{Z}_1 = \{0\}$.

Here is Zagier's conjecture on the dimension d_p of $\mathfrak{Z}_p$.

Conjecture *(Zagier). For $p \geq 3$ we have*

$$d_p = d_{p-2} + d_{p-3}.$$

$$(d_0,\ d_1,\ d_2, \ldots) = (1, 0, 1, 1, 1, 2, 2, \ldots).$$

This conjecture can be written

$$\sum_{p\ge 0} d_p X^p = \frac{1}{1-X^2-X^3}.$$

M. Hoffman conjectures: *a basis of* $\mathfrak{Z}_p$ *over* $\mathbf{Q}$ *is given by the numbers* $\zeta(s_1,\ldots,s_k)$, $s_1+\cdots+s_k=p$, *where each* s_i *is either* 2 *or* 3.

True for $p\le 16$ (Hoang Ngoc Minh)

A.G. Goncharov (2000) – *Multiple* ζ*-values, Galois groups and Geometry of Modular Varieties.*

T. Terasoma (2002) – *Mixed Tate motives and multiple zeta values.*

The numbers defined by the recurrence relation of Zagier's Conjecture

$$d_p = d_{p-2} + d_{p-3}.$$

with initial values $d_0 = 1$, $d_1 = 0$ *provide upper bounds for the actual dimension of* $\mathfrak{Z}_p$.