Families of cubic Thue equations with effective bounds for the solutions

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Abstract . To each non totally real cubic extension K of $\mathbf{Q}$ and to each generator α of the cubic field K, we attach a family of cubic Thue equations, indexed by the units of K, and we prove that this family of cubic Thue equations has only a finite number of integer solutions, by giving an effective upper bound for these solutions.

1 Statements

Let us consider an irreductible binary cubic form having rational integers coefficients

$$F(X,Y) = a_0X^3 + a_1X^2Y + a_2XY^2 + a_3Y^3 \in \mathbf{Z}[X,Y]$$

with the property that the polynomial $F(X,1)$ has exactly one real root α and two complex imaginary roots, namely α' and $\overline{\alpha'}$. Hence $\alpha \notin \mathbf{Q}$, $\alpha' \neq \overline{\alpha'}$ and

$$F(X,Y) = a_0(X-\alpha Y)(X-\alpha' Y)(X-\overline{\alpha'}Y).$$

Let K be the cubic number field $\mathbf{Q}(\alpha)$ which we view as a subfield of $\mathbf{R}$. Define $\sigma : K \to \mathbf{C}$ to be one of the two complex embeddings, the other one being the conjugate $\overline{\sigma}$. Hence $\alpha' = \sigma(\alpha)$ and $\overline{\alpha'} = \overline{\sigma}(\alpha)$. If τ is defined to be the complex conjugation, we have $\overline{\sigma} = \tau \circ \sigma$ and $\sigma \circ \tau = \sigma$.

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Let ε be a unit >1 of the ring $\mathbf{Z}_K$ of algebraic integers of K and let $\varepsilon' = \sigma(\varepsilon)$ and $\overline{\varepsilon'} = \overline{\sigma}(\varepsilon)$ be the two other algebraic conjugates of ε. We have

$$|\varepsilon'| = |\overline{\varepsilon'}| = \frac{1}{\sqrt{\varepsilon}} < 1.$$

For $n \in \mathbf{Z}$, define

$$F_n(X,Y) = a_0\bigl(X - \varepsilon^n \alpha Y\bigr)\bigl(X - \varepsilon'^n \alpha' Y\bigr)\bigl(X - \overline{\varepsilon'}^n\, \overline{\alpha'}\, Y\bigr).$$

Let $k \in \mathbf{N}$, where $\mathbf{N} = \{1,2,\ldots\}$. We plan to study the family of Thue inequations

$$0 < |F_n(x,y)| \le k, \tag{1}$$

where the unknowns n,x,y take values in $\mathbf{Z}$.

Theorem 1. *There exist effectively computable positive constants κ_1 and κ_2, depending only on F, such that, for all $k \in \mathbf{Z}$ with $k \ge 1$ and for all $(n,x,y) \in \mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}$ satisfying $\varepsilon^n\alpha \notin \mathbf{Q}$, $xy \neq 0$ and $|F_n(x,y)| \le k$, we have*

$$\max\Bigl\{\varepsilon^{|n|},\ |x|,\ |y|\Bigr\} \le \kappa_1 k^{\kappa_2}.$$

From this theorem, we deduce the following corollary.

Corollary 1 . *For $k \in \mathbf{Z}$, $k > 0$, the set*

$$\bigl\{(n,x,y) \in \mathbf{Z}\times\mathbf{Z}\times\mathbf{Z} \ \mid\ \varepsilon^n\alpha \notin \mathbf{Q}\ ;\ xy \neq 0\ ;\ |F_n(x,y)| \le k\bigr\}$$

is finite.

This corollary is a particular case of the main result of [2], but the proof in [2] is based on the Schmidt subspace theorem which does not allow to give an effective upper bound for the solutions (n,x,y).

Example. Let $D \in \mathbf{Z}$, $D \neq -1$. Let $\varepsilon := \bigl(\sqrt[3]{D^3+1} - D\bigr)^{-1}$. There exist two positive effectively computable absolute constants κ_3 and κ_4 with the following property. Define a sequence $(F_n)_{n\in\mathbf{Z}}$ of cubic forms in $\mathbf{Z}[X,Y]$ by

$$F_n(X,Y) = X^3 + a_n X^2 Y + b_n X Y^2 - Y^3,$$

where $(a_n)_{n\in\mathbf{Z}}$ is defined by the recurrence relation

$$a_{n+3} = 3Da_{n+2} + 3D^2 a_{n+1} + a_n$$

with the initial conditions $a_0 = 3D^2$, $a_{-1} = 3$ and $a_{-2} = -3D$, and where $(b_n)_{n\in\mathbf{Z}}$ is defined by $b_n = -a_{-n-2}$. Then, for x, y, n rational integers with $xy \neq 0$ and $n \neq -1$, we have

$$|F_n(x,y)| \ge \kappa_3 \max\{|x|,\ |y|,\ \varepsilon^{|n|}\}^{\kappa_4}.$$

This result follows from Theorem 1 with $\alpha = \varepsilon$ and

$$F(X,Y) \;=\; X^3 - 3DX^2Y - 3D^2XY^2 - Y^3.$$

Indeed, the irreducible polynomial of $\varepsilon^{-1} = \sqrt[3]{D^3+1} - D$ is

$$F_{-2}(X,1) \;=\; (X+D)^3 - D^3 - 1 \;=\; X^3 + 3DX^2 + 3D^2X - 1,$$

the irreducible polynomial of $\alpha \;=\; \varepsilon$ is

$$F(X,1) \;=\; F_0(X,1) \;=\; F_{-2}(1,X) \;=\; X^3 - 3D^2X^2 - 3DX - 1,$$

while

$$F_{-1}(X,Y) \;=\; (X-Y)^3 \;=\; X^3 - 3X^2Y + 3XY^2 - Y^3.$$

For $n \in \mathbf{Z}$, $n \neq -1$, $F_n(X,1)$ is the irreducible polynomial of $\alpha\varepsilon^n \;=\; \varepsilon^{n+1}$, while for any $n \in \mathbf{Z}$, $F_n(X,Y) \;=\; \mathrm{N}_{\mathbf{Q}(\varepsilon)/\mathbf{Q}}(X - \varepsilon^{n+1}Y)$. The recurrence relation for

$$a_n \;= \varepsilon^{n+1} + \varepsilon'^{n+1} + \overline{\varepsilon'}^{n+1}$$

follows from

$$\varepsilon^{n+3} \;=\; 3D\varepsilon^{n+2} + 3D^2\varepsilon^{n+1} + \varepsilon^n$$

and for b_n, from $F_{-n}(X,Y) \;=\; -F_{n-2}(Y,X)$.

2 Elementary estimates

For a given integer $k > 0$, we consider a solution (n,x,y) in $\mathbf{Z}^3$ of the Thue inequation (1) with $\varepsilon^n\alpha$ irrational and $xy \neq 0$. We will use $\kappa_5, \kappa_6, \ldots, \kappa_{55}$ to designate some constants depending only on α.

Let us firstly explain that in order to prove Theorem 1, we can assume $n \geq 0$ by eventually permuting x and y. Let us suppose that $n < 0$ and write

$$F(X,Y) \;=\; a_3(Y - \alpha^{-1}X)(Y - \alpha'^{-1}X)(Y - \overline{\alpha'}^{-1}X).$$

Then

$$F_n(X,Y) \;=\; a_3\big(Y - \varepsilon^{|n|}\alpha^{-1}X\big)\big(Y - \varepsilon'^{|n|}\alpha'^{-1}X\big)\big(Y - \overline{\varepsilon'}^{|n|}\,\overline{\alpha'}^{-1}X\big).$$

Now it is simply a matter of using the result for $|n|$ for the polynomial $G(X,Y) \;=\; F(Y,X)$.

Let us now check that, in order to prove the statements of §1, there is no restriction in assuming that α is an algebraic integer and that $a_0 \;=\; 1$. To achieve this goal, we define

$$\tilde{F}(T,Y) \;=\; T^3 + a_1T^2Y + a_0a_2TY^2 + a_0^2a_3Y^3 \in \mathbf{Z}[T,Y],$$

so that $a_0^2F(X,Y) \;=\; \tilde{F}(a_0X,Y)$. If we define $\tilde{\alpha} \;=\; a_0\alpha$ and $\tilde{\alpha}' \;=\; a_0\alpha'$, then $\tilde{\alpha}$ is a nonzero algebraic integer, and we have

$$\tilde{F}(T,Y) \;=\; (T-\tilde{\alpha}Y)(T-\tilde{\alpha}'Y)(T-\overline{\tilde{\alpha}'}Y).$$

For $n \in \mathbf{Z}$, the binary form

$$\tilde{F}_n(T,Y) \;=\; (T-\varepsilon^n\tilde{\alpha}Y)(T-\varepsilon'^n\,\tilde{\alpha}'Y)(T-\overline{\varepsilon'}^n\,\overline{\tilde{\alpha}'}Y)$$

satisfies

$$a_0^2F_n(X,Y) \;=\; \tilde{F}_n(a_0X,Y).$$

The condition (1) implies $0 < |\tilde{F}_n(a_0x,y)| \le a_0^2k.$ Therefore it suffices to prove the statements for $\tilde{F}_n$ instead of F_n, with α and α' replaced by $\tilde{\alpha}$ and $\tilde{\alpha}'$. This allows us, from now on, to suppose $\alpha \in \mathbf{Z}_K$ and $a_0 \;=\; 1$.

As already explained, we can assume $n \ge 0$. There is no restriction in supposing $k \ge 2$; (if we prove the result for a value of $k \ge 2$, we deduce it right away for smaller values of k, since we consider Thue inequations and not Thue equations). If k were asumed to be ≥ 2, we would not need κ_1, as is easily seen, and the conclusion would read

$$\max\{\varepsilon^{|n|},\ |x|,\ |y|\} \;\le\; k^{\kappa_2}.$$

Without loss of generality we can assume that n is sufficiently large. As a matter of fact, if n is bounded, we are led to some given Thue equations, and Theorem 1 follows from Theorem 5.1 of [3].

Let us recall that for an algebraic number γ, the house of γ, denoted $\overline{|\gamma|}$, is by definition the maximum of the absolute values of the conjugates of γ. Moreover, d is the degree of the algebraic number field K (namely $d \;=\; 3$ here) and R is the regulator of K (viz. $R = \log\varepsilon$), where, from now on, ε is the fundamental unit > 1 of the non totally real cubic field K. The next statement is Lemma A.6 of [3].

Lemma 1 *Let γ be a nonzero element of $\mathbf{Z}_K$ of norm $\le M$. There exists a unit $\eta \in \mathbf{Z}_K^\times$ such that the house $\overline{|\eta\gamma|}$ is bounded by an effectively computable constant which depends only on d, R and M.*

We need to make explicit the dependence upon M, and for this, it suffices to apply Lemma A.15 of [3], which we want to state, under the asumption that the d embeddings of the algebraic number field K in $\mathbf{C}$ are noted $\sigma_1,\dots,\sigma_d$.

Lemma 2 *Let K be an algebraic number field of degree d and let γ be a nonzero element of $\mathbf{Z}_K$ whose absolute value of the norm is m. Then there exists a unit $\eta \in \mathbf{Z}_K^\times$ such that*

$$\frac{1}{R}\max_{1\le j\le d}\Bigl|\log(m^{-1/d}|\sigma_j(\eta\gamma)|)\Bigr|$$

is bounded by an effectively computable constant which depends only on d.

Since $d = 3$, $K = \mathbf{Q}(\alpha)$ and the regulator R of K is an effectively computable constant (see for instance [1], §6.5), the conclusion of Lemma 2 is

$$-\kappa_5 \le \log(|\sigma_j(\eta\gamma)|/\sqrt[3]{m}) \le \kappa_5,$$

which can also be written as

$$\kappa_6\sqrt[3]{m} \le |\sigma_j(\eta\gamma)| \le \kappa_7\sqrt[3]{m},$$

with two effectively computable positive constants κ_6 and κ_7. We will use only the upper bound [1]: under the hypotheses of Lemma 1 with $d = 3$, when γ is a nonzero element of $\mathbf{Z}_K$ of norm $\le M$, there exists a unit η of $\mathbf{Z}_K^\times$ such that

$$\overline{|\eta\gamma|} \le \kappa_7\sqrt[3]{M}.$$

Since (n,x,y) satisfies (1), the element $\gamma = x - \varepsilon^n\alpha y$ of $\mathbf{Z}_K$ has a norm of absolute value $\le k$. It follows from Lemma 2 that γ can be written as

$$x - \varepsilon^n\alpha y = \varepsilon^\ell\xi_1 \tag{2}$$

with $\ell \in \mathbf{Z}$, $\xi_1 \in \mathbf{Z}_K$ and the house of ξ_1, $\overline{|\xi_1|} = \max\{|\xi_1|,\, |\xi_1'|\}$, satisfies

$$\overline{|\xi_1|} \le \kappa_8\sqrt[3]{k}.$$

We will not use the full force of this upper bound, but only the consequence

$$\max\left\{|\xi_1|^{-1}, |\xi_1'|^{-1}, \overline{|\xi_1|}\right\} \le k^{\kappa_9}. \tag{3}$$

Taking the conjugate of (2) by σ, we have

$$x - \varepsilon'^n\alpha' y = \varepsilon'^\ell\xi_1' \tag{4}$$

with $\xi_1' = \sigma(\xi_1)$.

Our strategy is to prove that $|\ell|$ is bounded by a constant times $\log k$, and that $|n|$ is also bounded by a constant times $\log k$; then we will show that $|y|$ is bounded by a a constant power of k and deduce that $|x|$ is also bounded by a constant power of k.

Let us eliminate x in (2) and (4) to obtain

$$y = -\frac{\varepsilon^\ell\xi_1 - \varepsilon'^\ell\xi_1'}{\varepsilon^n\alpha - \varepsilon'^n\alpha'}; \tag{5}$$

since we supposed $\varepsilon^n\alpha$ irrational, we did not divide by 0. The complex conjugate of (4) is written as

$$x - \overline{\varepsilon'}^n\,\overline{\alpha'}y = \overline{\varepsilon'}^\ell\,\overline{\xi_1'}. \tag{6}$$

[1] The lower bound follows from looking at the norm!

We eliminate x and y in the three equations (2), (4) and (6) to obtain a unit equation à la Siegel:

$$\varepsilon^\ell \xi_1(\alpha'\varepsilon'^n - \overline{\alpha'}\,\overline{\varepsilon'}^n) + \varepsilon'^\ell \xi_1'(\overline{\alpha'}\,\overline{\varepsilon'}^n - \alpha\varepsilon^n) + \overline{\varepsilon'}^\ell\,\overline{\xi_1'}(\alpha\varepsilon^n - \alpha'\varepsilon'^n) \;=\; 0. \tag{7}$$

In the remaining part of this section 2, we suppose

$$\varepsilon^n|\alpha| \;\geq\; 2|\varepsilon'^n\alpha'|. \tag{8}$$

Note that if this inequality is not satisfied, then we have

$$\varepsilon^{3n/2} \;<\; \frac{2|\alpha'|}{|\alpha|} \;<\kappa_{10},$$

and this leads to the inequality (18), and to the rest of the proof of Theorem 1 by using the argument following the inequality (18).

For $\ell > 0$, the absolute value of the numerator $\varepsilon^\ell\xi_1 - \varepsilon'^\ell\xi_1'$ in (5) is increasing like ε^ℓ and for $\ell < 0$ it is increasing like $\varepsilon^{|\ell|/2}$; for $n > 0$, the absolute value of the denominator $\varepsilon^n\alpha - \varepsilon'^n\alpha'$ is increasing like ε^n and for $n < 0$ it is increasing like $\varepsilon^{|n|/2}$. In order to extract some information from the equation (5), we write it in the form

$$y \;=\; \pm\frac{A-a}{B-b}$$

with

$$B \;=\; \varepsilon^n\alpha, \quad b \;=\; \varepsilon'^n\alpha', \qquad \{A,a\} \;=\; \left\{\varepsilon^\ell\xi_1\;,\;\varepsilon'^\ell\xi_1'\right\},$$

the choice of A and a being dictated by

$$|A| \;=\; \max\{\varepsilon^\ell|\xi_1|\;,\;|\varepsilon'^\ell\xi_1'|\}, \quad |a| \;=\; \min\{\varepsilon^\ell|\xi_1|\;,\;|\varepsilon'^\ell\xi_1'|\}.$$

Since $|A-a| \leq 2|A|$ and since $|b| \leq |B|/2$ because of (8), we have $|B-b| \geq |B|/2$, so we get

$$|y| \;\leq\; 4\frac{|A|}{|B|}\cdot$$

We will consider the two cases corresponding to the possible signs of ℓ, (remember that n is positive).

First case. Let $\ell \leq 0$. We have

$$|A| \;\leq\; \kappa_{11}\varepsilon^{|\ell|/2}k^{\kappa_9}.$$

We deduce from (5)

$$1 \;\leq\; |y| \leq 4\left|\frac{\xi_1'}{\alpha}\right|\varepsilon^{(|\ell|/2)-n} \;\leq\; \kappa_{12}\varepsilon^{(|\ell|/2)-n}k^{\kappa_9}. \tag{9}$$

Hence there exists κ_{13} such that

$$0 \ \leq\ \log|y| \ \leq\ \left(\frac{|\ell|}{2} - n\right)\log\varepsilon + \kappa_{13}\log k,$$

from which we deduce the inequality

$$n \leq \frac{|\ell|}{2} + \kappa_{14}\log k, \tag{10}$$

which will prove useful: n is roughly bounded by $|\ell|$. From (4) we deduce the existence of a constant κ_{15} such that

$$|x| \ \leq\ \varepsilon^{-n/2}|\alpha' y| + \kappa_{15}k^{\kappa_9}\varepsilon^{|\ell|/2}. \tag{11}$$

Second case. Let $\ell > 0$. We have

$$|A| \ \leq\ \kappa_{16}\varepsilon^{\ell}k^{\kappa_9}.$$

We deduce from (5) the upper bound

$$1 \ \leq\ |y| \ \leq\ 4\left|\frac{\xi_1}{\alpha}\right|\varepsilon^{\ell-n} \ \leq\ \kappa_{17}k^{\kappa_9}\varepsilon^{\ell-n}; \tag{12}$$

hence there exists κ_{18} such that

$$0 \ \leq\ \log|y| \ \leq\ (\ell - n)\log\varepsilon + \kappa_{18}\log k.$$

Consequently,

$$n \ \leq\ \ell + \kappa_{19}\log k. \tag{13}$$

From the relation (4) we deduce the existence of a constant κ_{20} such that

$$1 \ \leq\ |x| \ \leq\ \varepsilon^{-n/2}|\alpha' y| + \kappa_{20}k^{\kappa_9}\varepsilon^{-\ell/2}. \tag{14}$$

By taking into account the inequalities (9), (10) and (11) in the case $\ell \leq 0$, and the inequalities (12), (13) and (14) in the case $\ell > 0$, let us show that the existence of a constant κ_{21} satisfying $|\ell| \leq \kappa_{21}\log k$ allows to conclude the proof of Theorem 1. As a matter of fact, suppose

$$|\ell| \leq \kappa_{21}\log k. \tag{15}$$

Then (10) and (13) imply $n \leq \kappa_{22}\log k$, whereupon $|\ell|$ and n are effectively bounded by a constant times $\log k$. This implies that the elements ε^t, with t being $(|\ell|/2) - n$, $\ell - n$, $-n/2$, $|\ell|/2$ or $-\ell/2$, appearing in (9), (12), (11) and (14) are bounded from above by $k^{\kappa_{23}}$ for some constant κ_{23}. Therefore the upper bound of $|y|$ in the conclusion of Theorem 1 follows from (9) and (12) and the upper bound of $|x|$ is a consequence of (11) and (14). Our goal is to show that sooner or later, we end up with the inequality (15).

In the case $\ell > 0$, the lower bound $|x| \geq 1$ provides an extra piece of information. If the term $\varepsilon'^{\ell}\xi_1'$ on the right hand side of (4) does not have an absolute value $< 1/2$, then the upper bound (15) holds true and this suffices to claim the proof of Theorem 1. Suppose now $|\varepsilon'^{\ell}\xi_1'| < 1/2$. Since the relation (12) implies

$$\varepsilon^{-n/2}|\alpha' y| \;\leq\; 4\left|\frac{\xi_1\alpha'}{\alpha}\right|\varepsilon^{\ell-(3n/2)},$$

we have

$$1 \;\leq\; |x| \;\leq\; 4\left|\frac{\xi_1\alpha'}{\alpha}\right|\varepsilon^{\ell-(3n/2)}+\frac{1}{2}$$

and

$$1 \;\leq\; 8\left|\frac{\xi_1\alpha'}{\alpha}\right|\varepsilon^{\ell-(3n/2)}.$$

We deduce

$$\frac{3}{2}n \;\leq\; \ell+\kappa_{24}\log k. \tag{16}$$

The upper bound in (16) is sharper than the one in (13), but, amazingly, we used (13) to establish (16).

When $\ell<0$, we have $|\ell-n| \;=\; n+|\ell|\geq|\ell|$, while in the case $\ell\geq 0$ we have

$$|\ell-n| \;\geq\; \frac{1}{3}\ell+\frac{2}{3}\ell-n \;\geq\; \frac{1}{3}|\ell|-\kappa_{24}\log k,$$

because of (16). Therefore, if ℓ is positive (recall (16)), zero or negative (recall (10)), we always have

$$n \;\leq\; \frac{2}{3}|\ell|+\kappa_{25}\log k \quad\text{and}\quad |\ell-n| \;\geq\; \frac{1}{3}|\ell|-\kappa_{24}\log k \tag{17}$$

with $\kappa_{24}>0$ and $\kappa_{25}>0$.

3 Diophantine tool

Let us remind what we mean by the absolute logarithmic height $\mathrm{h}(\alpha)$ of an algebraic number α (cf. [4], Chap. 3). For L a number field and for $\alpha\in L$, we define

$$\mathrm{h}(\alpha) \;=\; \frac{1}{[L:Q]}\log H_L(\alpha),$$

with

$$H_L(\alpha) \;=\; \prod_{v}\max\{1,|\alpha|_v\}^{d_v}$$

where v runs over the set of places of L, with d_v being the local degree of the place v if v is ultrametric, $d_v \;=\; 1$ if v is real, $d_v \;=\; 2$ if v is complex. When $f(X)\in\mathbf{Z}[X]$ is the minimal polynomial of α and $f(X) \;=\; a_0\prod_{1\leq j\leq d}(X-\alpha_j)$, with $\alpha_1 \;=\; \alpha$, it happens that

$$\mathrm{h}(\alpha) \;=\; \frac{1}{d}\log M(f) \quad \text{with} \quad M(f) \;=\; |a_0|\prod_{1\le j\le d}\max\{1,|\alpha_j|\}.$$

We will use two particular cases of Theorem 9.1 of [4]. The first one is a lower bound for the linear form of logarithms $b_0\lambda_0+b_1\lambda_1+b_2\lambda_2$, and the second one is a lower bound for $\gamma_1^{b_1}\gamma_2^{b_2}-1$. Here is the first one.

Proposition 1. *There exists an explicit absolute constant $c_0>0$ with the following property. Let $\lambda_0,\lambda_1,\lambda_2$ be three logarithms of algebraic numbers and let b_0,b_1,b_2 be three rational integers such that $\Lambda \;=\; b_0\lambda_0+b_1\lambda_1+b_2\lambda_2$ be nonzero. Write*

$$\gamma_0 \;=\; e^{\lambda_0}, \quad \gamma_1 \;=\; e^{\lambda_1}, \quad \gamma_2 \;=\; e^{\lambda_2} \quad \textit{and} \quad D \;=\; [\mathbf{Q}(\gamma_0,\gamma_1,\gamma_2):\mathbf{Q}].$$

Let A_0, A_1, A_2 and B be real positive numbers satisfying

$$\log A_i \;\ge\; \max\left\{\mathrm{h}(\gamma_i),\frac{|\lambda_i|}{D},\frac{1}{D}\right\} \qquad (i \;=\; 0,1,2)$$

and

$$B \;\ge \max\left\{e,\;D,\;\frac{|b_2|}{D\log A_0}+\frac{|b_0|}{D\log A_2},\;\frac{|b_2|}{D\log A_1}+\frac{|b_1|}{D\log A_2}\right\}.$$

Then

$$|\Lambda| \;\ge\; \exp\bigl\{-c_0D^5(\log D)(\log A_0)(\log A_1)(\log A_2)(\log B)\bigr\}.$$

The second particular case of Theorem 9.1 in [4] that we will use is the next Proposition 2. It also follows from Corollary 9.22 of [4]. We could as well deduce it from Proposition 1.

Proposition 2. *Let D be a positive integer. There exists an explicit constant $c_1>0$, depending only on D with the following property. Let K be a number field of degree $\le D$. Let γ_1,γ_2 be nonzero elements in K and let b_1,b_2 be rational integers. Assume $\gamma_1^{b_1}\gamma_2^{b_2}\ne 1$. Set*

$$B \;=\; \max\{2,\,|b_1|,|b_2|\} \quad \textit{and, for } i \;=\; 1,2, \quad A_i \;=\; \exp\bigl(\max\{e,\,\mathrm{h}(\gamma_i)\}\bigr).$$

Then

$$|\gamma_1^{b_1}\gamma_2^{b_2}-1| \ge \exp\bigl\{-c_1(\log B)(\log A_1)(\log A_2)\bigr\}.$$

Proposition 2 will come into play via its following consequence.

Corollary 2 *Let δ_1 and δ_2 be two real numbers in the interval $[0,2\pi)$. Suppose that the numbers $e^{i\delta_1}$ and $e^{i\delta_2}$ are algebraic. There exists an explicit constant $c_2>0$, depending only upon δ_1 and δ_2, with the following property: for each $n\in\mathbf{Z}$ such that $\delta_1+n\delta_2\notin\mathbf{Z}\pi$, we have*

$$|\sin(\delta_1+n\delta_2)| \;\ge\; (|n|+2)^{-c_2}.$$

Proof. Write $\gamma_1 = e^{i\delta_1}$ and $\gamma_2 = e^{i\delta_2}$. By hypothesis, γ_1 and γ_2 are algebraic with $\gamma_1\gamma_2^n \neq 1$. Let us use Proposition 2 with $b_1 = 1$, $b_2 = n$. The parameters A_1 and A_2 depend only upon δ_1 and δ_2 and the number $B = \max\{2, |n|\}$ is bounded from above by $|n|+2$. Hence

$$|\gamma_1\gamma_2^n - 1| \geq (|n|+2)^{-c_3}$$

where c_3 depends only upon δ_1 and δ_2. Let ℓ be the nearest integer to $(\delta_1 + n\delta_2)/\pi$ (take the floor if there are two possible values) and let $t = \delta_1 + n\delta_2 - \ell\pi$. This real number t is in the interval $(-\pi/2, \pi/2]$. Now

$$|e^{it}+1| = |1+\cos(t)+i\,\sin(t)| = \sqrt{2(1+\cos(t))} \geq \sqrt{2}.$$

Since $e^{it} = (-1)^\ell\gamma_1\gamma_2^n$, we deduce

$$\begin{aligned}|\sin(\delta_1+n\delta_2)| = |\sin(t)| &= \frac{1}{2}\left|(-1)^{2\ell}e^{2it}-1\right|\\ &= \frac{1}{2}\left|(-1)^\ell e^{it}+1\right|\cdot\left|(-1)^\ell e^{it}-1\right| \geq \frac{\sqrt{2}}{2}\,|\gamma_1\gamma_2^n-1|.\end{aligned}$$

This secures the proof of Corollary 2.

The following elementary lemma makes clear that $e^t \sim 1$ for $t \to 0$. The first (resp. second) part follows from Exercice 1.1.a (resp. 1.1.b or 1.1.c) of [4]. We will use only the second part; the first one shows that the number t in the proof of Corollary 2 is close to 0, but we did not need it.

Lemma 3 *(a) For $t \in \mathbf{C}$, we have*

$$|e^t - 1| \leq |t|\max\{1, |e^t|\}.$$

(b) If a complex number z satisfies $|z-1| < 1/2$, then there exists $t \in \mathbf{C}$ such that $e^t = z$ and $|t| \leq 2|z-1|$. This t is unique and is the principal determination of the logarithm of z:

$$|\log z| \leq 2|z-1|.$$

4 Proof of Theorem 1

Let us define some real numbers θ, δ and ν in the interval $[0, 2\pi)$ by

$$\varepsilon' = \frac{1}{\varepsilon^{1/2}}e^{i\theta}, \quad \alpha' = |\alpha'|e^{i\delta}, \quad \xi_1' = |\xi_1'|e^{i\nu}.$$

By ordering the terms of (7), we can write this relation as

$$T_1 + T_2 + T_3 = 0,$$

and the three terms involved are

$$\begin{cases} T_1 := \varepsilon^\ell \xi_1(\alpha'\varepsilon'^n - \overline{\alpha'}\,\overline{\varepsilon'}^n) & = 2i\xi_1|\alpha'|\varepsilon^{\ell-n/2}\sin(\delta+n\theta),\\ T_2 := \alpha\varepsilon^n(\overline{\varepsilon'}^\ell\overline{\xi_1'} - \varepsilon'^\ell\xi_1') & = -2i|\xi_1'|\alpha\varepsilon^{n-\ell/2}\sin(\nu+\ell\theta),\\ T_3 := \xi_1'\varepsilon'^\ell\overline{\alpha'}\,\overline{\varepsilon'}^n - \overline{\xi_1'}\overline{\varepsilon'}^\ell\alpha'\varepsilon'^n & = 2i|\xi_1'\alpha'|\varepsilon^{-(n+\ell)/2}\sin(\nu-\delta+(\ell-n)\theta). \end{cases}$$

It turns out that these three terms are purely imaginary. We write this zero sum as

$$a+b+c \;=\; 0 \;\text{ with }\; |a| \;\geq\; |b| \;\geq\; |c|,$$

and we use the fact that this implies that $|a| \leq 2|b|$. Thanks to (17), Corollary 2 shows that a lower bound of the sinus terms is $|\ell|^{-\kappa_{26}}$ (and an obvious upper bound is 1!). Moreover,

– The T_1 term contains a constant factor and the factors:
 • $|\xi_1|$ with $k^{-\kappa_9} \leq |\xi_1| \leq k^{\kappa_9}$,
 • $\varepsilon^{\ell-(n/2)}$ (which is the main term),
 • a sinus with a parameter n (a lower bound of the absolute value of that sinus being $n^{-\kappa_{27}}$).

– Similarly, T_2 contains a constant factor and the factors:
 • $|\xi_1'|$, with $k^{-\kappa_9} \leq |\xi_1'| \leq k^{\kappa_9}$,
 • $\varepsilon^{n-(\ell/2)}$ (which the main term),
 • a sinus with a parameter ℓ (a lower bound of the absolute value of that sinus being $|\ell|^{-\kappa_{28}}$).

– Similarly, T_3 contains a constant factor and the factors:
 • $|\xi_1'|$, with $k^{-\kappa_9} \leq |\xi_1'| \leq k^{\kappa_9}$,
 • $\varepsilon^{-(n+\ell)/2}$ (which the main term),
 • a sinus with a parameter $\ell-n$ (a lower bound of the absolute value of that sinus being $|\ell-n|^{-\kappa_{29}}$).

We will consider three cases, and we will use the inequalities (3) and (17). This will eventually allow us to conclude that there is an upper bound for $|\ell|$ and n by an effective constant times $\log k$.

First case. If the two terms a and b with the largest absolute values are T_1 and T_2, from the inequalities $|T_1| \leq 2|T_2|$ and $|T_2| \leq 2|T_1|$ (which come from $|b| \leq |a| \leq 2|b|$), we deduce (thanks to (17))

$$k^{-\kappa_{30}}|\ell|^{-\kappa_{31}} \;\leq\; \varepsilon^{\frac{3}{2}(\ell-n)} \leq k^{\kappa_{32}}|\ell|^{\kappa_{33}},$$

whereupon, thanks again to (17), we have

$$-\kappa_{34}\log k + \frac{|\ell|}{3} \;\leq\; |\ell-n| \;\leq\; \kappa_{35}\log|\ell| + \kappa_{36}\log k,$$

which leads to $|\ell| \leq \kappa_{37}(\log k + \log|\ell|)$. This secures the upper bound (15), and ends the proof of Theorem 1.

Second case. Suppose that the two terms a and b with the largest absolute values are T_1 and T_3. By writing $|T_1| \le 2|T_3|$ and $|T_3| \le 2|T_1|$, we obtain (thanks to (17))

$$k^{-1/3}|\ell|^{-\kappa_{38}} \;\le\; \varepsilon^{3\ell/2} \;\le\; k^{1/3}|\ell|^{\kappa_{39}},$$

hence

$$|\ell| \;\le\; \kappa_{40}(\log k + \log|\ell|).$$

Once more, we have $\varepsilon^{|\ell|} \le k^{\kappa_{41}}$, and we saw that the upper bound (17) allows to draw the conclusion.

Third case. Let us consider the remaining case, namely, the two terms a and b with the largest absolute values being T_2 and T_3. Consequently, in the relation $T_1+T_2+T_3 \;=\; 0$, written in the form $a+b+c \;=\; 0$ with $|a| \ge |b| \ge |c|$, we have $c \;=\; T_1$. Writing $|T_2| \le 2|T_3|$ and $|T_3| \le 2|T_2|$, we obtain

$$k^{-1/3}|\ell|^{-\kappa_{42}} \;\le\; \varepsilon^{3n/2} \le k^{1/3}|\ell|^{\kappa_{43}}.$$

From the second of these inequalities, we deduce the existence of κ_{44} such that

$$n \;\le\; \kappa_{44}(\log k + \log|\ell|). \tag{18}$$

`Remark.` The upper bound (18) allows to proceed as in the usual proof of the Thue theorem where n is fixed.

From the upper bound $|T_1| \le |T_2|$, one deduces $n > \ell - \kappa_{45}\log k$, so that (18) leads right away to the conclusion if ℓ is positive.

Let us suppose now that ℓ is negative. Let us consider again the equation (7) that we write in the form

$$\rho_n\varepsilon^{\ell} + \mu_n\varepsilon'^{\ell} - \overline{\mu}_n\overline{\varepsilon'}^{\ell} \;=\; 0 \tag{19}$$

with

$$\rho_n \;=\; \xi_1(\alpha'\varepsilon'^n - \overline{\alpha'}\,\overline{\varepsilon'}^n) \quad\text{and}\quad \mu_n \;=\; \xi_1'(\overline{\alpha'}\,\overline{\varepsilon'}^n - \alpha\varepsilon^n).$$

We check (cf. Property 3.3 of [4])

$$\mathrm{h}(\mu_n) \le \kappa_{46}(n + \log k).$$

Let us divide each side of (19) by $-\mu_n\varepsilon'^{\ell}$:

$$\frac{\overline{\mu}_n\overline{\varepsilon'}^{\ell}}{\mu_n\varepsilon'^{\ell}} - 1 \;=\; \frac{\rho_n\varepsilon^{\ell}}{\mu_n\varepsilon'^{\ell}}.$$

We have

$$|\alpha'\varepsilon'^n - \overline{\alpha'}\,\overline{\varepsilon'}^n| \;\le\; |\alpha'\varepsilon'^n| + |\overline{\alpha'}\,\overline{\varepsilon'}^n| = 2\,|\varepsilon'^n\alpha'|$$

and, using (8),

$$|\overline{\alpha'}\,\overline{\varepsilon'}^n - \alpha\varepsilon^n| \;\ge\; \frac{1}{2}|\alpha|\varepsilon^n.$$

Since

$$\overline{\xi_1} \le k^{\kappa_9} \ \text{ and } \ |\xi_1'| > k^{-\kappa_9}$$

by (3), we come up with

$$|\rho_n| \le \kappa_{47} k^{\kappa_9} \varepsilon^{n/2}, \quad |\mu_n| \ge \kappa_{48} \varepsilon^n k^{-\kappa_9}.$$

Therefore, since $|\varepsilon'|^{-1} = \varepsilon^{1/2}$, we have

$$\left| \frac{\overline{\mu}_n \overline{\varepsilon'}^{\ell}}{\mu_n \varepsilon'^{\ell}} - 1 \right| = \left| \frac{\rho_n \varepsilon^{\ell}}{\mu_n \varepsilon'^{\ell}} \right| \le \kappa_{49} \varepsilon^{-(n+3|\ell|)/2} k^{\kappa_9}. \tag{20}$$

We denote by log the principal value of the logarithm and we set

$$\lambda_1 = \log\left(\frac{\overline{\varepsilon'}}{\varepsilon'}\right), \quad \lambda_2 = \log\left(\frac{\overline{\mu}_n}{\mu_n}\right) \quad \text{and} \quad \Lambda = \log\left(\frac{\overline{\mu}_n \overline{\varepsilon'}^{\ell}}{\mu_n \varepsilon'^{\ell}}\right).$$

We have

$$\lambda_1 = 2i\pi\nu \quad \lambda_2 = 2i\pi\theta_n,$$

where ν and θ_n are the real numbers in the interval $[0,1)$ defined by

$$\frac{\overline{\varepsilon'}}{\varepsilon'} = e^{2i\pi\nu} \quad \text{and} \quad \frac{\overline{\mu}_n}{\mu_n} = e^{2i\pi\theta_n}.$$

From $e^{\Lambda} = e^{\ell\lambda_1+\lambda_2}$ we deduce $\Lambda - \ell\lambda_1 - \lambda_2 = 2i\pi h$ with $h \in \mathbf{Z}$. From Lemma 3*b* we deduce $|\Lambda| \le 2|e^{\Lambda} - 1|$. Using $|\Lambda| < 2\pi$ and writing

$$2i\pi h = \Lambda - 2i\pi\ell\nu - 2i\pi\theta_n,$$

we deduce $|h| \le |\ell| + 2$.

In Proposition 1, let us take

$$b_0 = h, \quad b_1 = \ell, \quad b_2 = 1, \quad \gamma_0 = 1, \quad \lambda_0 = 2i\pi, \qquad \gamma_1 = \frac{\overline{\varepsilon'}}{\varepsilon'}, \quad \gamma_2 = \frac{\overline{\mu}_n}{\mu_n},$$

$$A_0 = A_1 = \kappa_{50}, \quad A_2 = (k\, \varepsilon^n)^{\kappa_{51}}, \quad B = e + \frac{|\ell|}{\log A_2} \cdot$$

Notice that the degree D of the field $\mathbf{Q}(\gamma_0, \gamma_1, \gamma_2)$ is ≤ 6. Then we obtain

$$\left| \frac{\overline{\mu}_n}{\mu_n} \left(\frac{\overline{\varepsilon'}}{\varepsilon'}\right)^{\ell} - 1 \right| = |e^{\Lambda} - 1| \ge \frac{1}{2}|\Lambda| \ge \exp\bigl\{-\kappa_{52}(\log A_2)(\log B)\bigr\}.$$

By combining this estimate with (20), we deduce

$$|\ell| \le \kappa_{53}(n + \log k)\log B,$$

which can also be written as $B \leq \kappa_{54} \log B$, hence B is bounded. This allows to obtain

$$|\ell| \;\leq\; \kappa_{55}(n + \log k).$$

We use (18) to deduce $\varepsilon^{|\ell|} \leq k^{\kappa_{41}}$ and we saw that the upper bound (15) leads to the conclusion of the main Theorem 1.

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