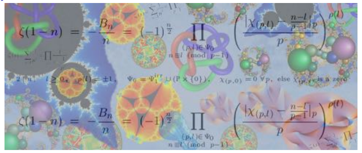
November 27, 2018

Department of Mathematics, Ramakrishna Mission Vivekananda University (RKMVU), Belur Math, Howrah, Kolkata (India).



On the Landau–Ramanujan constant

Michel Waldschmidt

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http://www.imj-prg.fr/~michel.waldschmidt/

Abstract

The Landau–Ramanujan constant α is defined as follows : for $N \to \infty$, the number of positive integers $\le N$ which are sums of two squares is asymptotically

$$\alpha \frac{N}{\sqrt{\log N}}.$$

In a joint work with Etienne Fouvry and Claude Levesque, we replace the quadratic form $\Phi_4(X,Y) = X^2 + Y^2$, which is the homogeneous version of the cyclotomic polynomial $\phi_4(t) = t^2 + 1$, with other binary forms.

This is a joint work with Étienne Fouvry and Claude Levesque

Étienne Fouvry

Claude Levesque

Representation of integers by cyclotomic binary forms.
Acta Arithmetica, **184**.1 (2018), 67 - 86.
Dedicated to Rob Tijdeman. `arXiv: 712.09019 [math.NT]`

November 6, 2017

Lecture on *Representation of positive integers by binary cyclotomic forms*
Joint work with Claude Levesque, in progress

Science Faculty, Mahidol University (Phrayathai campus), Bangkok (Thailand)
Invited by Chatchawan Panraksa

Chatchawan Panraksa

November 6, 2017

ALGEBRA AND NUMBER THEORY 7:5 (2013)
dx.doi.org/10.2140/ant.2013.7.1207

On binary cyclotomic polynomials

Étienne Fouvry

We study the number of nonzero coefficients of cyclotomic polynomials Φ_m, where m is the product of two distinct primes.

Joint work with Claude Levesque :
Representation of positive integers by binary cyclotomic forms

Étienne Fouvry

November 10-12, 2017 : ICMMEDC 2017

Mandalay (Myanmar)
The Tenth International Conference on Science and Mathematics Education in Developing Countries.

Claude Levesque

N.B. : The 11th International Conference on Mathematics and Mathematics Education in Developing Countries (ICMMEDC 2018) took place in Vientiane (Laos), October 31 - November 4, 2018.

The Landau–Ramanujan constant

Edmund Landau
1877 – 1938

Srinivasa Ramanujan
1887 – 1920

The number of positive integers $\leq N$ which are sums of two squares is asymptotically $\mathrm{C}_{\Phi_4} N(\log N)^{-\frac{1}{2}}$, where

$$\mathrm{C}_{\Phi_4} = \frac{1}{2^{\frac{1}{2}}} \cdot \prod_{p \equiv 3 \bmod 4} \left(1 - \frac{1}{p^2}\right)^{-\frac{1}{2}}.$$

Online Encyclopedia of Integer Sequences https://oeis.org/A064533

[OEIS A064533] `Decimal expansion of Landau-Ramanujan constant.`

$$\mathrm{C}_{\Phi_4} = 0.764\,223\,653\,589\,220\ldots$$

- Ph. Flajolet and I. Vardi, Zeta function expansions of some classical constants, Feb 18 1996.
- Xavier Gourdon and Pascal Sebah, Constants and records of computation.
- David E. G. Hare, $125\,079$ digits of the Landau-Ramanujan constant.

The Landau–Ramanujan constant

References : https://oeis.org/A064533

- B. C. Berndt, Ramanujan's notebook part IV, Springer-Verlag, 1994
- S. R. Finch, Mathematical Constants, Cambridge, 2003, pp. 98-104.
- G. H. Hardy, "Ramanujan, Twelve lectures on subjects suggested by his life and work", Chelsea, 1940.
- Institute of Physics, Constants - Landau-Ramanujan Constant
- Simon Plouffe, Landau Ramanujan constant
- Eric Weisstein's World of Mathematics, Ramanujan constant
- https://en.wikipedia.org/wiki/Landau-Ramanujan_constant

Sums of two squares

A prime number is a sum of two squares if and only if it is either 2 or else congruent to 1 modulo 4.

Pierre de Fermat
1607 (?) – 1665

Identity of Brahmagupta :

$$(a^2+b^2)(c^2+d^2)=e^2+f^2$$

with

$$e=ac-bd,\ f=ad+bc.$$

Brahmagupta
598 – 668

Sums of two squares

If a and q are two integers, we denote by $N_{a,q}$ any integer ≥ 1 satisfying the condition

$$p \mid N_{a,q} \Longrightarrow p \equiv a \bmod q.$$

An integer $m \geq 1$ can be written as

$$m=\Phi_4(x,y)=x^2+y^2$$

if and only if there exist integers $a \geq 0$, $N_{3,4}$ and $N_{1,4}$ such that

$$m=2^a\, N_{3,4}^2\, N_{1,4}.$$

Positive definite quadratic forms

Let $F \in \mathbb{Z}[X,Y]$ be a positive definite quadratic form. There exists a positive constant C_F such that, for $N \to \infty$, the number of positive integers $m \in \mathbb{Z}$, $m \leq N$ which are represented by F is asymptotically $\mathsf{C}_F N(\log N)^{-\frac{1}{2}}$.

Paul Bernays
1888 – 1977

P. Bernays, *Über die Darstellung von positiven, ganzen Zahlen durch die primitiven, binären quadratischen Formen einer nicht quadratischen Diskriminante*, Ph.D. dissertation, Georg-August-Universität, Göttingen, Germany, 1912.

http://www.ethlife.ethz.ch/archive_articles/120907_bernays_fm/

Paul Bernays (1888 – 1977)

https://www.thefamouspeople.com/profiles/paul-bernays-7244.php

• 1912, Ph.D. in mathematics, University of Göttingen, *On the analytic number theory of binary quadratic forms* (Advisor : Edmund Landau).
• 1913, Habilitation, University of Zürich, *On complex analysis and Picard's theorem*, advisor Ernst Zermelo.
• 1912 – 1917, Zürich ; work with Georg Pólya, Albert Einstein, Hermann Weyl.
• 1917 – 1933, Göttingen, with David Hilbert. Studied with Emmy Noether, Bartel Leendert van der Waerden, Gustav Herglotz.
• 1935 – 1936, Institute for Advanced Study, Princeton. Lectures on mathematical logic and axiomatic set theory.
• 1936 —, ETH Zürich.
• With David Hilbert, "Grundlagen der Mathematik" (1934 – 39) 2 vol. — Hilbert–Bernays paradox.
• Axiomatic Set Theory (1958). —
Von Neumann–Bernays–Gödel set theory.

Specific binary forms

• Sums of cubes, biquadrates,...

Notice that $X^3 + Y^3 = (X+Y)(X^2 - XY + Y^2)$

We start with the quadratic form $\Phi_3(X,Y) = X^2 + XY + Y^2$ which is the homogeneous version of the cyclotomic polynomial $\phi_3(t) = t^2 + t + 1$.
Notice that

$$\Phi_6(X,Y) = \Phi_3(X,-Y) = X^2 - XY + Y^2$$

Also

$$\Phi_8(X,Y) = X^4 + Y^4.$$

The quadratic form $x^2 + xy + y^2$

A prime number is represented by the quadratic form $x^2 + xy + y^2$ if and only if it is either 3 or else congruent to 1 modulo 3.
Product of two numbers represented by the quadratic form $x^2 + xy + y^2$:

$$(a^2 + ab + b^2)(c^2 + cd + d^2) = e^2 + ef + f^2$$

with

$$e = ac - bd, \; f = ad + bd + bc.$$

The quadratic cyclotomic field $\mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta_3)$,
$1 + \zeta_3 + {\zeta_3}^2 = 0$:

$$a^2 + ab + b^2 = \mathrm{Norm}_{\mathbb{Q}(\zeta_3)/\mathbb{Q}}(a - \zeta_3 b).$$

Loeschian numbers : $m = x^2 + xy + y^2$

An integer $m \geq 1$ can be written as

$$m = \Phi_3(x,y) = \Phi_6(x,-y) = x^2 + xy + y^2$$

if and only if there exist integers $b \geq 0$, $N_{2,3}$ and $N_{1,3}$ such that

$$m = 3^b \, N_{2,3}^2 \, N_{1,3}.$$

The number of positive integers $\leq N$ which are represented by the quadratic form $x^2 + xy + y^2$ is asymptotically $\mathsf{C}_{\Phi_3} N(\log N)^{-\frac{1}{2}}$, where

$$\mathsf{C}_{\Phi_3} = \frac{1}{2^{\frac{1}{2}} 3^{\frac{1}{4}}} \cdot \prod_{p \equiv 2 \bmod 3} \left(1 - \frac{1}{p^2}\right)^{-\frac{1}{2}}$$

Online Encyclopedia of Integer Sequences OEIS A301429

[OEIS A301429] `Decimal expansion of an analog of the Landau-Ramanujan constant for Loeschian numbers.`

The first decimal digits of C_{Φ_3} are

$$\mathsf{C}_{\Phi_3} = 0.638\,909\,405\,44\ldots$$

April 17-21, 2018 : Roma (Italia)

Lecture on
Representation of integers by cyclotomic binary forms

4th Mini Symposium of the Roman Number Theory Association

http://www.rnta.eu/ms.html

Zeta function expansions of some classical constants, Feb 18 1996.

Philippe Flajolet

Ilan Vardi

Bill Allombert

$$\begin{aligned}\mathsf{C}_{\Phi_3} = {} & 0.638909405445343882\\ & 225494267492824509375\\ & 49755080291233454216\\ & 92365708076310027649\\ & 65824689717911252866\\ & 43881416875191074 24\ldots\end{aligned}$$

Loeschian numbers which are sums of two squares

An integer $m \geq 1$ is simultaneously of the forms

$$m = \Phi_4(x,y) = x^2 + y^2 \text{ and } m = \Phi_3(u,v) = u^2 + uv + v^2$$

if and only if there exist integers a, $b \geq 0$, $N_{5,12}$, $N_{7,12}$, $N_{11,12}$ and $N_{1,12}$ such that

$$m = \left(2^a\, 3^b\, N_{5,12}\, N_{7,12}\, N_{11,12}\right)^2 N_{1,12}.$$

The number of Loeschian integers $\leq N$ which are sums of two squares is asymptotically $\beta N (\log N)^{-3/4}$, where

$$\beta = \frac{3^{\frac{1}{4}}}{2^{\frac{5}{4}}} \cdot \pi^{\frac{1}{2}} \cdot (\log(2+\sqrt{3}))^{\frac{1}{4}} \cdot \frac{1}{\Gamma(1/4)} \cdot \prod_{p \equiv 5,7,11 \bmod 12} \left(1 - \frac{1}{p^2}\right)^{-\frac{1}{2}}.$$

OEIS A301430 $\qquad \beta = 0.302\,316\,142\,35\ldots$

[OEIS A301430] `Decimal expansion of an analog of the Landau-Ramanujan constant for Loeschian numbers which are sums of two squares.`

$$\beta = \frac{3^{\frac{1}{4}}}{2^{\frac{5}{4}}} \cdot \pi^{\frac{1}{2}} \cdot (\log(2+\sqrt{3}))^{\frac{1}{4}} \cdot \frac{1}{\Gamma(1/4)} \cdot \prod_{p \equiv 5,7,11 \bmod 12} \left(1 - \frac{1}{p^2}\right)^{-\frac{1}{2}}$$

Bill Allombert

$$\beta = \begin{array}{l} 0.30231614235706563794 \\ 77699004801997156024 12 \\ 79518936964545886784 12 \\ 88865448752410510899 48 \\ 74678139792727085677 65 \\ 91327259106668371358 63\ldots \end{array}$$

Cyclotomic polynomials

Definition by induction :

$$\phi_1(t) = t - 1, \qquad t^n - 1 = \prod_{d \mid n} \phi_d(t).$$

For p prime,

$$t^p - 1 = (t-1)(t^{p-1} + t^{p-2} + \cdots + t + 1) = \phi_1(t)\phi_p(t),$$

so

$$\phi_p(t) = t^{p-1} + t^{p-2} + \cdots + t + 1.$$

For instance

$$\phi_2(t) = t+1, \quad \phi_3(t) = t^2+t+1, \quad \phi_5(t) = t^4+t^3+t^2+t+1.$$

Cyclotomic polynomials

$$\phi_n(t) = \frac{t^n - 1}{\displaystyle\prod_{\substack{d \neq n \\ d \mid n}} \phi_d(t)}.$$

For instance

$$\phi_4(t) = \frac{t^4 - 1}{t^2 - 1} = t^2 + 1 = \phi_2(t^2),$$

$$\phi_6(t) = \frac{t^6 - 1}{(t^3 - 1)(t+1)} = \frac{t^3 + 1}{t + 1} = t^2 - t + 1 = \phi_3(-t).$$

The degree of $\phi_n(t)$ is $\varphi(n)$, where φ is the Euler totient function.

Cyclotomic polynomials and roots of unity

For $n \geq 1$, if ζ is a primitive n–th root of unity,

$$\phi_n(t) = \prod_{\gcd(j,n)=1} (t - \zeta^j).$$

For $n \geq 1$, $\phi_n(t)$ is the irreducible polynomial over $\mathbb{Q}$ of the primitive n–th roots of unity,

Let K be a field and let n be a positive integer. Assume that K has characteristic either 0 or else a prime number p prime to n. Then the polynomial $\phi_n(t)$ is separable over K and its roots in K are exactly the primitive n–th roots of unity which belong to K.

Properties of $\phi_n(t)$

- For $n \geq 2$ we have

$$\phi_n(t) = t^{\varphi(n)}\phi_n(1/t)$$

- Let $n = 2^{e_0}p_1^{e_1}\cdots p_r^{e_r}$ where $p_1, \dots, p_r$ are different odd primes, $e_0 \geq 0$, $e_i \geq 1$ for $i = 1, \dots, r$ and $r \geq 1$. Denote by R the radical of n, namely

$$R = \begin{cases} 2p_1\cdots p_r & \text{if } e_0 \geq 1, \\ p_1\cdots p_r & \text{if } e_0 = 0. \end{cases}$$

Then,

$$\phi_n(t) = \phi_R(t^{n/R}).$$

- Let $n = 2m$ with m odd ≥ 3. Then

$$\phi_n(t) = \phi_m(-t).$$

$\phi_n(1)$

For $n \geq 2$, we have $\phi_n(1) = e^{\Lambda(n)}$, where the von Mangoldt function is defined for $n \geq 1$ as

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^r \text{ with } p \text{ prime and } r \geq 1\,; \\ 0 & \text{otherwise.} \end{cases}$$

In other terms we have

$$\phi_n(1) = \begin{cases} p & \text{if } n = p^r \text{ with } p \text{ prime and } r \geq 1\,; \\ 1 & \text{otherwise.} \end{cases}$$

$\phi_n(-1)$

For $n \geq 3$,

$$\phi_n(-1) = \begin{cases} 1 & \text{if } n \text{ is odd}\,; \\ \phi_{n/2}(1) & \text{if } n \text{ is even.} \end{cases}$$

In other terms, for $n \geq 3$,

$$\phi_n(-1) = \begin{cases} p & \text{if } n = 2p^r \text{ with } p \text{ a prime and } r \geq 1\,; \\ 1 & \text{otherwise.} \end{cases}$$

Hence $\phi_n(-1) = 1$ when n is odd or when $n = 2m$ where m has at least two distinct prime divisors.

Lower bound for $\phi_n(t)$

For $n \geq 3$, the polynomial $\phi_n(t)$ has real coefficients and no real root, hence it takes only positive values (and its degree $\varphi(n)$ is even).
For $n \geq 3$ and $t \in \mathbb{R}$, we have

$$\phi_n(t) \geq 2^{-\varphi(n)}.$$

Consequence : from

$$\phi_n(t) = t^{\varphi(n)}\phi_n(1/t)$$

we deduce, for $n \geq 3$ and $t \in \mathbb{R}$,

$$\phi_n(t) \geq 2^{-\varphi(n)}\max\{1, |t|\}^{\varphi(n)}.$$

$\phi_n(t) \geq 2^{-\varphi(n)}$ for $n \geq 3$ and $t \in \mathbb{R}$

Proof.
Let ζ_n be a primitive n-th root of unity in $\mathbb{C}$;

$$\phi_n(t) = \mathrm{N}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(t - \zeta_n) = \prod_\sigma (t - \sigma(\zeta_n)),$$

where σ runs over the embeddings $\mathbb{Q}(\zeta_n) \to \mathbb{C}$. We have

$$|t - \sigma(\zeta_n)| \geq |\Im\mathrm{m}(\sigma(\zeta_n))| > 0,$$

$$(2i)\Im\mathrm{m}(\sigma(\zeta_n)) = \sigma(\zeta_n) - \overline{\sigma(\zeta_n)} = \sigma(\zeta_n - \overline{\zeta_n}).$$

Now $(2i)\Im\mathrm{m}(\zeta_n) = \zeta_n - \overline{\zeta_n} \in \mathbb{Q}(\zeta_n)$ is an algebraic integer :

$$2^{\varphi(n)}\phi_n(t) \geq |\mathrm{N}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}((2i)\Im\mathrm{m}(\zeta_n))| \geq 1.$$

Generalization to CM fields

K. Győry

L. Lovász

K. Győry & L. Lovász, *Representation of integers by norm forms II*, Publ. Math. Debrecen **17**, 173–181, (1970).
K. Győry, *Représentation des nombres entiers par des formes binaires*, Publ. Math. Debrecen **24**, 363–375, (1977).

Refinement (FLW)

Let $c_n = \inf_{t\in\mathbb{R}} \phi_n(t)$.
Refinement of the lower bound $c_n \geq 2^{-\varphi(n)}$:

For $n \geq 3$

$$c_n \geq \left(\frac{\sqrt{3}}{2}\right)^{\varphi(n)}.$$

Equality for $n = 3$ and $n = 6$.

For n a power of 2, $c_n = 1$.
Otherwise, if n has r distinct primes $p_1, \ldots, p_r$ with p_1 the smallest, then

$$c_n = c_{p_1\cdots p_r} \geq p_1{}^{-2^{r-2}}.$$

The cyclotomic binary forms

For $n \geq 2$, define

$$\Phi_n(X, Y) = Y^{\varphi(n)}\phi_n(X/Y).$$

This is a binary form in $\mathbb{Z}[X, Y]$ of degree $\varphi(n)$.
Consequence of the lower bound $c_n \geq 2^{-\varphi(n)}$:
for $n \geq 3$ and $(x, y) \in \mathbb{Z}^2$,

$$\Phi_n(x, y) \geq 2^{-\varphi(n)} \max\{|x|, |y|\}^{\varphi(n)}.$$

Therefore, if $\Phi_n(x, y) = m$, then

$$\max\{|x|, |y|\} \leq 2m^{1/\varphi(n)}.$$

If $\max\{|x|, |y|\} \geq 3$, then n is bounded :

$$\varphi(n) \leq \frac{\log m}{\log(3/2)}.$$

Binary cyclotomic forms (EF–CL–MW 2018)

Let m be a positive integer and let n, x, y be rational integers satisfying $n \geq 3$, $\max\{|x|, |y|\} \geq 2$ and $\Phi_n(x, y) = m$.Then

$$\max\{|x|, |y|\} \leq \frac{2}{\sqrt{3}} m^{1/\varphi(n)}, \quad \text{hence} \quad \varphi(n) \leq \frac{2}{\log 3} \log m.$$

These estimates are optimal, since for $\ell \geq 1$,

$$\Phi_3(\ell, -2\ell) = 3\ell^2.$$

If we assume $\varphi(n) > 2$, namely $\varphi(n) \geq 4$, then

$$\varphi(n) \leq \frac{4}{\log 11} \log m$$

which is best possible since $\Phi_5(1, -2) = 11$.

The sequence $(a_m)_{m\geq 1}$

For each integer $m \geq 1$, the set

$$\{(n, x, y) \in \mathbb{N}\times\mathbb{Z}^2 \mid n \geq 3, \max\{|x|, |y|\} \geq 2, \Phi_n(x, y) = m\}$$

is finite. Let a_m the number of its elements.

The sequence of integers $m \geq 1$ such that $a_m \geq 1$ starts with the following values of a_m

m	3	4	5	7	8	9	10	11	12	13	16	17
a_m	8	16	8	24	4	16	8	8	12	40	40	16

OEIS A299214

https://oeis.org/A299214
Number of representations of integers by cyclotomic binary forms.

The sequence $(a_m)_{m\geq 1}$ starts with
$0, 0, 8, 16, 8, 0, 24, 4, 16, 8, 8, 12, 40, 0, 0, 40, 16, 4, 24, 8, 24,$
$0, 0, 0, 24, 8, 12, 24, 8, 0, 32, 8, 0, 8, 0, 16, 32, 0, 24, 8, 8, 0, 32,$
$0, 8, 0, 0, 12, 40, 12, 0, 32, 8, 0, 8, 0, 32, 8, 0, 0, 48, 0, 24, 40,$
$16, 0, 24, 8, 0, 0, 0, 4, 48, 8, 12, 24, \ \dots$

OEIS A296095

https://oeis.org/A296095
Integers represented by cyclotomic binary forms.

$a_m \neq 0$ for $m =$
$3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 16, 17, 18, 19, 20, 21, 25, 26, 27,$
$28, 29, 31, 32, 34, 36, 37, 39, 40, 41, 43, 45, 48, 49, 50, 52, 53,$
$55, 57, 58, 61, 63, 64, 65, 67, 68, 72, 73, 74, 75, 76, 79, 80, 81,$
$82, 84, 85, 89, 90, 91, 93, 97, 98, 100, 101, 103, 104, 106, 108,$
$109, 111, 112, 113, 116, 117, 121, 122, \ \dots$

OEIS A293654

https://oeis.org/A293654
Integers not represented by cyclotomic binary forms.

$a_m = 0$ for $m =$
$1, 2, 6, 14, 15, 22, 23, 24, 30, 33, 35, 38, 42, 44, 46, 47, 51, 54,$
$56, 59, 60, 62, 66, 69, 70, 71, 77, 78, 83, 86, 87, 88, 92, 94, 95,$
$96, 99, 102, 105, 107, 110, 114, 115, 118, 119, 120, 123, 126,$
$131, 132, 134, 135, 138, 140, 141, 142, 143, 150, \ \ldots$

Numbers represented by a cyclotomic binary form of degree ≥ 2

For $N \geq 1$, the number of $m \leq N$ for which there exists $n \geq 3$ and $(x, y) \in \mathbb{Z}^2$ with $\max(|x|, |y|) \geq 2$ and $m = \Phi_n(x, y)$, is asymptotically

$$(\mathsf{C}_{\Phi_4} + \mathsf{C}_{\Phi_3}) \frac{N}{(\log N)^{\frac{1}{2}}} - \beta \frac{N}{(\log N)^{\frac{3}{4}}} + O\left(\frac{N}{(\log N)^{\frac{3}{2}}}\right)$$

as $N \to \infty$.

$\mathsf{C}_{\Phi_4} + \mathsf{C}_{\Phi_3} = 1.403\,133\,059\,034\ \ldots \quad \beta = 0.302\,316\,142\,35\ldots$

Etienne Fouvry, Claude Levesque & M.W. ; *Representation of integers by cyclotomic binary forms.* Acta Arithmetica, **184**.1 (2018), 67 - 86.
Dedicated to Rob Tijdeman. arXiv: 712.09019 [math.NT]

Higher degree

The situation for quadratic forms of degree ≥ 3 is different for several reasons.

• If a positive integer m is represented by a positive definite quadratic form, it usually has many such representations ; while if a positive integer m is represented by an irreducible binary form of degree $d \geq 3$, it usually has few such representations.

• If F is a positive definite quadratic form, the number of (x, y) with $F(x, y) \leq N$ is asymptotically a constant times N, but the number of $F(x, y)$ is much smaller.

• If F is an irreducible binary form of degree $d \geq 3$, the number of (x, y) with $F(x, y) \leq N$ is asymptotically a constant times $N^{\frac{2}{d}}$, the number of $F(x, y)$ is also asymptotically a constant times $N^{\frac{2}{d}}$.

Higher degree

A quadratic form has infinitely many automorphisms, an irreducible binary form of higher degree has a finite group of automorphisms.

Stanley Yao Xiao

S. Yao Xiao, *On the representation of integers by binary quadratic forms.*
arXiv:1704.00221

Sums of k–th powers

If a positive integer m is a sum of two squares, there are many such representations.
Indeed, the number of (x, y) in $\mathbb{Z} \times \mathbb{Z}$ with $x^2 + y^2 \leq N$ is asymptotic to πN, while the number of values $\leq N$ taken by the quadratic form Φ_4 is asymptotic to $\mathsf{C}_{\Phi_4} N/\sqrt{\log N}$ where C_{Φ_4} is the Landau–Ramanujan constant. Hence Φ_4 takes each of these values with a high multiplicity, on the average $(\pi/\mathsf{C}_{\Phi_4})\sqrt{\log N}$.
On the opposite, given an integer $k \geq 3$, that a positive integer is a sum of two k–th powers in more than one way (not counting symmetries) is

- rare for $k = 3$,
- extremely rare for $k = 4$,
- maybe impossible for $k \geq 5$.

1729 : the taxicab number

The smallest positive integer which is sum of two cubes in two essentially different ways :

$$1729 = 10^3 + 9^3 = 12^3 + 1^3.$$

Godfrey Harold Hardy
1877–1947

Srinivasa Ramanujan
1887 – 1920

1657 : Frénicle de Bessy (1605 ? – 1675)

The sequence of Taxicab numbers

[OEIS A001235] `Taxi-cab numbers: sums of` 2 `cubes in more than` 1 `way.`

$$1729 = 10^3 + 9^3 = 12^3 + 1^3, \quad 4104 = 2^3 + 16^3 = 9^3 + 15^3, \ldots$$

$$1729, 4104, 13832, 20683, 32832, 39312, 40033, 46683, 64232, 65728, 110656, 110808, 134379, 149389, 165464, 171288, 195841, 216027, 216125, 262656, 314496, 320264, 327763, 373464, 402597, 439101, 443889, 513000, 513856, 515375, 525824, 558441, 593047, \ldots$$

If n is in this sequence, then nk^3 also, hence this sequence is infinite.

Another sequence of Taxicab numbers (Fermat)

[OEIS A011541] `Hardy-Ramanujan numbers: the smallest number that is the sum of` 2 `positive integral cubes in` n `ways.`
`http://mathworld.wolfram.com/TaxicabNumber.html`
$T_a(1) = 2,$

$T_a(2) = 1729 = 10^3 + 9^3 = 12^3 + 1^3,$

$T_a(3) = 87\,539\,319 = 167^3 + 436^3 = 228^3 + 423^3 = 255^3 + 414^3,$

$T_a(4) = 6\,963\,472\,309\,248 = 2421^3 + 19\,083^3 = 5436^3 + 18\,948^3 = 10\,200^3 + 18\,072^3 = 13\,322^3 + 16\,630^3,$

$T_a(5) = 48\,988\,659\,276\,962\,496,$
$T_a(6) = 24\,153\,319\,581\,254\,312\,065\,344.$

Hardy and Wright,
An Introduction of Theory of Numbers

Fermat proved that numbers expressible as a sum of two positive integral cubes in n different ways exist for any n.

Pierre de Fermat
1607 (?) – 1665

2003 : C. S. Calude, E. Calude and M. J. Dinneen,
With high probability,

$$T_a(6) = 24153319581254312065344.$$

Cubefree taxicab numbers

$$15\,170\,835\,645 = 517^3+2468^3 = 709^3+2456^3 = 1733^3+2152^3.$$

The smallest cubefree taxicab number with three representations was discovered by Paul Vojta (unpublished) in 1981 while he was a graduate student.

Paul Vojta

Cubefree taxicab numbers

Stuart Gascoigne and Duncan Moore (2003) :
$1\,801\,049\,058\,342\,701\,083 = 92227^3 + 1216500^3 = 136635^3 + 1216102^3 = 341995^3 + 1207602^3 = 600259^3 + 1165884^3$

[OEIS A080642] `Cubefree taxicab numbers: the smallest cubefree number that is the sum of` 2 `cubes in` n `ways.`

https://en.wikipedia.org/wiki/Taxicab_number

Taxicabs and Sums of Two Cubes

If the sequence (a_n) of cubefree taxicab numbers with n representations is infinite, then the Mordell-Weil rank of the elliptic curve $x^3 + y^3 = a_n$ tends to infinity with n.

Joseph Silverman

J. H. Silverman, Taxicabs and Sums of Two Cubes, Amer. Math. Monthly, **100** (1993), 331-340.

$635\,318\,657 = 158^4 + 59^4 = 134^4 + 133^4$.

Leonhard Euler
1707 – 1783

The smallest integer represented by $x^4 + y^4$ in two essentially different ways was found by Euler, it is $635\,318\,657 = 41 \times 113 \times 241 \times 569$.

[OEIS A216284] Number of solutions to the equation $x^4 + y^4 = n$ with $x \geq y > 0$.
An infinite family with one parameter is known for non trivial solutions to $x_1^4 + x_2^4 = x_3^4 + x_4^4$.
http://mathworld.wolfram.com/DiophantineEquation4thPowers.html

Sums of k–th powers

One conjectures that given $k \geq 5$, if an integer can be written as $x^k + y^k$, there is essentially a unique such representation. But there is no value of k for which this has been proved.

Binary cyclotomic forms of higher degree

The situation for binary cyclotomic forms is different when the degree is 2 or when it is > 2 also for the following reason.
A necessary and sufficient condition for a number m to be represented by one of the quadratic forms Φ_3, Φ_4, is given by a congruence.
By contrast, consider the quartic binary form $\Phi_8(X, Y) = X^4 + Y^4$. On the one hand, an odd integer represented by Φ_8 is of the form

$$N_{1,8}(N_{3,8}N_{5,8}N_{7,8})^4.$$

On the other hand, there are many integers of this form which are not represented by Φ_8.

[OEIS A004831] Numbers that are the sum of at most 2 nonzero 4th powers.

$$0, 1, 2, 16, 17, 32, 81, 82, 97, 162, 256, 257, 272, 337, 512, 625, \dots$$

Quartan primes

[OEIS A002645] Quartan primes: primes of the form $x^4 + y^4$, $x > 0$, $y > 0$.

The list of prime numbers represented by Φ_8 start with
$2, 17, 97, 257, 337, 641, 881, 1297, 2417, 2657, 3697, 4177, 4721, 6577, 10657, 12401, 14657, 14897, 15937, 16561, 28817, 38561, 39041, 49297, 54721, 65537, 65617, 66161, 66977, 80177, 83537, 83777, 89041, 105601, 107377, 119617, \dots$

It is not known whether this list is finite or not.

The largest known quartan prime is currently the largest known generalized Fermat prime: The $1\,353\,265$-digit $(145\,310^{65\,536})^4 + 1^4$.

Primes of the form $x^{2^k} + y^{2^k}$

[OEIS A002313] primes of the form $x^2 + y^2$,
[OEIS A002645] primes of the form $x^4 + y^4$,
[OEIS A006686] primes of the form $x^8 + y^8$,
[OEIS A100266] primes of the form $x^{16} + y^{16}$,
[OEIS A100267] primes of the form $x^{32} + y^{32}$.

Primes of the form $X^2 + Y^4$

John Friedlander

Henryk Iwaniec

However, it is known that there are infinitely many prime numbers of the form $X^2 + Y^4$.

Friedlander, J. & Iwaniec, H. *The polynomial $X^2 + Y^4$ captures its primes*, Ann. of Math. (2) **148** (1998), no. 3, 945–1040.

https://arxiv.org/pdf/math/9811185.pdf [A028916]

K. Mahler (1933)

Let F be a binary form of degree $d \geq 3$ with nonzero discriminant.
Denote by A_F the area (Lebesgue measure) of the domain

$$\{(x, y) \in \mathbb{R}^2 \ \mid \ F(x, y) \leq 1\}.$$

For $Z > 0$ denote by $N_F(Z)$ the number of $(x, y) \in \mathbb{Z}^2$ such that $0 < |F(x, y)| \leq Z$.
Then

$$N_F(Z) = A_F Z^{\frac{2}{d}} + O(Z^{\frac{1}{d-1}})$$

as $Z \to \infty$.

Kurt Mahler

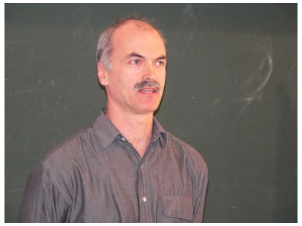
Kurt Mahler
1903 – 1988

Über die mittlere Anzahl der Darstellungen grosser Zahlen durch binäre Formen,
Acta Math. **62** (1933), 91-166.
https://carma.newcastle.edu.au/mahler/biography.html

C.L. Stewart - S.Y. Xiao

Let F be a binary form of degree $d \geq 3$ with nonzero discriminant.
There exists a positive constant $C_F > 0$ such that the number of integers of absolute value at most N which are represented by $F(X,Y)$ is asymptotic to $C_F N^{\frac{2}{d}} + O(N^{\beta_d})$ with $\beta_d < \frac{2}{d}$.

Cam Stewart and Stanley Yao Xiao


Cam Stewart


Stanley Yao Xiao

C.L. Stewart and S. Yao Xiao, *On the representation of integers by binary forms*,
`arXiv:1605.03427v2`

Cyclotomic binary forms of degree 4

(*Joint work with Étienne Fouvry - in progress*).

$\Phi_5(X,Y) = X^4 + X^3Y + X^2Y^2 + XY^3 + Y^4$.
$\Phi_8(X,Y) = X^4 + Y^4$.
$\Phi_{12}(X,Y) = X^4 - X^2Y^2 + Y^4$.
Also
$\Phi_{10}(X,Y) = \Phi_5(X,-Y) = X^4 - X^3Y + X^2Y^2 - XY^3 + Y^4$.

For $n \in \{5, 8, 12\}$, the number of positive integers $m \leq N$ which can be written as $m = \Phi_n(x,y)$ is asymptotic to $C_{\Phi_n} N^{\frac{1}{2}}$.

Numbers represented by two cyclotomic binary forms of degree 4

The number of integers $\leq N$ which are represented by two of the three quartic cyclotomic binary forms Φ_5, Φ_8 and Φ_{12} is bounded by $O_\epsilon(N^{\frac{3}{8}+\epsilon})$.

Consequence : the number of integers $\leq N$ which are represented by a cyclotomic binary form of degree 4 is asymptotic to

$$C_4 N^{\frac{1}{2}} + O_\epsilon(N^{\frac{3}{8}+\epsilon}),$$

where

$$C_4 = C_{\Phi_5} + C_{\Phi_8} + C_{\Phi_{12}}.$$

Numbers represented by a cyclotomic binary form of degree $\geq d$

Any prime number p is represented by a cyclotomic binary form : $\Phi_p(1,1) = p$.

Given an integer $d \geq 2$, we consider the set of positive integers m which can be written as $m = \Phi_n(x,y)$ with $n \geq d$ and $(x,y) \in \mathbb{Z}^2$ satisfying $\max(|x|,|y|) \geq 2$.

Numbers represented by a cyclotomic binary form of degree $\geq d$

Let $d \geq 6$. The number of integers $m \leq N$ which can be written $m = \Phi_n(x,y)$ with $n \geq d$ and $(x,y) \in \mathbb{Z}^2$ satisfying $\max(|x|,|y|) \geq 2$ is asymptotic to

$$C_d N^{\frac{2}{d}} + O_d(N^{\frac{2}{d+2}}),$$

with

$$C_d = \sum_n \mathsf{C}_{\Phi_n},$$

where the sum is over the set of integers n such that $\varphi(n) = d$ and n is not congruent to 2 modulo 4.

Isomorphic cyclotomic binary forms

Recall that the cyclotomic polynomials $\phi_n(t) \in \mathbb{Z}[t]$ satisfy $\phi_{2n}(t) = \phi_n(-t)$ for odd $n \geq 3$.

For n_1 and n_2 positive integers with $n_1 < n_2$, the following conditions are equivalent :
(1) $\varphi(n_1) = \varphi(n_2)$ and the two binary forms Φ_{n_1} et Φ_{n_2} are isomorphic.
(2) The two binary forms Φ_{n_1} and Φ_{n_2} represent the same integers.
(3) n_1 is odd and $n_2 = 2n_1$.

Even integers not represented by Euler totient function

The list of even integers which are not values of Euler φ function (i.e., for which $C_d = 0$) starts with

$14, 26, 34, 38, 50, 62, 68, \mathbf{74}, \mathbf{76}, 86, 90, 94, 98, 114, 118, \mathbf{122}, \mathbf{124}, 134, 142, 146, \mathbf{152}, \mathbf{154}, 158, 170, 174, 182, \mathbf{186}, \mathbf{188}, 194, 202, 206, 214, 218, 230, \mathbf{234}, \mathbf{236}, \mathbf{242}, \mathbf{244}, \mathbf{246}, \mathbf{248}, 254, 258, 266, 274, 278, \mathbf{284}, \mathbf{286}, 290, 298, \mathbf{302}, \mathbf{304}, 308, 314, 318, \ldots$

[`OEIS A005277`] `Nontotients: even` n `such that` $\varphi(m) = n$ `has no solution.`

Numbers represented by two cyclotomic binary forms of the same degree

Given two binary cyclotomic forms of the same degree and not isomorphic, and given $\epsilon > 0$, for $N \to \infty$ the number of positive integers $\le N$ which are represented by these two forms is bounded by

$$\begin{cases} O_\epsilon(N^{\frac{3}{d\sqrt{d}}+\epsilon}) \text{ for } d = 4, 6, 8, \\ \\ O_{d,\epsilon}(N^{\frac{1}{d}+\epsilon}) \text{ for } d \ge 10. \end{cases}$$

A weak but uniform bound

For $d \ge 2$ and $N \to \infty$, the number of $m \le N$ for which there exists $n \ge d$ and $(x, y) \in \mathbb{Z}^2$ with $\max(|x|, |y|) \ge 2$ and $m = \Phi_n(x, y)$ is bounded by

$$29N^{\frac{2}{d}}(\log N)^{1.161}.$$

Further developments (work in progress)

Representation of integers by other binary forms

• Representation of integers by the binary forms $X^n + Y^n$, $X^n - Y^n$ and $F_n(X, Y)$, where

$$F_n(X, Y) = X^n + X^{n-1}Y + \cdots + XY^{n-1} + Y^n.$$

Suggestion of *Florian Luca* (RNTA 2018)

Study the representation of integers by the polynomials *Dickson polynomials of the first and second kind*

• The sequence of *Dickson polynomials of the first kind* $(D_n)_{n\ge0}$ (resp. *second kind* $(E_n)_{n\ge0}$) is defined by

$$D_n(X + Y, XY) = X^n + Y^n$$

(resp.

$$E_n(X + Y, XY) = F_n(X, Y)).$$

Dickson polynomials : representation of integers by $X^n + Y^n$ and $X^n - Y^n$ when $x + y$ and xy are integers (x and y are quadratic integers).

Cyclotomic Dickson polynomials

- For $n \geq 2$, define

$$\Psi_n(X+Y, XY) = \Phi_n(X, Y).$$

Study the representation of integers by the polynomials Ψ_n.

Representation of integers by $\Phi_n(X, Y)$ where $x+y$ and xy are integers.

Dickson polynomials are not homogeneous.

Work in progress...

November 27, 2018

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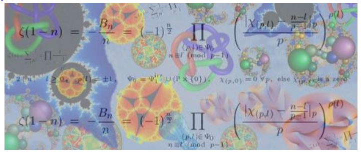

On the Landau–Ramanujan constant

Michel Waldschmidt

Sorbonne Université, Institut de Mathématiques de Jussieu
http://www.imj-prg.fr/~michel.waldschmidt/