The Legacy of Srinivasa Ramanujan
Viceregal Lodge, University of Delhi
December 17 - 22, 2012

From highly composite numbers to transcendental number theory

by

Michel Waldschmidt

http://www.math.jussieu.fr/∼miw/

Abstract

In his well–known paper in Proc. London Math. Soc., 1915, Ramanujan defined and studied *highly composite numbers*. A highly composite number is a positive integer m with more divisors than any positive integer smaller than m.

This work was pursued in 1944 by L. Alaoglu and P. Erdős (*On highly composite and similar numbers*, Trans. Amer. Math. Soc.), who raised a question which is related with this topic and which belongs to transcendental number theory.

A simple instance is the following open question : *does there exist a real irrational number t such that 2^t and 3^t are integers ?*

We plan to survey the development of this problem.

Srinivasa Ramanujan

On highly composite and similar numbers
Proc. London Math. Soc. 1915 (2) 14, 347–409
JFM 45.0286.02

Proc. London Math. Soc. 1915 (2) 14, 347–409

HIGHLY COMPOSITE NUMBERS. 347

HIGHLY COMPOSITE NUMBERS

By S. RAMANUJAN.

Communicated by G. H. HARDY.*

[Read June 11th, 1914.—Received November 1st, 1914.—
Received, in revised form, March 10th, 1915.]

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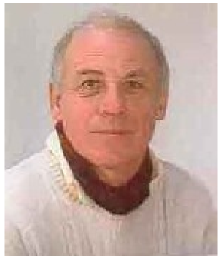
CONTENTS.

* I am indebted to Mr. Hardy for valuable suggestions and assistance in preparing this paper.

Jean-Louis Nicolas and Guy Robin

Ramanujan, Srinivasa. *Highly composite numbers.* Annotated by Jean-Louis Nicolas and Guy Robin. Ramanujan J. 1, No. 2, 119-153 (1997).



Jean-Louis Nicolas and Guy Robin

Ramanujan, Srinivasa. *Highly composite numbers.* Annotated by Jean-Louis Nicolas and Guy Robin. Ramanujan J. 1, No. 2, 119-153 (1997).

In 1915, the London Mathematical Society published in its Proceedings a paper of Ramanujan entitled "Highly composite numbers" [Proc. Lond. Math. Soc. (2) 14, 347–409 (1915; JFM 45.0286.02)]. But it was not the whole work on the subject, and in "The lost notebook and other unpublished papers", one can find a manuscript, handwritten by Ramanujan, which is the continuation of the paper published by the London Mathematical Society.
This paper is the typed version of the above mentioned manuscript with some notes, mainly explaining the link between the work of Ramanujan and the works published after 1915 on the subject.

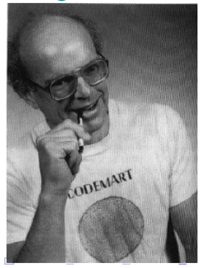
The sequence of highly composite numbers

$$1,\ 2,\ 4,\ 6,\ 12,\ 24,\ 36,\ 48,\ 60,\ 120,\ 180,\ 240,\ 360,\ 720,\ \ldots$$

Number of divisors

$$1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 9,\ 10,\ 12,\ 16,\ 18,\ 20,\ 24,\ 30,\ 32,\ \ldots$$

http://oeis.org/A002182 http://oeis.org/A002183



Neil J. A. Sloane

The divisor function $d(n)$

The divisor function $d(n) = \sum_{d|n} 1$:

$$\begin{array}{rccccccccc} n \ = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ d(n) = & 1 & 2 & 2 & 3 & 2 & 4 & 2 & 4 & 3 \end{array}$$

http://oeis.org/A000005
$1, 2, 2, 3, 2, 4, 2, 4, 3, 4, 2, 6, 2, 4, 4, 5, 2, 6, 2, 6, 4, 4, 2, 8, 3, 4, 4, 6, \ldots$

The divisor function $d(n)$

The divisor function $d(n) = \sum_{d|n} 1$:

$$\begin{array}{rccccccccc} n \ = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ d(n) = & 1 & 2 & 2 & 3 & 2 & 4 & 2 & 4 & 3 \end{array}$$

Prime numbers : $d(n) = 2$:

$$\begin{array}{rccccccccc} n \ = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ d(n) = & & 2 & 2 & & 2 & & 2 & & \end{array}$$

http://oeis.org/A000040
$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, \ldots$

The divisor function $d(n)$

The divisor function $d(n) = \sum_{d|n} 1$:

$$\begin{array}{rccccccccc} n \ = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ d(n) = & 1 & 2 & 2 & 3 & 2 & 4 & 2 & 4 & 3 \end{array}$$

• Highly composite numbers : *A highly composite number is a positive integer m with more divisors than any positive integer smaller than m.*

$$\begin{array}{rccccccccc} n \ = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ d(n) = & 1 & 2 & & 3 & & 4 & & & \end{array}$$

The divisor function $d(n)$

At

$$n = p_1^{a_1} \cdots p_s^{a_s} \quad = \quad \prod_p p^{a(p)},$$

the divisor function $d(n) = \sigma_0(n)$ takes the value

$$d(n) = (a_1 + 1) \cdots (a_s + 1) \quad = \quad \prod_p (a(p) + 1).$$

Highly composite numbers

• Highly composite numbers : *A highly composite number is a positive integer with more divisors than any smaller positive integer.*
Sequence of highly composite numbers

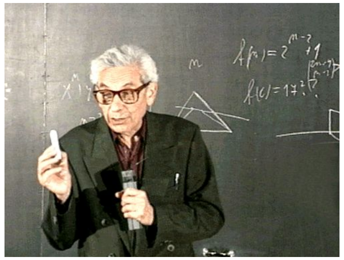
$$\begin{array}{rccccccccc} n = & 1 & 2 & 4 & 6 & 12 & 24 & 36 & 48 & 60\ldots \\ d(n) = & 1 & 2 & 3 & 4 & 6 & 8 & 9 & 10 & 12\ldots \end{array}$$

http://oeis.org/A002182
$1, 2, 4, 6, 12, 24, 36, 48, 60, 120, 180, 240, 360, 720, 840, \ldots$
This sequence is infinite : since $2n$ has more divisors than n, if n is a highly composite number, then there is a highly composite number larger than n and at most $2n$.

The smallest highly composite numbers

n			$d(n)$
1	=	1	1
2	=	2	2
4	=	2^2	3
6	=	$2 \cdot 3$	4
12	=	$2^2 \cdot 3$	6
24	=	$2^3 \cdot 3$	8
36	=	$2^2 \cdot 3^2$	9
48	=	$2^4 \cdot 3$	10
60	=	$2^2 \cdot 3 \cdot 5$	12
120	=	$2^3 \cdot 3 \cdot 5$	16
180	=	$2^2 \cdot 3^2 \cdot 5$	18
240	=	$2^4 \cdot 3 \cdot 5$	20
360	=	$2^3 \cdot 3^2 \cdot 5$	24
720	=	$2^4 \cdot 3^2 \cdot 5$	30
840	=	$2^3 \cdot 3 \cdot 5 \cdot 7$	32

Pál Erdős and Jean-Louis Nicolas



Let $Q(x)$ be the number of highly composite numbers $\leq x$.
Pál Erdős (1944) :

$$Q(x) \geq (\log x)^a.$$

Jean-Louis Nicolas (1988) :

$$Q(x) \leq (\log x)^b.$$

Smooth, abundant, practical numbers

• A B*–smooth number* is an integer which factors completely into prime numbers $\leq B$. (Used in cryptography).
Highly composite numbers are smooth numbers.

• An *abundant number* is a number for which the sum of its proper divisors is greater than the number itself.
A highly composite number higher than 6 *is also an abundant number.*
http://oeis.org/A005101
$12, 18, 20, 24, 30, 36, 40, 42, 48, 54, 56, 60, 66, 70, 72, 78, \ldots$

• *A practical number* is a positive integer n such that all smaller positive integers can be represented as sums of distinct divisors of n.
Every highly composite number is a practical number.
http://oeis.org/A005153
$1, 2, 4, 6, 8, 12, 16, 18, 20, 24, 28, 30, 32, 36, 40, 42, 48, 54, \ldots$

Sum of divisors

Sum of divisors :

$$\sigma(n) = \sigma_1(n) = \sum_{d|n} d.$$

http://oeis.org/A000203
$1, 3, 4, 7, 6, 12, 8, 15, 13, 18, 12, 28, 14, 24, 24, 31, 18, 39, \dots$

$$\text{For} \quad n = \prod_p p^{a(p)}, \quad \sigma(n) = \prod_p \frac{p^{a(p)+1} - 1}{p - 1}.$$

- *Perfect number* : $\sigma(n) = 2n$.
- *Abundant number* : $\sigma(n) > 2n$.

Superabundant numbers

- *Superabundant number* :

$$\frac{\sigma(n)}{n} > \frac{\sigma(k)}{k} \quad \text{for} \quad k < n.$$

Sequence http://oeis.org/A004394 in OEIS:
$1, 2, 4, 6, 12, 24, 36, 48, 60, 120, 180, 240, 360, 720, 840, \dots$
Leonidas Alaoglu and Pál Erdős (1944).

http://en.wikipedia.org/wiki/Superabundant_number
Unknown to Alaoglu and Erdős, about 30 pages of Ramanujan's 1915 paper "Highly Composite Numbers" were suppressed. Those pages were finally published in The Ramanujan Journal 1 (1997), 119-153. In section 59 of that paper, Ramanujan defines generalized highly composite numbers, which include the superabundant numbers.

Alaoglu and Erdős

Ramanujan (1915) listed 102 highly composite numbers up to 6 746 328 388 800, but omitted 293 318 625 600.

Alaoglu and Erdős,
On highly composite and similar numbers, Trans. AMS **56** (3), 1944, 448–469.

highly abundant numbers,
super abundant numbers,
colossally abundant numbers.

Alaoglu and Erdős

THEOREM 10. *If n_ϵ is the colossally abundant number associated with ϵ, and if $k_q(\epsilon)$ is the exponent of the prime q, then*

$$k_q(\epsilon) = [\log \{(q^{1+\epsilon} - 1)/(q^\epsilon - 1)\}/\log q] - 1.$$

This shows that the error term in Theorem 4 is nearly the best possible. Here $[x]$ denotes the greatest integer less than x.

The numbers n_ϵ and $k_q(\epsilon)$ do not decrease as ϵ decreases. Since $\log \{(q^{1+\epsilon}-1)/(q^\epsilon-1)\}/\log q$ is a continuous function of ϵ, $k_q(\epsilon)$ will increase by steps of at most 1, and this will occur when $\log \{(q^{1+\epsilon}-1)/(q^\epsilon-1)\}/\log q$ is an integer. But this makes q^ϵ rational. It is very likely that q^x and p^x can not be rational at the same time except if x is an integer. This would show that the quotient of two consecutive colossally abundant numbers is a prime. At present we can not show this. Professor Siegel has communicated to us the result that q^x, r^x and s^x can not be simultaneously rational except if x is an integer. Hence *the quotient of two consecutive colossally abundant numbers is either a prime or the product of two distinct primes.*

Alaoglu and Erdős

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Colossally abundant numbers

• *Colossally abundant number* : a positive integer n for which there exists $\varepsilon > 0$ such that, for all $k > 1$,

$$\frac{\sigma(n)}{n^{1+\varepsilon}} \geq \frac{\sigma(k)}{k^{1+\varepsilon}}.$$

Sequence `A004490 in OEIS:`
$2, 6, 12, 60, 120, 360, 2520, 5040, 55440, 720720, 1441440, \dots$
Successive quotients : $3, 2, 5, 2, 3, 7, 2, 11, 13, 2, \dots$

$2 \quad 6 \quad 12 \quad 60 \quad 120$
$\quad 3 \quad 2 \quad 5 \quad 2 \quad 3$

$360 \quad 2520 \quad 5040 \quad 55440 \quad 720720$
$\quad 7 \quad 2 \quad 11 \quad 13$

2^t may be algebraic and t not an integer

Example : For $t = \dfrac{\log 1729}{\log 2}$, we have $2^t = 1729 \in \mathbf{Z}$, but

$$3^t = \exp((\log 3)(\log 1729)/\log 2) = 135451.44\dots$$

is not an integer.
Also 3^t is an integer for some values of t not in $\mathbf{Z}$ – for instance for $t = \dfrac{\log 691}{\log 3}\cdot$ But then

$$2^t = \exp((\log 2)(\log 691)/\log 3) = 61.87\dots$$

is not an integer.
Can these values of t be the same producing integer values for both 2^t and 3^t ?
Let t be a real number such that 2^t and 3^t are integers. Does it follow that t is a positive integer ?

Special case of the four exponentials Conjecture

Conjecture [Special case of the four exponentials Conjecture] *Let p and q be two distinct prime numbers. If t is a real number such that p^t and q^t are both algebraic, then t is a rational number.*

Theorem [Special case of the Six Exponentials Theorem] *If the three numbers p^t, q^t and r^t are algebraic for three distinct primes p, q, r, then t is a rational number.*

Corollary : *The quotient of two consecutive colossally abundant numbers is either a prime or a product of two distinct primes.*

1972 Putnam Prize Competition

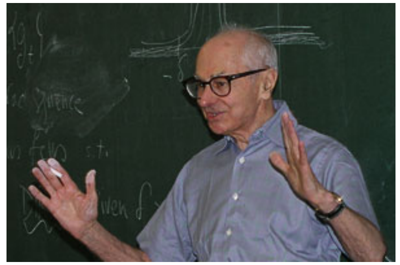
Using the *calculus of finite differences*, show that, if $t \in \mathbf{R}$ is such that $n^t \in \mathbf{Z}$ for all $n \geq 1$, then $t \in \mathbf{N}$.

First method H. Halberstam. – Transcendental numbers ; The Mathematical Gazette **58** (1976), 276–284.

Second method R. Balasubramanian. cf. M. Waldschmidt. – *Linear independence of logarithms of algebraic numbers.* The Institute of Mathematical Sciences, Madras, IMSc Report N° **116**, (1992), 168 pp.

Carl Ludwig Siegel and Serge Lang

Carl Ludwig Siegel
(1896 - 1981)



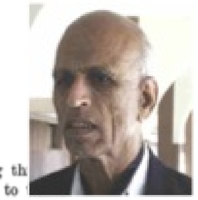
Serge Lang
(1927 - 2005)

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Siegel.html

Mathematicians/Lang.html

Ramachandra's contributions to the theory of transcendental numbers

[1968].– *Contributions to the theory of transcendental numbers* (I) ; Acta Arith., **14** (1968), 65–72 ; (II), id., 73–88.



It is a pleasure to thank professor C. L. Siegel for going th[...] the manuscript in detail and suggesting this presentation, and to [...] professor K. G. Ramanathan for many helpful discussions in connection with the preparation of the manuscript.

(²) After writing this manuscript I came to know from professor C. L. Siegel that this is a result first due to Schneider and Siegel. Their result is unpublished. This result is also to be found in a recent paper by S. Lang, *Algebraic values of meromorphic functions*, Topology 5 (4), (1966), pp. 363-370. The results of this paper have something in common with Lang's results.

http://matwbn.icm.edu.pl/tresc.php?wyd=6&tom=14

ACTA ARITHMETICA
XIV (1968)

Contributions to the theory of transcendental numbers (I)

by

K. Ramachandra (Bombay)

Dedicated to the memory of
Jacques Hadamard (1865-1963)

§ 1. Introduction. In this paper we prove the main theorem relating to the set (or a subset) of complex numbers at which a given set of algebraically independent meromorphic functions assume values in a fixed algebraic number field (we actually prove a more general result which may be useful elsewhere). We state a few deductions in § 2 and it is interesting to note that the Main Theorem gives significant results in the case (overlooked by Gelfond) *where the functions concerned do not satisfy algebraic differential equations of the first order with algebraic number coefficients.* Since some of the deductions require lengthy preparations we postpone the proofs of these and other deductions to part II, which is a continuation of this paper. We give a brief history of this theorem in this section.

In the year 1929, A. O. Gelfond made the important discovery that $a^b = e^{b \log a}$ is transcendental for every imaginary quadratic irrationality b and every algebraic a different from zero except for $\log a = 0$ (¹). Assuming the result to be false Gelfond applied the interpolation formula

K. Ramachandra (1933–2011)

Ramachandra's house in Bangalore

Photos taken on April 26, 2011, after a colloquium talk I gave in Bangalore in memory of K. Ramachandra.

Atle Selberg

Atle Selberg
1917 - 2007

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Selberg.html

Four exponentials Conjecture

Set $2^t = a$ and $3^t = b$. Then the determinant

$$\begin{vmatrix} \log 2 & \log 3 \\ \log a & \log b \end{vmatrix}$$

vanishes.

Four exponentials Conjecture. *Let*

$$\begin{pmatrix} \log \alpha_1 & \log \alpha_2 \\ \log \beta_1 & \log \beta_2 \end{pmatrix}$$

be a 2×2 *matrix whose entries are logarithms of algebraic numbers. Assume the two columns are* $\mathbf{Q}$ *-linearly independent and the two rows are also* $\mathbf{Q}$ *-linearly independent. Then the matrix is regular.*

Four exponentials Conjecture and first problem of Schneider

Problem 1 :

$$\frac{\log \alpha_1}{\log \alpha_2} = \frac{\log \alpha_3}{\log \alpha_4}$$

Problem 8 : *One at least of the two numbers*

$$e^e, \; e^{e^2}.$$

is transcendental.

Introduction aux Nombres Transcendants

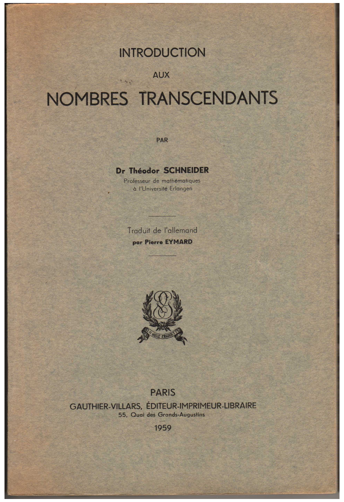
Theodor Schneider

Einführung in die Transzendenten Zahlen

Springer, 1957.

Traduction française par Pierre Eymard

Gauthier-Villars, 1959



Matrices and exponentials

A 2×2 matrix with complex entries has rank ≤ 1 if and only if it can be written

$$\begin{pmatrix} x_1 y_1 & x_1 y_2 \\ x_2 y_1 & x_2 y_2 \end{pmatrix}$$

with x_1, x_2, y_1, y_2 in $\mathbf{C}$.

For such a matrix, the two rows are linearly independent over $\mathbf{Q}$ if and only if x_1 and x_2 are linearly independent over $\mathbf{Q}$ (means : x_2/x_1 is irrational), while the two colums are linearly independent over $\mathbf{Q}$ if and only if y_1 and y_2 are linearly independent over $\mathbf{Q}$ (means : y_2/y_1 is irrational).

Connection with the four exponentials Conjecture

Set $\alpha_{ij} = e^{x_i y_j}$ for $i, j = 1, 2$. Then the matrix

$$\begin{pmatrix} \log \alpha_{11} & \log \alpha_{12} \\ \log \alpha_{21} & \log \alpha_{22} \end{pmatrix} = \begin{pmatrix} x_1 y_1 & x_1 y_2 \\ x_2 y_1 & x_2 y_2 \end{pmatrix}$$

has rank ≤ 1.

Four exponentials Conjecture

Conjecture. *Let* x_1, x_2 *be* $\mathbf{Q}$*–linearly independent complex numbers and* y_1, y_2 *be also* $\mathbf{Q}$*–linearly independent complex numbers. Then one at least of the four numbers*

$$e^{x_1y_1},\ e^{x_1y_2},\ e^{x_2y_1},\ e^{x_2y_2}$$

is transcendental.

Six exponentials Theorem

Theorem (Siegel, Lang, Ramachandra). *Let* x_1, x_2 *be two* $\mathbf{Q}$*–linearly independent complex numbers and* y_1, y_2, y_3 *be also* $\mathbf{Q}$*–linearly independent complex numbers. Then one at least of the* 6 *numbers*

$$e^{x_iy_j}, \quad (i = 1, 2,\ j = 1, 2, 3)$$

is transcendental.

Six exponentials Theorem

Theorem (Siegel, Lang, Ramachandra). *Let*

$$\begin{pmatrix} \log \alpha_1 & \log \alpha_2 & \log \alpha_3 \\ \log \beta_1 & \log \beta_2 & \log \beta_3 \end{pmatrix}$$

be a 2×3 *matrix whose entries are logarithms of algebraic numbers. Assume the three columns are linearly independent over* $\mathbf{Q}$ *and the two rows are also linearly independent over* $\mathbf{Q}$*. Then the matrix has rank* 2*.*

Six exponentials Theorem

Let d and ℓ be positive integers with $d\ell > d + \ell$. This means $d \geq 2$ and $\ell \geq 3$ or $d \geq 3$ and $\ell \geq 2$.

Theorem. *Let* $x_1, \ldots, x_d$ *be* $\mathbf{Q}$*–linearly independent complex numbers and* $y_1, \ldots, y_\ell$ *be also* $\mathbf{Q}$*–linearly independent complex numbers. Then one at least of the* $d\ell$ *numbers*

$$e^{x_iy_j}, \quad (1 \leq i \leq d,\ 1 \leq j \leq \ell)$$

is transcendental.

Equivalently :

If the entries of a $d \times \ell$ *matrix*

$$\bigl(\log \alpha_{ij}\bigr)_{1\leq i\leq d,\ 1\leq j\leq \ell}$$

are logarithms of algebraic numbers, if the d *rows are* $\mathbf{Q}$ *-linearly independent and the* ℓ *columns are also* $\mathbf{Q}$ *-linearly independent, then the matrix has rank* ≥ 2*.*

Transcendence of 2^t, 2^{t^2}, 2^{t^3}

In the six exponentials Theorem, select

$$x_1 = 1, \quad x_2 = t$$

and

$$y_1 = \log 2, \quad y_2 = t \log 2, \quad y_3 = t^2 \log 2$$

where t is an irrational number. The numbers $e^{x_i y_j}$ are 2, 2^t, 2^{t^2}, 2^{t^3}, hence one at least of the three numbers 2^t, 2^{t^2}, 2^{t^3} is transcendental.

In case t is algebraic, these three numbers are transcendental (Gel'fond–Schneider's Theorem).

Transcendence of 2^{t^k}

With

$$x_1 = 1, \quad x_2 = t^a$$

and

$$y_1 = \log 2, \quad y_2 = t^b \log 2, \quad y_3 = t^c \log 2$$

where t is a transcendental number and a, $c > b$ are positive integers, one deduce the transcendence of one at least of the numbers

$$2^{t^a}, \quad 2^{t^b}, \quad 2^{t^c}, \quad 2^{t^{a+b}} \quad 2^{t^{a+c}}.$$

For instance with $c = a + b$ it follows that for a and b positive integers, one at least of the numbers

$$2^{t^a}, \quad 2^{t^b}, \quad 2^{t^{a+b}} \quad 2^{t^{2a+b}}.$$

is transcendental.

Is 2^π transcendental ?

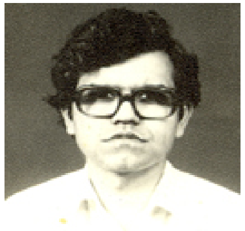
One conjectures that all numbers 2^{π^n} $(n \geq 1)$ are transcendental.

A special case of the four exponentials Conjecture is that one at least of the two numbers 2^π, 2^{π^2} is transcendental.

According to the six exponentials Theorem, one at least of the three numbers 2^π, 2^{π^2}, 2^{π^3} is transcendental.

Algebraic approximations to 2^{π^k} (T.N. Shorey)

When α_1, α_2, α_3 are algebraic numbers, T.N. Shorey (1974) gave a lower bound for $|2^\pi - \alpha_1| + |2^{\pi^2} - \alpha_2| + |2^{\pi^3} - \alpha_3|$ in terms of the heights and degrees of α_1, α_2, α_3.



[T.N. Shorey, 1974].– *On the sum* $\sum_{k=1}^{3} |2^{\pi^k} - \alpha_k|$, α_k *algebraic numbers*, J. Number Theory **6** (1974), 248-260.

Algebraic approximations to 2^{π^k} (S. Srinivasan)

S. Srinivasan : Using a theorem of Szemeredi, investigates the number of algebraic numbers among the numbers 2^{π^k}, $(1 \le k \le N)$. :

$$O(\sqrt{N}).$$

[Srinivasan, 1974].– *On algebraic approximations to* 2^{π^k} $(k = 1, 2, 3, \ldots)$; Indian J. Pure Appl. Math., **5** (1974), 513–523.

[Srinivasan, 1979].– *On algebraic approximations to* 2^{π^k} $(k = 1, 2, 3, \ldots)$, (II) ; J. Indian Math. Soc., **43** (1979), 53–60.

Algebraic numbers among 2^{π^k}

Number of algebraic numbers among the numbers 2^{π^k}, $(1 \le k \le N)$:

$$\le (2 + \epsilon)\sqrt{N}$$

.

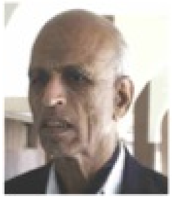
[K. Ramachandra and S. Srinivasan, 1983].– *A note to a paper by Ramachandra on transcendental numbers* ; Hardy-Ramanujan Journal, **6** (1983), 37–44.

Algebraic numbers among 2^{π^k}

Number of algebraic numbers among the numbers 2^{π^k}, $(1 \le k \le N)$:

$$\le (\sqrt{2} + \epsilon)\sqrt{N}$$

.



[K. Ramachandra and R. Balasubramanian, 1982].– *Transcendental numbers and a lemma in combinatorics* ; Proc. Sem. Combinatorics and Applications, Indian Stat. Inst., (1982), 57–59.

The five exponentials Theorem (1985)

Theorem *Let* x_1, x_2 *be two* $\mathbf{Q}$*–linearly independent complex numbers and* y_1, y_2 *be also two* $\mathbf{Q}$*–linearly independent complex numbers. Then one at least of the* 5 *numbers*

$$e^{x_1y_1},\ e^{x_1y_2},\ e^{x_2y_1},\ e^{x_2y_2},\ e^{x_2/x_1}$$

is transcendental.

The five exponentials Theorem (matrix form)

Theorem (1985). *Let M be a 2×3 matrix whose entries are either algebraic numbers or logarithms of algebraic numbers. Assume the three columns are linearly independent over $\overline{\mathbf{Q}}$ and the two rows are also linearly independent over $\overline{\mathbf{Q}}$. Then M has rank 2.*

Assume $e^{x_i y_j} = \alpha_{ij}$ are algebraic for $i, j = 1, 2$ and that also $e^{x_2/x_1} = \gamma$ is algebraic. Then the 2×3 matrix

$$\begin{pmatrix} \log \alpha_{11} & \log \alpha_{12} & 1 \\ \log \alpha_{21} & \log \alpha_{22} & \log \gamma \end{pmatrix} = \begin{pmatrix} x_1 y_1 & x_1 y_2 & 1 \\ x_2 y_1 & x_2 y_1 & x_2/x_1 \end{pmatrix}$$

has rank 1.

The field $\overline{\mathbf{Q}}$, the $\mathbf{Q}$–space $\mathcal{L}$ and the $\overline{\mathbf{Q}}$–space $\widetilde{\mathcal{L}}$

We denote by $\overline{\mathbf{Q}}$ the field of algebraic numbers.

Denote by $\mathcal{L}$ the $\mathbf{Q}$-vector subspace of $\mathbf{C}$ of logarithms of algebraic numbers : it consists of the complex numbers λ for which e^{λ} is algebraic (say $\lambda = \log \alpha$).

Further denote by $\widetilde{\mathcal{L}}$ the $\overline{\mathbf{Q}}$-vector space spanned by 1 and $\mathcal{L}$: hence $\widetilde{\mathcal{L}}$ is the set of linear combinations with algebraic coefficients of logarithms of algebraic numbers :

$$\widetilde{\mathcal{L}} = \{\Lambda = \beta_0 + \beta_1 \lambda_1 + \cdots + \beta_n \lambda_n \; ; \; n \geq 0, \beta_i \in \overline{\mathbf{Q}}, \; \lambda_i \in \mathcal{L}\}.$$

Notice that $\widetilde{\mathcal{L}} \supset \overline{\mathbf{Q}} \cup \mathcal{L}$.

The Strong Six Exponentials Theorem

Theorem (D.Roy, 1992). *If x_1, x_2 are $\overline{\mathbf{Q}}$–linearly independent complex numbers and y_1, y_2, y_3 are $\overline{\mathbf{Q}}$–linearly independent complex numbers, then one at least of the six numbers*

$$x_1 y_1, \; x_1 y_2, \; x_1 y_3, \; x_2 y_1, \; x_2 y_2, \; x_2 y_3$$

is not in $\widetilde{\mathcal{L}}$.

The Strong Four Exponentials Conjecture

Conjecture. *If x_1, x_2 are $\overline{\mathbf{Q}}$–linearly independent complex numbers and y_1, y_2 are $\overline{\mathbf{Q}}$–linearly independent complex numbers, then one at least of the four numbers*

$$x_1 y_1, \; x_1 y_2, \; x_2 y_1, \; x_2 y_2$$

is not in $\widetilde{\mathcal{L}}$.

Lower bound for the rank of matrices

- Rank of matrices. An alternative form of the strong Six Exponentials Theorem (resp. the strong four exponentials Conjecture) is the fact that *a* 2×3 *(resp.* 2×2*) matrix with entries in* $\widetilde{\mathcal{L}}$

$$\begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \end{pmatrix} \qquad \left(\text{resp. } \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix} \right),$$

the rows of which are linearly independent over $\overline{\mathbf{Q}}$ *and the columns of which are also linearly independent over* $\overline{\mathbf{Q}}$*, has maximal rank* 2*.*

The strong Six Exponentials Theorem

References :

D. Roy – « Matrices whose coefficients are linear forms in logarithms », *J. Number Theory* **41** (1992), no. 1, p. 22–47.

M. Waldschmidt – *Diophantine approximation on linear algebraic groups*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. **326**, Springer-Verlag, Berlin, 2000.

Rank of matrices with entries in $\mathcal{L}$

The six exponentials Theorem has been generalized in 1980 to lower bounds for the rank of matrices of any size.

Under suitable assumptions, the rank r of a $d \times \ell$ matrix with entries in $\mathcal{L}$ (logarithms of algebraic numbers) satisfies

$$r \geq \frac{d\ell}{d+\ell},$$

which is half of what is expected.

Hence, when $d = \ell$,

$$r \geq \frac{d}{2}.$$

p– adic analog : Leopoldt's Conjecture, ℓ–adic representations (cf lecture by C. Khare).

Rank of matrices with entries in $\widetilde{\mathcal{L}}$

In 1992, D. Roy extended this result to matrices with entries $\widetilde{\mathcal{L}}$ (linear combinations of 1 and logarithms of algebraic numbers).

More precisely, D. Roy defines the structural rank $r_{str}(M)$ of a matrix M with entries in $\widetilde{\mathcal{L}}$ and proves that the rank r of M satisfies :

$$r \geq \frac{1}{2} r_{str}(M).$$

Schanuel's Conjecture

Let $x_1, \ldots, x_n$ be $\mathbf{Q}$-linearly independent complex numbers. Then at least n of the $2n$ numbers $x_1, \ldots, x_n,\ e^{x_1}, \ldots, e^{x_n}$ are algebraically independent.

In other terms, the conclusion is

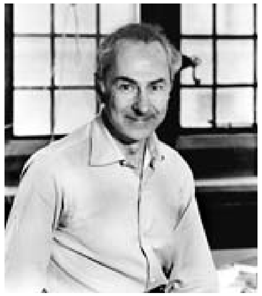
$$\operatorname{tr}\deg_{\mathbf{Q}} \mathbf{Q}(x_1, \ldots, x_n,\ e^{x_1}, \ldots, e^{x_n}) \geq n.$$

`Remark` : For almost all tuples (with respect to the Lebesgue measure) the transcendence degree is $2n$.

Dale Brownawell and Stephen Schanuel

Origin of Schanuel's Conjecture

Course given by Serge Lang (1927–2005) at Columbia in the 60's



S. Lang – *Introduction to transcendental numbers*, Addison-Wesley 1966.

Easy consequence of Schanuel's Conjecture

According to Schanuel's Conjecture, the following numbers are algebraically independent :

$$e+\pi,\ e\pi,\ \pi^e,\ e^e, e^{e^2}, \ldots,\ e^{e^e}, \ldots,\ \pi^\pi, \pi^{\pi^2}, \ldots\ \pi^{\pi^\pi} \ldots$$

$$\log\pi,\ \log(\log 2),\ \pi\log 2,\ (\log 2)(\log 3),\ 2^{\log 2},\ (\log 2)^{\log 3} \ldots$$

Proof : Use Schanuel's Conjecture several times.

Further consequences of Schanuel's Conjecture

Ram Murty

Kumar Murty

N. Saradha

Transcendental values of class group L–functions, of Gamma values, of log Gamma values, ...
Joint works of R. Murty with S. Gun and P. Rath (2008, 2009).

Ubiquity of Schanuel's Conjecture

Other contexts : p–adic numbers, Leopoldt's Conjecture on the p–adic rank of the units of an algebraic number field
Non-vanishing of Regulators
Non–degenerescence of heights
Conjecture of B. Mazur on rational points
Diophantine approximation on tori

Dipendra Prasad

Gopal Prasad

Schanuel's Conjecture for $n = 1$

For $n = 1$, Schanuel's Conjecture is the Hermite–Lindemann Theorem :

> *If x is a non–zero complex numbers, then one at least of the 2 numbers x, e^x is transcendental.*

Equivalently, *if x is a non–zero algebraic number, then e^x is a transcendental number.*

Another equivalent statement is that *if α is a non–zero algebraic number and $\log \alpha$ any non–zero logarithm of α, then $\log \alpha$ is a transcendental number.*

Consequence : transcendence of numbers like

$$e, \quad \pi, \quad \log 2, \quad e^{\sqrt{2}}.$$

Charles Hermite et Ferdinand Lindemann

(1822 – 1901)

Hermite (1873) :
Transcendance de e
$e = 2,718\,281\ \ldots$

(1852 – 1939)

Lindemann (1882) :
Transcendance de π
$\pi = 3,141\,592\ \ldots$

Schanuel's Conjecture for $n = 2$

For $n = 2$ Schanuel's Conjecture is not yet known :

? If x_1, x_2 are $\mathbf{Q}$–linearly independent complex numbers, then among the 4 numbers x_1, x_2, e^{x_1}, e^{x_2}, at least 2 are algebraically independent.

A few consequences :
With $x_1 = 1$, $x_2 = i\pi$: algebraic independence of e and π.
With $x_1 = 1$, $x_2 = e$: algebraic independence of e and e^e.
With $x_1 = \log 2$, $x_2 = (\log 2)^2$: algebraic independence of $\log 2$ and $2^{\log 2}$.
With $x_1 = \log 2$, $x_2 = \log 3$: algebraic independence of $\log 2$ and $\log 3$.

Lindemann Weierstraß

Lindemann–Weierstraß Theorem = case where $x_1, \ldots, x_n$ are algebraic.

Let $\beta_1, \ldots, \beta_n$ be algebraic numbers which are linearly independent over $\mathbf{Q}$. Then the numbers $e^{\beta_1}, \ldots, e^{\beta_n}$ are algebraically independent over $\mathbf{Q}$.

Algebraic independence of logarithms of algebraic numbers

It is not known that *there exist two logarithms of algebraic numbers which are algebraically independent.*

Even the non–existence of non–trivial quadratic relations among logarithms of algebraic numbers is not yet established.

According to the *four exponentials Conjecture*, *any quadratic relation $(\log \alpha_1)(\log \alpha_4) = (\log \alpha_2)(\log \alpha_3)$ is trivial : either $\log \alpha_1$ and $\log \alpha_2$ are linearly dependent, or else $\log \alpha_1$ and $\log \alpha_3$ are linearly dependent.*

Algebraic independence of logarithms of algebraic numbers

One of the main problems in transcendental number theory is to prove that $\mathbf{Q}$ *–linearly independent logarithms of algebraic numbers are algebraically independent*. Such a result would solve the question of the rank of matrices having entries in the space of logarithms of algebraic numbers.

Baker's results provide a satisfactory answer for the *linear independence of such numbers over the field of algebraic numbers*. But he says nothing about algebraic independence.

Roy's Theorem (1999)

The conjecture on algebraic independence of logarithms is equivalent to the question of the rank of matrices with entries logarithms of algebraic numbers.

A result of Damien Roy

Let k be a field and $P \in k[X_1, \ldots, X_n]$ a polynomial in n variables. Then there exists a square matrix M, whose entries are linear polynomials in $1, X_1, \ldots, X_n$, such that P is the determinant of M.

Towards Schanuel's Conjecture

Ch. Hermite, F. Lindemann, C.L. Siegel, A.O. Gel'fond, Th. Schneider, A. Baker, S. Lang, W.D. Brownawell, D.W. Masser, D. Bertrand, G.V. Chudnovsky, P. Philippon, G. Wüstholz, Yu.V. Nesterenko, D. Roy.

Damien Roy

Strategy suggested by D. Roy in 1999, Journées Arithmétiques, Roma :

Conjecture equivalent to Schanuel's Conjecture.

Transcendence of p–adic numbers – open problems

The p–adic analog of Lindemann–Weierstrass's Theorem on independence of the exponentials of algebraic numbers is not known.

p–adic analog of Dirichlet's unit Theorem : p–adic rank of the units of an algebraic number field, Leopoldt's Conjecture, p–adic regulator.

Leopoldt's Conjecture

Leopoldt's Conjecture (1962)

$$\text{rank}\left(\log_p \sigma_i(\epsilon_j)\right) = r$$

$$r = r_1 + r_2 - 1$$

Heinrich-Wolfgang Leopoldt
(August 22, 1927 – July 28, 2011)

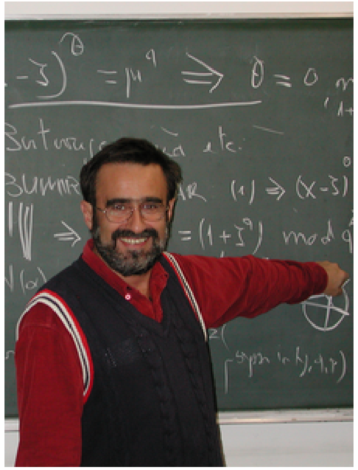
Preda Mihăilescu

arXiv:0905.1274

Date : Fri, 8 May 2009 14 :52 :57 GMT (16kb)

Title : *On Leopoldt's conjecture and some consequences*

Author : *Preda Mihăilescu*



http ://arxiv.org/abs/0905.1274

The conjecture of Leopoldt states that the p - adic regulator of a number field does not vanish. It was proved for the abelian case in 1967 by Brumer, using Baker theory. If the Leopoldt conjecture is false for a galois field $\mathbf{K}$, there is a *phantom* $\mathbf{Z}_p$ - extension of $\mathbf{K}_\infty$ arising. We show that this is strictly correlated to some infinite Hilbert class fields over $\mathbf{K}_\infty$, which are generated at intermediate levels by roots from units from the base fields. It turns out that the extensions of this type have bounded degree. This implies the Leopoldt conjecture for arbitrary finite number fields.

Preda Mihăilescu

http ://arxiv.org/abs/0905.1274v2

Replaced with revised version

Date : *Sat, 27 Jun 2009 17 :57 :24 GMT (33kb)*

Title : *The T and T^* components of Λ - modules and Leopoldt's conjecture*

Author : *Preda Mihăilescu*

Comments : *Modified second version. Added many details of proofs. The final argument is modified and the proof now extends also to the conjecture of Gross.*

http ://arxiv.org/abs/0905.1274v3

Replaced with revised version

Date : *Tue, 15 Sep 2009 08 :01 :02 GMT (69kb)*

Title : *The T and T^* components of Λ - modules and Leopoldt's conjecture*

Author : *Preda Mihăilescu*

Comments : *Modified third version. Largely extended the build up and proofs and added an Appendix for the technical details. Contains now the Kummer theory of class field radicals needed for the proofs of the Leopoldt and Gross-Kuz'min Conjectures.*

http ://arxiv.org/abs/0905.1274v4

Cite as : arXiv :0905.1274v4

Date : *Sat, 27 Jun 2009 17 :57 :24 GMT (33kb)*
(Submitted on 8 May 2009 (v1), last revised 20 Sep 2010 (this version, v4))

Title : *The T and T^* components of Λ - modules and Leopoldt's conjecture*

Author : *Preda Mihăilescu*

Comments : *In the fourth version there is a modification of Proposition 5 which supports the final argument of proof of Leopoldt's conjecture. Please note also the new series "Seminar Notes on Open Questions in Iwasawa Theory" (snoqit).*

http ://arxiv.org/abs/0909.2738

Date : *Tue, 15 Sep 2009 08 :09 :10 GMT (11kb)*
Title : *Applications of Baker Theory to the Conjecture of Leopoldt*
Authors : *Preda Mihăilescu*
Comments : *A proof variant for the Leopoldt conjecture, using Diophantine approximation. The final step of the proof uses class field theory and for this we draw back on some results from the third version of arXiv :0905.1274*
In this paper we use Baker theory for giving an alternative proof of Leopoldt's Conjecture for totally real extensions $\mathbf{K}$. This approach uses a formulation of the Conjecture for relative extensions which can be proved by Diophantine approximation and reduces the problem to the fact that the module of classes containing products of p - units, is finite. The proof of this fact is elementary, but requires class field theory. The methods used here are a sharpening of the ones presented at the SANT meeting in Göttingen, 2008 and exposed in [M1] and [M2]

http ://arxiv.org/abs/1105.5989

Date : *Submitted on 30 May 2011*

Title : *SNOQIT I : Growth of Λ–modules and Kummer theory*

Authors : *Preda Mihăilescu*

Comments : *The paper contains at the end a proof of the conjecture of Gross - Kuz'min, for CM extensions of $\mathbf{Q}$. The main topic of the paper is the investigation of the growth of order and ranks at finite levels of some Lambda modules (p-parts of ideal class groups).*

SNOQIT : *Seminar Notes on Open Questions in Iwasawa Theory*

Problem of K. Mahler et Yu Manin

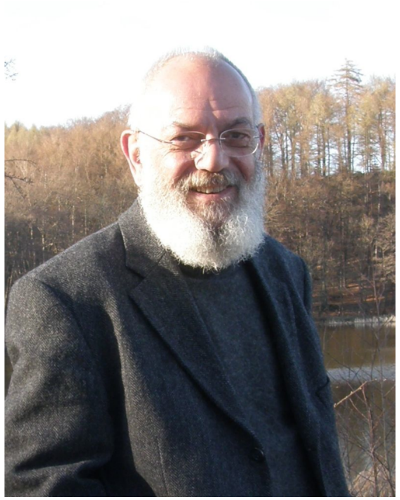
Transcendence of $J(q)$ for $q = e^{2i\pi\tau}$ algebraic .

Kurt Mahler

Yuri Manin

Le théorème stéphanois

A special case of a conjecture by Ramachandra generalizing the four exponentials conjecture to elliptic functions : problem of Mahler and Manin, solved by K. Barré–Siriex, G. Diaz, F. Gramain and G. Philibert (1996).



François Gramain

Transcendence and modular forms

Yu.V.Nesterenko (1996)
Algebraic independence of $\Gamma(1/4)$, π and e^{π}.
Also : Algebraic independence of $\Gamma(1/3)$, π and $e^{\pi\sqrt{3}}$.

Corollary : *The numbers* $\pi = 3.141\,592\,653\,5\ldots$ *and* $e^{\pi} = 23.140\,692\,632\,7\ldots$ *are algebraically independent.*

Transcendence of values of Dirichlet's L–functions : Sanoli Gun, Ram Murty and Purusottam Rath (2009).

Mazur's problem

Density of rational points on varieties
Special case of algebraic groups : generalization of the four exponentials conjecture.

Periods of $K3$ surfaces (2003 with H. Shiga)

Barry Mazur

References on density questions related to the four exponentials Conjecture and the six exponentials Theorem

- C. Bertolin – « Périodes de 1-motifs et transcendance », *J. Number Theory* **97** (2002), no. 2, p. 204–221.
- J.-L. Colliot-Thélène, A. N. Skorobogatov & P. Swinnerton-Dyer – « Double fibres and double covers : paucity of rational points », *Acta Arith.* **79** (1997), no. 2, p. 113–135.
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- — , « Questions of decidability and undecidability in number theory », *J. Symbolic Logic* **59** (1994), no. 2, p. 353–371.
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- D. Prasad – « An analogue of a conjecture of Mazur : a question in Diophantine approximation on tori », in *Contributions to automorphic forms, geometry, and number theory*, Johns Hopkins Univ. Press, Baltimore, MD, 2004, p. 699–709.
- G. Prasad & A. S. Rapinchuk – « Zariski-dense subgroups and transcendental number theory », *Math. Res. Lett.* **12** (2005), no. 2-3, p. 239–249.

D. Roy – « Simultaneous approximation in number fields », *Invent. Math.* **109** (1992), no. 3, p. 547–556.

M. Waldschmidt – « Densité des points rationnels sur un groupe algébrique », *Experiment. Math.* **3** (1994), no. 4, p. 329–352.

— , « On Ramachandra's contributions to transcendental number theory », *Ramanujan Mathematical Society Lecture Notes Series* **2** (2006), p. 175–179.

Density of an additive subgroup of $\mathbf{R}$

Kronecker : The additive group

$$\mathbf{Z}+\mathbf{Z}\theta=\left\{a+b\theta\;;\;(a,b)\in\mathbf{Z}^2\right\}$$

is dense in $\mathbf{R}$ if and only if θ is irrational (means : 1 and θ are $\mathbf{Q}$ linearly independent).

Example : $\mathbf{Z}+\mathbf{Z}e$ and $\mathbf{Z}+\mathbf{Z}\pi$ are dense in $\mathbf{R}$.

Also $\mathbf{Z}+\mathbf{Z}e\pi+\mathbf{Z}(e+\pi)$ is dense in $\mathbf{R}$. Hence there exists a subgroup of rank 2 which is also dense. But no one knows how to produce one explicitly.

Density of an additive subgroup of $\mathbf{R}^n$

Kronecker : Let $\theta_1,\ldots,\theta_n$ be real numbers. *Then the subgroup*

$$\mathbf{Z}^n+\mathbf{Z}(\theta_1,\ldots,\theta_n)$$

of $\mathbf{R}^n$ *is dense if and only if the numbers* $1,\theta_1,\ldots,\theta_n$ *are linearly independent over* $\mathbf{Q}$.

Footnote : According to his own taste, the reader will find a reference either in
N. Bourbaki, *Eléments de Mathématique*, Topologie Générale, Herman 1974, Chap. VII, § 1, N°1, Prop. 2 ;
or else in
G.H. Hardy and A.M. Wright, *An Introduction to the Theory of Numbers*, Oxford Sci. Publ., 1938, Chap. XXIII.

Additive subgroups of $\mathbf{R}^n$

$$\mathbf{Z}^n+\mathbf{Z}(\theta_1,\ldots,\theta_n)\subset\mathbf{R}^n$$

is the set of tuples

$$(a_1+a_0\theta_1,\ldots,a_n+a_0\theta^n)$$

with $(a_0,a_1,\ldots,a_n)\in\mathbf{Z}^{n+1}$.

Multiplicative groups of $(\mathbf{R}_+^\times)^n$

Multiplicative analog : for positive real numbers α_{ij}, $(1 \le i \le n,\ 1 \le j \le n+1)$ consider the set of tuples

$$\left(\alpha_{i,1}^{a_1} \cdots \alpha_{i,n+1}^{a_{n+1}}\right)_{1\le i\le n} \in (\mathbf{R}_+^\times)^n$$

with $(a_1, a_2, \dots, a_{n+1}) \in \mathbf{Z}^{n+1}$.

Additive vs multiplicative groups :
Take exponential or logarithm and change basis.

Density of a multiplicative subgroup of $\mathbf{R}_+^\times$

The multiplicative subgroup of rank 1 of $\mathbf{R}_+^\times$ generated by 2 :

$$\left\{\dots, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 16, \dots\right\}$$

is not dense in $\mathbf{R}_+^\times$,

but the subgroup of rank 2 generated by $2, 3$, namely

$$\{2^a 3^b \ ;\ (a,b) \in \mathbf{Z}^2\}$$

is dense in $\mathbf{R}_+^\times$.

Explanation : *the number* $(\log 2)/(\log 3)$ *is irrational*.

Density of a multiplicative subgroup of $(\mathbf{R}_+^\times)^2$

Let α_i and β_i be positive real numbers. The multiplicative subgroup generated by (α_1, β_1), (α_2, β_2), (α_3, β_3) in $(\mathbf{R}_+^\times)^2$, namely the set of

$$\left(\alpha_1^{a_1}\alpha_2^{a_2}\alpha_3^{a_3},\ \beta_1^{a_1}\beta_2^{a_2}\beta_3^{a_3}\right)$$

for $(a_1, a_2, a_3) \in \mathbf{Z}^2$, is dense in $(\mathbf{R}_+^\times)^2$ if and only if, for any $(s_1, s_2, s_3) \in \mathbf{Z}^3 \setminus \{0\}$, the 3×3 matrix

$$\begin{pmatrix} \log\alpha_1 & \log\alpha_2 & \log\alpha_3 \\ \log\beta_1 & \log\beta_2 & \log\beta_3 \\ s_1 & s_2 & s_3 \end{pmatrix}$$

has maximal rank 3.

A question from transcendental number theory

Equivalent :
the matrix

$$\begin{pmatrix} s_3\log\alpha_1 - s_1\log\alpha_3 & s_3\log\alpha_2 - s_2\log\alpha_3 \\ s_3\log\beta_1 - s_1\log\beta_3 & s_3\log\beta_2 - s_2\log\beta_3 \end{pmatrix}$$

has maximal rank 2.

A fundamental problem is to study the rank of matrices with entries which are logarithms of algebraic numbers.

Example of an open question

For $\alpha = a + b\sqrt{2} \in \mathbf{Q}(\sqrt{2})$, write $\overline{\alpha} = a - b\sqrt{2}$.

Define

$$\alpha_1 := 2\sqrt{2} - 1, \quad \alpha_2 := 3\sqrt{2} - 1, \quad \alpha_2 := 4\sqrt{2} - 1,$$

and let Γ be the set of elements in $(\mathbf{R}^\times)^2$ of the form

$$\left(\alpha_1^{a_1}\alpha_2^{a_2}\alpha_3^{a_3},\ \overline{\alpha}_1^{a_1}\overline{\alpha}_2^{a_2}\overline{\alpha}_3^{a_3}\right)$$

with $(a_1, a_2, a_3) \in \mathbf{Z}^3$.

Question : *Is Γ dense in $(\mathbf{R}^\times)^2$?*

This is a special case of the four exponentials Conjecture !