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Multivariate Lidstone Interpolation

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Abstract

According to the classical theory of Lidstone interpolation, an entire function of a single complex variable having exponential type $< \pi$ is determined by it derivatives of even order at 0 and 1. This theory can be generalized to several variables. In this lecture, we will give an introduction to this generalization by explaining how it works for two variables.

We first review the one variable theory.

In December 2021, I give a series of four courses on this topic at The Institute of Mathematical Sciences (IMSc) Chennai.
The courses are recorded; see

https://sites.google.com/view/profmichelwaldschmidthlectures/home

The link is also available on my website

http://www.imj-prg.fr/~michel.waldschmidt/

Lidstone interpolation problem

We denote by $2\mathbb{N}$ the set of even nonnegative integers.
The following interpolation problem was considered by G.J. Lidstone in 1930.
Given two sequences of complex numbers $(a_t)_{t\in 2\mathbb{N}}$ *and* $(b_t)_{t\in 2\mathbb{N}}$, *does there exist an entire function* f *satisfying*

$$f^{(t)}(0) = a_t, \quad f^{(t)}(1) = b_t \text{ for } t \in 2\mathbb{N} \quad ?$$

Is such a function f *unique?*

The answer to unicity is plain: the function $\sin(\pi z)$ satisfies these conditions with $a_t = b_t = 0$, hence there is no unicity, unless we restrict the question to entire functions satisfying some extra condition. Such a condition is a bound on the growth of f.

We start with unicity ($a_t = b_t = 0$) and polynomials.

Even derivatives at 0 and 1

Lemma.
Let f be a polynomial satisfying

$$f^{(t)}(0) = f^{(t)}(1) = 0 \text{ for all } t \in 2\mathbb{N}.$$

Then $f = 0$.

Proof.

By induction on the degree of the polynomial f.
If f has degree ≤ 1, say $f(z) = a_0 z + a_1$, the conditions $f(0) = f(1) = 0$ imply $a_0 = a_1 = 0$, hence $f = 0$.
If f has degree $\leq d$ with $d \geq 2$ and satisfies the hypotheses, then f'' also satisfies the hypotheses and has degree $< d$, hence by induction $f'' = 0$ and therefore f has degree ≤ 1.
The result follows. $\square$

An isomorphism

Let $T \in 2\mathbb{N}$. The space $\mathbb{C}[z]_{\leq T+1}$ of polynomials of degree $\leq T+1$ has dimension $T+2$. All elements $f \in \mathbb{C}[z]_{\leq T+1}$ satisfy $f^{(k)} = 0$ for $k \geq T+2$.
The previous lemma shows that the linear map

$$\begin{array}{ccc} \mathbb{C}[z]_{\leq T+1} & \longrightarrow & \mathbb{C}^{T+2} \\ f & \longmapsto & \left(f^{(t)}(0),\ f^{(t)}(1)\right)_{0 \leq t \leq T, t \in 2\mathbb{N}} \end{array}$$

is injective. Hence it is an isomorphism.

Solution of Lidstone interpolation problem

Given numbers a_t and b_t ($t \in 2\mathbb{N}$), all but finitely many of them are 0, there is a unique polynomial f such that

$$f^{(t)}(0) = a_t \text{ and } f^{(t)}(1) = b_t \text{ for all } t \in 2\mathbb{N}.$$

A polynomial f is uniquely determined by the numbers

$$f^{(t)}(0) \text{ and } f^{(t)}(1) \text{ for } t \in 2\mathbb{N}.$$

Lidstone expansion of a polynomial

Theorem (G. J. Lidstone (1930)).

There exist two sequences of polynomials, $\big(\Lambda_{t,0}(z)\big)_{t\in 2\mathbb{N}}$, $\big(\Lambda_{t,1}(z)\big)_{t\in 2\mathbb{N}}$, *such that any polynomial* f *can be written as a finite sum*

$$f(z) = \sum_{t\in 2\mathbb{N}} f^{(t)}(0)\Lambda_{t,0}(z) + \sum_{t\in 2\mathbb{N}} f^{(t)}(1)\Lambda_{t,1}(z).$$

Using Kronecker symbol, this is equivalent to

$$\Lambda_{t,1}^{(\tau)}(0) = 0 \text{ and } \Lambda_{t,1}^{(\tau)}(1) = \delta_{\tau t} \text{ for } \tau \in 2\mathbb{N} \text{ and } t \in 2\mathbb{N},$$

$$\Lambda_{t,0}^{(\tau)}(1) = 0 \text{ and } \Lambda_{t,0}^{(\tau)}(0) = \delta_{\tau t} \text{ for } \tau \in 2\mathbb{N} \text{ and } t \in 2\mathbb{N}.$$

Classical notation for Lidstone polynomials: Λ_k

Usually, the polynomials that we denote $\Lambda_{t,1}(z)$ with t even, are denoted $\Lambda_k(z)$ when $t = 2k$:

$$f(z) = \sum_{k\geq 0} f^{(2k)}(0)\Lambda_k(1-z) + \sum_{k\geq 0} f^{(2k)}(1)\Lambda_k(z).$$

The involution: $z \mapsto 1 - z$ shows that

$$\Lambda_{t,0}(z) = \Lambda_{t,1}(1-z).$$

For $t = 2k$, our polynomials $\Lambda_{2k,0}(z)$ are nothing else than $\Lambda_k(1-z)$ in the classical notation.

Differential equations

$\Lambda_{0,1}(z) = z$: $\qquad \Lambda_{0,1}(0) = 0$, $\Lambda_{0,1}(1) = 1$,

$$\Lambda_{0,1}^{(t)}(0) = \Lambda_{0,1}^{(t)}(1) = 0 \text{ for } t \geq 2,\ t \in 2\mathbb{N}.$$

The sequence of Lidstone polynomials $\left(\Lambda_{t,1}\right)_{t \in 2\mathbb{N}}$ is determined by $\Lambda_{0,1}(z) = z$ and

$$\Lambda_{t,1}'' = \Lambda_{t-2,1} \text{ for } t \geq 2,\ t \in 2\mathbb{N},$$

with the initial conditions $\Lambda_{t,1}(0) = \Lambda_{t,1}(1) = 0$ for $t \geq 2$, $t \in 2\mathbb{N}$.

Lidstone polynomials

For $t \in 2\mathbb{N}$, the polynomial $\Lambda_{t,1}$ is odd, it has degree $t+1$ and leading term $\frac{1}{(t+1)!}z^{t+1}$. For instance

$$\Lambda_{0,1}(z) = z,$$

$$\Lambda_{2,1}(z) = \frac{1}{6}(z^3 - z) = \frac{1}{6}z(z-1)(z+1),$$

$$\Lambda_{4,1}(z) = \frac{1}{120}z^5 - \frac{1}{36}z^3 + \frac{7}{360}z = \frac{1}{360}z(z^2-1)(3z^2-7).$$

Inductive formula

For $t \in 2\mathbb{N}$, the polynomial $f_t(z) = z^{t+1}$ satisfies

$$f_t{}^{(\tau)}(0) = 0 \text{ for } \tau \in 2\mathbb{N},$$

$$f_t{}^{(\tau)}(1) = \begin{cases} \frac{(t+1)!}{(t-\tau+1)!} & \text{for } 0 \le \tau \le t,\ \tau \in 2\mathbb{N} \\ 0 & \text{for } \tau \ge t+2,\ \tau \in 2\mathbb{N}. \end{cases}$$

Hence

$$z^{t+1} = \sum_{\substack{0 \le \tau \le t \\ \tau \in 2\mathbb{N}}} \frac{(t+1)!}{(t-\tau+1)!} \Lambda_{\tau,1}(z),$$

which yields the induction formula

$$\Lambda_{t,1}(z) = \frac{1}{(t+1)!} z^{t+1} - \sum_{\substack{0 \le \tau \le t-2 \\ \tau \in 2\mathbb{N}}} \frac{1}{(t-\tau+1)!} \Lambda_{\tau,1}(z).$$

$\Lambda_{t,0}(z)$

The same proof gives, for t even,

$$\frac{z^t}{t!} = \Lambda_{t,0}(z) + \sum_{\substack{0 \le \tau \le t \\ \tau \in 2\mathbb{N}}} \frac{1}{(t-\tau)!} \Lambda_{\tau,1}(z)$$

giving a formula for $\Lambda_{t,0}(z)$ in terms of the $\Lambda_{\tau,1}(z)$.

Order and exponential type

Order of an entire function :

$$\varrho(f) = \limsup_{r\to\infty} \frac{\log\log|f|_r}{\log r} \quad \text{where} \quad |f|_r = \sup_{|z|=r} |f(z)|.$$

Exponential type of an entire function :

$$\tau(f) = \limsup_{r\to\infty} \frac{\log|f|_r}{r}.$$

If the exponential type is finite, then f has order ≤ 1. If f has order < 1, then the exponential type is 0.

For $\zeta \in \mathbb{C} \setminus \{0\}$, the function $e^{\zeta z}$ has order 1 and exponential type $|\zeta|$.

Exponential type $< \pi$: unicity

Theorem (H. Poritsky, 1932).
Let f be an entire function of exponential type $< \pi$ satisfying $f^{(t)}(0) = f^{(t)}(1) = 0$ for all sufficiently large $t \in 2\mathbb{N}$. Then f is a polynomial.

This is best possible: the entire function $\sin(\pi z)$ has exponential type π and satisfies $f^{(t)}(0) = f^{(t)}(1) = 0$ for all $t \in 2\mathbb{N}$.

Proof of Poritsky's unicity Theorem

Let $\tilde{f} = f - P$, where P is the polynomial satisfying

$$P^{(t)}(0) = f^{(t)}(0) \text{ and } P^{(t)}(1) = f^{(t)}(1) \text{ for } t \in 2\mathbb{N}.$$

We have $\tilde{f}^{(t)}(0) = \tilde{f}^{(t)}(1) = 0$ for all $t \in 2\mathbb{N}$.

The functions $\tilde{f}(z)$ and $\tilde{f}(1-z)$ are odd, hence $\tilde{f}(z)$ is periodic of period 2. Therefore there exists an entire function g such that $\tilde{f}(z) = g(\mathrm{e}^{\pi \mathrm{i} z})$. Since $\tilde{f}(z)$ has exponential type $< \pi$, we deduce $\tilde{f} = 0$ and $f = P$.

Exponential type $< \pi$: existence of the expansion

Theorem (H. Poritsky, 1932).
For any entire function f of exponential type $< \pi$, we have

$$f(z) = \sum_{t \in 2\mathbb{N}} f^{(t)}(0)\Lambda_{t,0}(z) + \sum_{t \in 2\mathbb{N}} f^{(t)}(1)\Lambda_{t,1}(z),$$

with absolutely convergent series for each $z \in \mathbb{C}$.

Solution of the Lidstone interpolation problem

Consequence of Poritsky's expansion formula: Let $(a_t)_{t\in 2\mathbb{N}}$ and $(b_t)_{t\in 2\mathbb{N}}$ be two sequences of complex numbers satisfying

$$\limsup_{t\to\infty} |a_t|^{1/t} < \pi \text{ and } \limsup_{t\to\infty} |b_t|^{1/t} < \pi.$$

Then the function

$$f(z) = \sum_{t\in 2\mathbb{N}} a_t \Lambda_{t,0}(z) + \sum_{t\in 2\mathbb{N}} b_t \Lambda_{t,1}(z)$$

is the unique entire function of exponential type $< \pi$ satisfying

$$f^{(t)}(0) = a_t \text{ and } f^{(t)}(1) = b_t \text{ for all } t \in 2\mathbb{N}.$$

Poritsky's expansion for $\mathrm{e}^{\zeta z}$ with $0 < |\zeta| < \pi$

Assume Poritsky's expansion formula holds for $f_\zeta(z) := \mathrm{e}^{\zeta z}$.
Since $f_\zeta^{(t)}(0) = \zeta^t$ and $f_\zeta^{(t)}(1) = \mathrm{e}^\zeta \zeta^t$, we deduce

$$\mathrm{e}^{\zeta z} = \sum_{t \in 2\mathbb{N}} \Lambda_{t,0}(z)\zeta^t + \mathrm{e}^\zeta \sum_{t \in 2\mathbb{N}} \Lambda_{t,1}(z)\zeta^t.$$

Replacing ζ with $-\zeta$ yields

$$\mathrm{e}^{-\zeta z} = \sum_{t \in 2\mathbb{N}} \Lambda_{t,0}(z)\zeta^t + \mathrm{e}^{-\zeta} \sum_{t \in 2\mathbb{N}} \Lambda_{t,1}(z)\zeta^t.$$

Hence

$$\mathrm{e}^{\zeta z} - \mathrm{e}^{-\zeta z} = (\mathrm{e}^\zeta - \mathrm{e}^{-\zeta}) \sum_{t \in 2\mathbb{N}} \Lambda_{t,1}(z)\zeta^t.$$

Generating series

For $\zeta \in \mathbb{C}$, $\zeta \notin \pi \mathrm{i}\mathbb{Z}$, the entire function

$$f(z) = \frac{\sinh(\zeta z)}{\sinh(\zeta)} = \frac{\mathrm{e}^{\zeta z} - \mathrm{e}^{-\zeta z}}{\mathrm{e}^{\zeta} - \mathrm{e}^{-\zeta}}$$

satisfies

$$f'' = \zeta^2 f, \quad f(0) = 0, \quad f(1) = 1,$$

hence $f^{(t)}(0) = 0$ and $f^{(t)}(1) = \zeta^t$ for all $t \in 2\mathbb{N}$.
For $0 < |\zeta| < \pi$ and $z \in \mathbb{C}$, we deduce

$$\boxed{\frac{\sinh(\zeta z)}{\sinh(\zeta)} = \sum_{t \in 2\mathbb{N}} \Lambda_{t,1}(z)\zeta^t}$$

Notice that

$$\mathrm{e}^{\zeta z} = \frac{\sinh((1-z)\zeta)}{\sinh(\zeta)} + \mathrm{e}^{\zeta} \, \frac{\sinh(\zeta z)}{\sinh(\zeta)}.$$

Proof of Poritsky's expansion formula

Once we know Poritsky's expansion formula for the functions $e^{\zeta z}$ when $0 < |\zeta| < \pi$, we deduce the general case for an entire function of exponential type $\tau(f) < \pi$ by means of Laplace transform:

$$f(z) = \sum_{k \geq 0} a_k \frac{z^k}{k!}, \qquad F(\zeta) = \sum_{k \geq 0} a_k \zeta^{-k-1}.$$

For $r > \tau(f)$ we use the inverse Laplace transform formula

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta|=r} e^{\zeta z} F(\zeta) d\zeta$$

with

$$f^{(t)}(z) = \frac{1}{2\pi i} \int_{|\zeta|=r} \zeta^t e^{\zeta z} F(\zeta) d\zeta.$$

Entire function of finite exponential type

Proposition (I.J. Schoenberg, 1936).

Let f *be an entire function of finite exponential type* $\tau(f)$ *satisfying* $f^{(t)}(0) = f^{(t)}(1) = 0$ *for all* $t \in 2\mathbb{N}$*. Then there exist complex numbers* $C_1, \ldots, C_K$ *with* $K \leq \tau(f)/\pi$ *such that*

$$f(z) = \sum_{k=1}^{K} C_k \sin(k\pi z).$$

Theorem (R.C. Buck, 1955).

Let K *be a positive integer. Let* f *be an entire function of finite exponential type* $\tau(f) < (K+1)\pi$*. Then there exists entire functions* g_t *(*$t \in 2\mathbb{N}$*) and constants* $C_1, \ldots, C_K$ *such that, for* $z \in \mathbb{C}$*, we have*

$$f(z) = \sum_{t \in 2\mathbb{N}} f^{(t)}(0) g_t(1-z) + \sum_{t \in 2\mathbb{N}} f^{(t)}(1) g_t(z) + \sum_{k=1}^{K} C_k \sin(k\pi z).$$

Two variables

We now work with two variables $\underline{z} = (z_1, z_2) \in \mathbb{C}^2$. We write $|\underline{z}|$ for $\max\{|z_1|, |z_2|\}$. We keep the notation

$$|f|_r = \sup_{|\underline{z}|=r} |f(\underline{z})|.$$

For $\underline{t} = (t_1, t_2) \in \mathbb{N}^2$, we set $\|\underline{t}\| = t_1 + t_2$ and we define

$$D^{\underline{t}} = \left(\frac{\partial}{\partial z_1}\right)^{t_1} \left(\frac{\partial}{\partial z_2}\right)^{t_2}.$$

We also write $\underline{z}^{\underline{t}} = {z_1}^{t_1} {z_2}^{t_2}$.

Lidstone approximation on the triangle

• 2005 Francesco Aldo Costabile & Francesco Dell'Accio. *Lidstone approximation on the triangle Appl. Numer. Math.*
The authors use the univariate theory to cover the triangle with corners $(0,0),(1,0),(0,1)$: they write the expansion of a function on each segment $[(t,0),(0,t)]$, $0 \leq t \leq 1$, by means of Lidstone interpolation in a single variable.

• 2005 Teodora Cătinaș. *The combined Shepard-Lidstone bivariate operator.*

• 2008 Francesco Aldo Costabile, Francesco Dell'Accio, & Luca Guzzardi. *New bivariate polynomial expansion with boundary data on the simplex*

• 2012 Rosanna Caira, Francesco Aldo Costabile, & Filomena Di Tommaso. *On the bivariate Shepard-Lidstone operators*

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Appl. Numer. Math., 52(4):339–361, 2005. MR Zbl

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The combined Shepard-Lidstone bivariate operator.
In *Trends and applications in constructive approximation*, volume 151 of *Internat. Ser. Numer. Math.*, pages 77–89. Birkhäuser, Basel, 2005. MR Zbl

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New bivariate polynomial expansion with boundary data on the simplex.
Calcolo, 45(3):177–192, 2008. MR Zbl

Rosanna Caira, Francesco Aldo Costabile, & Filomena Di Tommaso.
On the bivariate Shepard-Lidstone operators.
J. Comput. Appl. Math., 236(7):1691–1707, 2012. MR Zbl

Using Lidstone polynomials in one variable

Using Lidstone polynomials in one variable yields an expansion of a polynomial in two variables in terms of the derivatives $D^{\underline{t}}$ with t_1 and t_2 even at the four points $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$. However such an expansion is not unique.

Let $T \geq 0$ be even. For a polynomial f of total degree $\leq T+1$, we have $D^{\underline{t}} f = 0$ as soon as $\|\underline{t}\| \geq T+2$. The dimension of the space $\mathbb{C}[\underline{z}]_{\leq T+1}$ of polynomials of total degree $\leq T+1$ is $\frac{1}{2}(T+2)(T+3)$.

The number of $\underline{t} \in \mathbb{N}^2$ with t_1 and t_2 even and $\|\underline{t}\| \leq T$ is $\frac{1}{8}(T+2)(T+4)$.

With four points, we have too many values; with three points, we do not have enough values.

Example: $T = 2$

The space of polynomials of total degree ≤ 3 in 2 variables has dimension 10, a basis is

$$1,\ z_1,\ z_2,\ {z_1}^2,\ z_1 z_2,\ {z_2}^2,\ {z_1}^3,\ {z_1}^2 z_2,\ z_1 {z_2}^2,\ {z_2}^3.$$

The derivatives $D^{(t_1,t_2)}$ with t_1 and t_2 even and $t_1 + t_2 \leq 2$ are $D^{(0,0)}$, $D^{(2,0)}$, $D^{(0,2)}$. With 4 points this gives 12 values, which is too big, larger than 10. With 3 points this gives 9 values, which is too small, less than 10.

If we keep three points and add $D^{(1,1)}$, again we have 12 values, too much.

Our solution: at $(0,0)$ we take 4 derivatives,

$$D^{(0,0)},\ D^{(2,0)},\ D^{(0,2)},\ D^{(1,1)},$$

at $(1,0)$ and $(0,1)$ we take only 3 derivatives,

$$D^{(0,0)},\ D^{(2,0)},\ D^{(0,2)}.$$

Checking the dimensions

Our conditions at $(0,0)$ involve all $D^{\underline{t}}$ with $t_1 + t_2$ even, while the conditions at $(1,0)$ and $(0,1)$ involve only the $D^{\underline{t}}$ with both t_1 and t_2 even.

The number of $\underline{t} \in \mathbb{N}^2$ with $\|\underline{t}\|$ even and $\|\underline{t}\| \leq T$ is $\frac{1}{4}(T+2)^2$.

The number of $\underline{t} \in \mathbb{N}^2$ with t_1 and t_2 even and $\|\underline{t}\| \leq T$ is $\frac{1}{8}(T+2)(T+4)$.

We have

$$\frac{1}{4}(T+2)^2 + \frac{1}{4}(T+2)(T+4) = \frac{1}{2}(T+2)(T+3).$$

The set $\mathcal{T}$

We introduce the following subset of $\mathbb{N}^2 \times \{0,1,2\}$:

$$\mathcal{T} = \{(\underline{t},0) \mid \|\underline{t}\| \in 2\mathbb{N}, \bigcup \{(\underline{t},i) \in \mathbb{N}^2 \times \{1,2\} \mid t_1, t_2 \in 2\mathbb{N}\}.$$

For $T \geq 0$ even, there is a natural bijective map between the set of $\underline{k} \in \mathbb{N}^2$ with $\|\underline{k}\| \leq T+1$ and the set $\{(\underline{t},i) \in \mathcal{T} \mid \|\underline{t}\| \leq T\}$: the image of (k_1,k_2) with k_1+k_2 even is $((k_1,k_2),0)$, the image of (k_1,k_2) with k_1 odd and k_2 even is $((k_1-1,k_2),1)$, and finally the image of (k_1,k_2) with k_1 even and k_2 odd is $((k_1,k_2-1),2)$.

For the inverse bijective map, the image of $((t_1,t_2),0)$ is (t_1,t_2), the image of $((t_1,t_2),1)$ is (t_1+1,t_2) and the image of $((t_1,t_2),2)$ is (t_1,t_2+1).

Unicity

Define $\underline{e}_0 = (0,0)$, $\underline{e}_1 = (1,0)$, $\underline{e}_2 = (0,1)$.

Here is the corresponding generalization of Lemma 1:

Lemma.
Let $f \in \mathbb{C}[\underline{z}]$ be a polynomial satisfying

$$D^{\underline{t}} f(\underline{e}_i) = 0 \text{ for all } \quad (\underline{t}, i) \in \mathcal{T}.$$

Then $f = 0$.

There are several proofs of this lemma; one of them is to use the theory in one variable.

An isomorphism

One deduces:

Lemma.
For T even, the map $f \mapsto \left((D^{\underline{t}} f)(\underline{e}_i)\right)_{\substack{(\underline{t},i)\in\mathcal{T} \\ \|\underline{t}\|\leq T}}$ is an isomorphism from the space of polynomials of total degree $\leq T+1$ to the space of complex tuples $\left(a_{\underline{t},i}\right)_{(\underline{t},i)\in\mathcal{T},\|\underline{t}\|\leq T}$.

From this Lemma we deduce:

Theorem.
For each $(\underline{t}, i) \in \mathcal{T}$, there exists a unique polynomial $\Lambda_{\underline{t},i}$ satisfying, for all $(\underline{\tau}, j) \in \mathcal{T}$,

$$(D^{\underline{\tau}}\Lambda_{\underline{t},i})(\underline{e}_j) = \delta_{\underline{\tau},\underline{t}}\delta_{ij}.$$

The polynomials $\Lambda_{\underline{t},i}(z_1, z_2)$

An equivalent formulation is the following:

Corollary.

Any polynomial $f \in \mathbb{C}[z_1, z_2]$ can be expanded as a finite sum

$$f(z_1, z_2) = \sum_{(\underline{t},i)\in\mathcal{T}} (D^{\underline{t}} f)(\underline{e}_i)\Lambda_{\underline{t},i}(z_1, z_2).$$

This formula can be written

$$\begin{aligned} f(z_1, z_2) = & \sum_{\|\underline{t}\|\in 2\mathbb{N}} (D^{\underline{t}} f)(0,0)\Lambda_{\underline{t},0}(z_1, z_2) \\ & + \sum_{t_1,t_2\in 2\mathbb{N}} (D^{\underline{t}} f)(1,0)\Lambda_{\underline{t},1}(z_1, z_2) \\ & + \sum_{t_1,t_2\in 2\mathbb{N}} (D^{\underline{t}} f)(0,1)\Lambda_{\underline{t},2}(z_1, z_2). \end{aligned}$$

Examples

From the corollary one deduces, for t_1 and t_2 even,

$$\frac{{z_1}^{t_1+1}}{(t_1+1)!}\frac{{z_2}^{t_2}}{t_2!} = \sum_{\substack{0\le\tau_1\le t_1\\ \tau_1\in 2\mathbb{N}}} \frac{1}{(t_1-\tau_1+1)!}\Lambda_{(\tau_1,t_2),1}(\underline{z}),$$

$$\frac{{z_1}^{t_1}}{t_1!}\frac{{z_2}^{t_2+1}}{(t_2+1)!} = \sum_{\substack{0\le\tau_2\le t_2\\ \tau_2\in 2\mathbb{N}}} \frac{1}{(t_2-\tau_2+1)!}\Lambda_{(t_1,\tau_2),2}(\underline{z}),$$

$$\frac{{z_1}^{t_1}}{t_1!}\ \frac{{z_2}^{t_2}}{t_2!} = \Lambda_{\underline{t},0}(\underline{z}) + \sum_{\substack{0\le\tau_1\le t_1\\ \tau_1\in 2\mathbb{N}}} \frac{1}{(t_1-\tau_1)!}\Lambda_{(\tau_1,t_2),1}(\underline{z}) + \sum_{\substack{0\le\tau_2\le t_2\\ \tau_2\in 2\mathbb{N}}} \frac{1}{(t_2-\tau_2)!}\Lambda_{(t_1,\tau_2),2}(\underline{z}),$$

while for t_1 and t_2 odd we have

$$\frac{{z_1}^{t_1}}{t_1!}\ \frac{{z_2}^{t_2}}{t_2!} = \Lambda_{\underline{t},0}(\underline{z}).$$

Inductive formulae

This yields recurrence formulae producing the polynomials $\Lambda_{\underline{t},i}$ by induction on $\|\underline{t}\|$: for t_1 and t_2 even, we have

$$\Lambda_{\underline{t},1}(\underline{z}) = \frac{{z_1}^{t_1+1}}{(t_1+1)!}\frac{{z_2}^{t_2}}{t_2!} - \sum_{\substack{0\le\tau_1\le t_1-2\\ \tau_1\in2\mathbb{N}}} \frac{1}{(t_1-\tau_1+1)!}\Lambda_{(\tau_1,t_2),1}(\underline{z}),$$

$$\Lambda_{\underline{t},2}(\underline{z}) = \frac{{z_1}^{t_1}}{t_1!}\ \frac{{z_2}^{t_2}}{t_2!} - \sum_{\substack{0\le\tau_2\le t_2-2\\ \tau_2\in2\mathbb{N}}} \frac{1}{(t_2-\tau_2+1)!}\Lambda_{(t_1,\tau_2),2}(\underline{z}),$$

$$\Lambda_{\underline{t},0}(\underline{z}) = \frac{{z_1}^{t_1}}{t_1!}\frac{{z_2}^{t_2}}{t_2!} - \sum_{\substack{0\le\tau_1\le t_1\\ \tau_1\in2\mathbb{N}}} \frac{1}{(t_1-\tau_1)!}\Lambda_{(\tau_1,t_2),1}(\underline{z})$$
$$- \sum_{\substack{0\le\tau_2\le t_2\\ \tau_2\in2\mathbb{N}}} \frac{1}{(t_2-\tau_2)!}\Lambda_{(t_1,\tau_2),2}(\underline{z}).$$

For t_1 and t_2 odd, we have

$$\Lambda_{\underline{t},0}(\underline{z}) = \frac{{z_1}^{t_1}}{t_1!}\frac{{z_2}^{t_2}}{t_2!}.$$

Exponential type

We will say that an entire function f of two variables has *exponential type* $\leq \tau$ *in both variables* if for each $z_1 \in \mathbb{C}$, the function $z_2 \mapsto f(z_1, z_2)$ has exponential type $\leq \tau$ and for each $z_2 \in \mathbb{C}$, the function $z_1 \mapsto f(z_1, z_2)$ has exponential type $\leq \tau$:

$$\limsup_{r \to \infty} \frac{1}{r} \log \sup_{|z_2| \leq r} |f(z_1, z_2)| \leq \tau \text{ for all } \; z_1 \in \mathbb{C}$$

and

$$\limsup_{r \to \infty} \frac{1}{r} \log \sup_{|z_1| \leq r} |f(z_1, z_2)| \leq \tau \text{ for all } \; z_2 \in \mathbb{C}.$$

Unicity for entire functions of two variables

We extend the result of H. Poritsky on the unicity of the expansion to two variables.

Proposition.

Let f be an entire function in $\mathbb{C}^2$ having exponential type $< \pi$ in both variables. If

$$(D^{\underline{t}} f)(\underline{e}_i) = 0 \quad \textit{for all} \quad (\underline{t}, i) \in \mathcal{T},$$

then $f = 0$.

One can prove this result by using the one variable result. It also follows from the next theorem.

Expansion for entire functions of two variables

We extend the result of H. Poritsky on the existence of an expansion to two variables.

Theorem.
Let f be an entire function in $\mathbb{C}^2$ having exponential type $< \pi$ in both variables. Then

$$f(\underline{z}) = \sum_{(\underline{t},i)\in\mathcal{T}} (D^{\underline{\tau}} f)(\underline{e}_i)\Lambda_{\underline{t},i}(\underline{z}).$$

For each $\underline{z} \in \mathbb{C}^2$ the series is absolutely convergent.

A special case

Consider the function $e^{\underline{\zeta}\underline{z}}$ where $\underline{\zeta} \in \mathbb{C}^2$ satisfies $0 < |\zeta_1|, |\zeta_2| < \pi$. We claim:

$$e^{\underline{\zeta}\underline{z}} = \sum_{\|\underline{t}\| \in 2\mathbb{N}} \Lambda_{\underline{t},0}(\underline{z})\underline{\zeta}^{\underline{t}} + \sum_{t_1,\, t_2 \in 2\mathbb{N}} \Lambda_{\underline{t},1}(\underline{z}) e^{\zeta_1} \underline{\zeta}^{\underline{t}} + \sum_{t_1,\, t_2 \in 2\mathbb{N}} \Lambda_{\underline{t},2}(\underline{z}) e^{\zeta_2} \underline{\zeta}^{\underline{t}}.$$

which can be written

$$e^{\underline{\zeta}\underline{z}} = \sum_{(\underline{t},i) \in \mathcal{T}} \Lambda_{\underline{t},i}(\underline{z}) e^{\zeta_i} \underline{\zeta}^{\underline{t}}$$

by setting $\zeta_0 = 0$.
We wish to use this formula by replacing ζ_1 with $-\zeta_1$ and/or ζ_2 with $-\zeta_2$. However the first sum does not behave well under these substitutions. So we split it into two parts.

Four generating series

Let us introduce the four generating series M_{ij}

$$M_{00}(\underline{\zeta}, \underline{z}) = \sum_{t_1, t_2 \text{ both even}} \Lambda_{(t_1,t_2),2}(z_1, z_2){\zeta_1}^{t_1}{\zeta_2}^{t_2},$$

$$M_{01}(\underline{\zeta}, \underline{z}) = \sum_{t_1, t_2 \text{ both even}} \Lambda_{(t_1,t_2),1}(z_1, z_2){\zeta_1}^{t_1}{\zeta_2}^{t_2},$$

$$M_{10}(\underline{\zeta}, \underline{z}) = \sum_{t_1, t_2 \text{ both even}} \Lambda_{(t_1,t_2),0}(z_1, z_2){\zeta_1}^{t_1}{\zeta_2}^{t_2},$$

$$M_{11}(\underline{\zeta}, \underline{z}) = \sum_{t_1, t_2 \text{ both odd}} \Lambda_{(t_1,t_2),0}(z_1, z_2){\zeta_1}^{t_1}{\zeta_2}^{t_2},$$

so that

$$\sum_{(\underline{t},i)\in\mathcal{T}} \Lambda_{\underline{t},i}(\underline{z})\underline{\zeta}^{\underline{t}} = M_{00}(\underline{\zeta}, \underline{z}) + M_{01}(\underline{\zeta}, \underline{z}) + M_{10}(\underline{\zeta}, \underline{z}) + M_{11}(\underline{\zeta}, \underline{z}).$$

Four generating series

We get

$$
\begin{aligned}
\mathrm{e}^{\zeta_1 z_1+\zeta_2 z_2} &= M_{1,0}(\underline{\zeta},\underline{z}) + M_{1,1}(\underline{\zeta},\underline{z}) + M_{0,1}(\underline{\zeta},\underline{z})\mathrm{e}^{\zeta_1} + M_{0,0}(\underline{\zeta},\underline{z})\mathrm{e}^{\zeta_2},\\
\mathrm{e}^{-\zeta_1 z_1+\zeta_2 z_2} &= M_{1,0}(\underline{\zeta},\underline{z}) - M_{1,1}(\underline{\zeta},\underline{z}) + M_{0,1}(\underline{\zeta},\underline{z})\mathrm{e}^{-\zeta_1} + M_{0,0}(\underline{\zeta},\underline{z})\mathrm{e}^{\zeta_2},\\
\mathrm{e}^{\zeta_1 z_1-\zeta_2 z_2} &= M_{1,0}(\underline{\zeta},\underline{z}) - M_{1,1}(\underline{\zeta},\underline{z}) + M_{0,1}(\underline{\zeta},\underline{z})\mathrm{e}^{\zeta_1} + M_{0,0}(\underline{\zeta},\underline{z})\mathrm{e}^{-\zeta_2},\\
\mathrm{e}^{-\zeta_1 z_1-\zeta_2 z_2} &= M_{1,0}(\underline{\zeta},\underline{z}) + M_{1,1}(\underline{\zeta},\underline{z}) + M_{0,1}(\underline{\zeta},\underline{z})\mathrm{e}^{-\zeta_1} + M_{0,0}(\underline{\zeta},\underline{z})\mathrm{e}^{-\zeta_2}.
\end{aligned}
$$

This is a system of 4 linear equations in 4 unknowns, namely the generating series M_{ij}.

Solving the system

We obtain :

$$\begin{aligned} M_{00}(\underline{\zeta}, \underline{z}) &= \cosh(\zeta_1 z_1) \frac{\sinh(\zeta_2 z_2)}{\sinh(\zeta_2)}, \\ M_{01}(\underline{\zeta}, \underline{z}) &= \frac{\sinh(\zeta_1 z_1)}{\sinh(\zeta_1)} \cosh(\zeta_2 z_2), \\ M_{11}(\underline{\zeta}, \underline{z}) &= \sinh(\zeta_1 z_1) \sinh(\zeta_2 z_2) \end{aligned}$$

and

$$\begin{aligned} M_{10}(\underline{\zeta}, \underline{z}) = \cosh(\zeta_1 z_1) \cosh(\zeta_2 z_2) &- \frac{\sinh(\zeta_1 z_1) \cosh(\zeta_2 z_2)}{\tanh(\zeta_1)} \\ &- \frac{\cosh(\zeta_1 z_1) \sinh(\zeta_2 z_2)}{\tanh(\zeta_2)}. \end{aligned}$$

Explicit formulae for the polynomials $\Lambda_{\underline{t},i}$

For t_1 and t_2 both in $2\mathbb{N}$,

$$\Lambda_{(t_1,t_2),1}(z_1,z_2) = \Lambda_{t_1,1}(z_1)\frac{{z_2}^{t_2}}{t_2!},$$
$$\Lambda_{(t_1,t_2),2}(z_1,z_2) = \frac{{z_1}^{t_1}}{t_1!}\Lambda_{t_2,1}(z_2),$$
$$\Lambda_{(t_1,t_2),0}(z_1,z_2) = \frac{{z_1}^{t_1}}{t_1!}\Lambda_{t_2,0}(z_2) + \Lambda_{t_1,0}(z_1)\frac{{z_2}^{t_2}}{t_2!} - \frac{{z_1}^{t_1}{z_2}^{t_2}}{t_1!t_2!},$$

while for t_1 and t_2 both odd,

$$\Lambda_{(t_1,t_2),0}(z_1,z_2) = \frac{{z_1}^{t_1}}{t_1!}\frac{{z_2}^{t_2}}{t_2!}.$$

These formulae can also be deduced from the recurrence formulae, or else checked directly using the definition of the $\Lambda_{\underline{t},i}$.

Back to entire functions of exponential type $< \pi$

Once we know the explicit expressions of the four generating series, one deduces

$$\mathrm{e}^{\underline{\zeta}\underline{z}} = \sum_{(\underline{t},i)\in\mathcal{T}} \Lambda_{\underline{t},i}(\underline{z})\mathrm{e}^{\zeta_i}\underline{\zeta}^{\underline{t}}.$$

The existence and unicity of the expansion

$$f(\underline{z}) = \sum_{(\underline{t},i)\in\mathcal{T}} (D^{\underline{\tau}} f)(\underline{e}_i)\Lambda_{\underline{t},i}(\underline{z}).$$

for entire functions of exponential type $< \pi$ in both variables follows by means of Laplace transform in two variables.

Entire functions of finite exponential type

Here is an extension to two variables of the result of I.J. Schoenberg:

Proposition.

Let f be an entire function having exponential type $\leq \tau$ in both variables, with $\tau < (K+1)\pi$. Assume $(D^{\underline{t}} f)(\underline{e}_i) = 0$ for all $(\underline{t}, i) \in \mathcal{T}$. Then there exist even entire functions of a single variable $h_{k,1}$ and $h_{k,2}$ ($k = 1, 2, \ldots, K$) having exponential type $\leq \tau$ such that

$$f(z_1, z_2) = \sum_{k=1}^{K} \big(h_{k,1}(z_1) \sin(k\pi z_2) + \sin(k\pi z_1) h_{k,2}(z_2)\big).$$

Entire functions of finite exponential type

Here is an extension to two variables of the result of R.C. Buck:

Theorem.
Let K be a nonnegative integer. Let f be an entire function in $\mathbb{C}^2$ of finite exponential type $\leq \tau$ in both variables, with $\tau < (K+1)\pi$. Then for $\underline{z} \in \mathbb{C}^2$ we have

$$f(\underline{z}) = \sum_{(\underline{t},i)\in\mathcal{T}} g_{\underline{t},i}(\underline{z})(D^{\underline{t}}f)(\underline{e}_i) + \sum_{k=1}^{K} \left(h_{k,1}(z_1)\sin(k\pi z_2) + \sin(k\pi z_1)h_{k,2}(z_2)\right),$$

where the functions $g_{\underline{t},i}(\underline{z})$ are entire functions in $\mathbb{C}^2$, the series is absolutely convergent and $h_{k,1}$, $h_{k,2}$ ($k = 1, 2, \ldots, K$) are even entire functions of a single variable of exponential type $\leq \tau$.

Several variables

In 2 variables, we had 3 points $\underline{e}_0 = (0,0)$, $\underline{e}_1 = (1,0)$ and $\underline{e}_2 = (0,1)$, we took the derivatives $D^{\underline{t}}$ with $\|\underline{t}\|$ even at $\underline{e}_0$ and only with t_1, t_2 both even at $\underline{e}_1$ and $\underline{e}_2$.

In n variables, we take $n+1$ points, first $\underline{e}_0 = 0$ (the origine), next the canonical basis $\underline{e}_1, \dots, \underline{e}_n$ of $\mathbb{C}^n$.

We take the derivatives $D^{\underline{t}}$ with $\|\underline{t}\|$ even at all $n+1$ points, without restriction at $\underline{e}_0$, with the restriction t_1 even at $\underline{e}_1$, and more generally for $1 \le i \le n$ with the restriction that $t_1, \dots, t_i$ are even at $\underline{e}_i$. Notice that at $\underline{e}_{n-1}$ and $\underline{e}_n$ the restriction is the same: we only take all t_i even.

The set $\mathcal{T}$ and the polynomials $\Lambda_{\underline{t},i}$

We denote by $\mathcal{T}$ the set of $(\underline{t}, i)$ with $\underline{t} \in \mathbb{N}^n$, $\|\underline{t}\|$ even, $i \in \{0, 1, \dots, n\}$, which satisfy the additional condition, for $i \geq 1$, that $t_1, \dots, t_i$ are even.

Theorem.
For each $(\underline{t}, i) \in \mathcal{T}$, there exists a unique polynomial $\Lambda_{\underline{t},i} \in \mathbb{C}[\underline{z}]$ satisfying, for all $(\underline{\tau}, j) \in \mathcal{T}$,

$$(D^{\underline{\tau}} \Lambda_{\underline{t},i})(\underline{e}_j) = \delta_{\underline{\tau},\underline{t}} \delta_{ij}.$$

The total degree of $\Lambda_{\underline{t},i}$ is $\leq \|\underline{t}\| + 1$.

Expansion for polynomials

Corollary.

Any polynomial $f \in \mathbb{C}[\underline{z}]$ *can be expanded as a finite sum*

$$f(\underline{z}) = \sum_{(\underline{t},i)\in\mathcal{T}} (D^{\underline{t}} f)(\underline{e}_i)\Lambda_{\underline{t},i}(\underline{z}).$$

Consequence: Applying this result to the functions $\frac{\underline{z}^{\underline{k}}}{\underline{k}!}$ yields inductive formulae for $\Lambda_{\underline{t},i}$.
One deduces explicit formulae for these polynomials in terms of the one variable Lidstone polynomials.

Expansion for entire functions of exponential type $< \pi$

Theorem.
Any entire function f in $\mathbb{C}^n$ of exponential type $< \pi$ in each of the variables can be written in a unique way as the sum of a series

$$f(z) = \sum_{(\underline{t},i)\in\mathcal{T}} (D^{\underline{t}} f)(\underline{e}_i)\Lambda_{\underline{t},i}(z)$$

which is absolutely convergent for all $\underline{z} \in \mathbb{C}^n$.

This result for the functions $\mathrm{e}^{\underline{\zeta z}}$ yields explicit formulae for the generating series of 2^n families of polynomials $\Lambda_{\underline{t},i}(z)$.

Expansion for entire functions of finite exponential type

Theorem.

Let K be a nonnegative integer. Let f be an entire function in $\mathbb{C}^n$ of finite exponential type $\leq \tau$ in all variables, with $\tau < (K+1)\pi$. Then for $\underline{z} \in \mathbb{C}^n$ we have

$$f(\underline{z}) = \sum_{(\underline{t},i)\in\mathcal{T}} g_{\underline{t},i}(\underline{z})(D^{\underline{t}} f)(\underline{e}_i)$$

$$+ \sum_{k=1}^{K} \sum_{i=1}^{n} h_{k,i}(z_1, \ldots, z_{i-1}, z_{i+1} \ldots, z_n) \sin(k\pi z_i),$$

where the functions $g_{\underline{t},i}(\underline{z})$ are entire functions in $\mathbb{C}^n$, the series is absolutely convergent and $h_{k,i}$, ($k = 1, 2, \ldots, K$, $i = 1, \ldots, n$) are even entire functions of $n-1$ variable of exponential type $\leq \tau$ in all $n-1$ variables.

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Multivariate Lidstone Interpolation

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