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# Linear recurrence sequences, exponential polynomials and Diophantine approximation

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# Abstract

Linear recurrence sequences are ubiquitous. They occur in biology, economics, computer science (analysis of algorithms), digital signal processing. We give a survey of this subject, including connections with linear combinations of powers and with exponential polynomials, with an emphasis on arithmetic questions. This lecture will include new results, arising from a joint work with Claude Levesque, involving families of Diophantine equations, with explicit examples related to some units of L. Bernstein and H. Hasse.

# Leonardo Pisano (Fibonacci)

Fibonacci sequence $(F_n)_{n\geq 0}$

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,$$

$$34,\ 55,\ 89,\ 144,\ 233\dots$$

is defined by

$$F_0 = 0,\ F_1 = 1,$$

$$F_{n+2} = F_{n+1} + F_n \quad (n \geq 0).$$

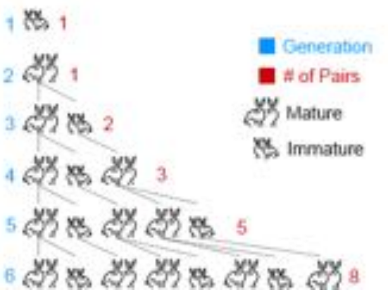
http://oeis.org/A000045

Leonardo Pisano (Fibonacci)
(1170–1250)

# Fibonacci rabbits

Fibonacci considers the growth of a rabbit population.

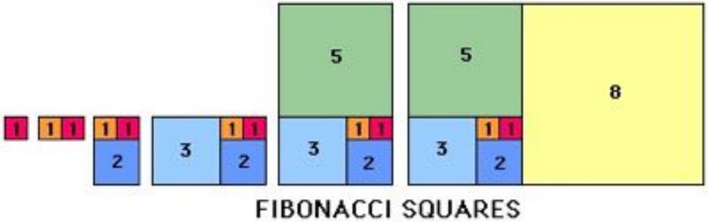
A newly born pair of rabbits, one male, one female, are put in a field. Rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits ; rabbits never die and a mating pair always produces




one new pair (one male, one female) every month from the second month on. The puzzle that Fibonacci posed was : how many pairs will there be in one year ?

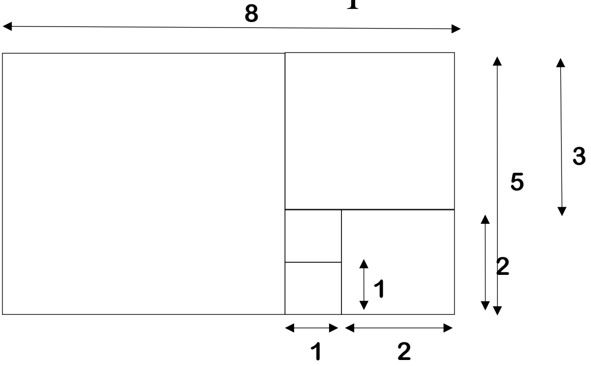
*Answer* : $F_{12} = 144$.

# Fibonacci squares



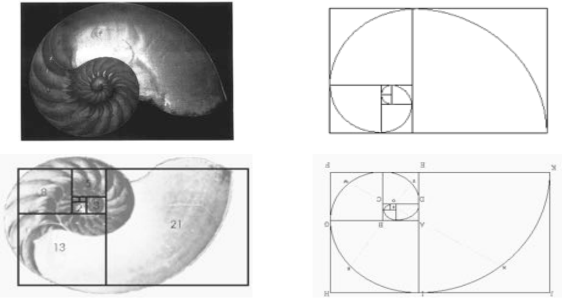

http://mathforum.org/dr.math/faq/faq.golden.ratio.html

# Geometric construction of the Fibonacci sequence



# Fibonacci numbers in nature

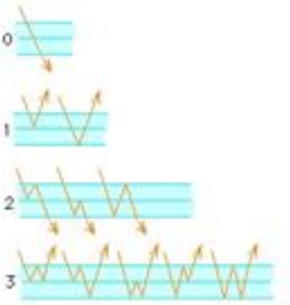
***Ammonite (Nautilus shape)***



# Reflections of a ray of light

Consider three parallel sheets of glass and a ray of light which crosses the first sheet. Each time it touches one of the sheets, it can cross it or reflect on it.

Denote by $p_n$ the number of different paths with the ray going out of the system after $n$ reflections.




$p_0 = 1,$

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$p_2 = 3,$

$p_3 = 5.$

In general, $p_n = F_{n+2}$.

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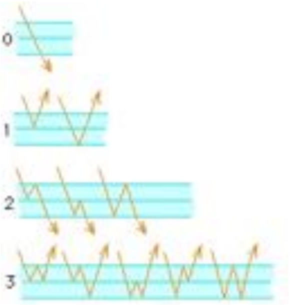


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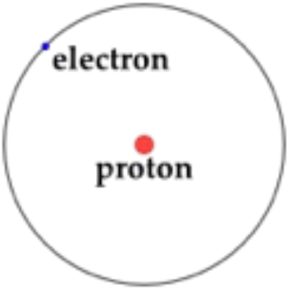
# Levels of energy of an electron of an atom of hydrogen

An atom of hydrogen can have three levels of energy, $0$ at the ground level when it does not move, $1$ or $2$. At each step, alternatively, it gains or losses $1$ or $2$ levels of energy, without going below $0$ nor above $2$. Let $\ell_n$. be the number of different possible histories of this electron are there after $n$ steps.




We have $\ell_0 = 1$ (initial state level $0$)

$\ell_1 = 2$ : state $1$ or $2$, histories $01$ or $02$.

$\ell_2 = 3$ : histories $010$, $021$ or $020$.

$\ell_3 = 5$ : histories $0101$, $0102$, $0212$, $0201$ or $0202$.

In general, $\ell_n = F_{n+2}$.

# Rhythmic patterns

The Fibonacci sequence appears in Indian mathematics, in connection with Sanskrit prosody. Several Indian scholars, Pingala (200 BC), Virahanka (c. 700 AD), Gopāla (c. 1135), and the Jain scholar Hemachandra (c. 1150) studied rhythmic patterns that are formed from one-beat notes (or short syllables, `ti` in Morse Alphabet) : • and two-beat notes (or long syllables, `ta ta` in Morse) : ■■.

$1$ beat, $1$ pattern : •

$2$ beats, $2$ patterns : •• and ■■

$3$ beats, $3$ patterns : • • •, •■■ and ■■•

$4$ beats, $5$ patterns :

• • ••, ■■ • •, •■■•, • • ■■, ■■■■

$n$ beats, $F_{n+1}$ patterns

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$2$ beats, $2$ patterns : ●● and ■■

$3$ beats, $3$ patterns : ●●●, ●■■ and ■■●

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●●●●, ■■●●, ●■■●, ●●■■, ■■■■

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# Fibonacci sequence and the Golden ratio

For $n \geq 0$, the Fibonacci number $F_n$ is the nearest integer to

$$\frac{1}{\sqrt{5}}\Phi^n,$$

where $\Phi$ is the *Golden Ratio* :

$$\Phi = \frac{1+\sqrt{5}}{2} = \lim_{n\to\infty}\frac{F_{n+1}}{F_n} = 1.6180339887499\ldots$$

which satisfies

$$\Phi = 1 + \frac{1}{\Phi}.$$

# Binet's formula

For $n \geq 0$,

$$F_n = \frac{\Phi^n - (-\Phi)^{-n}}{\sqrt{5}}$$

$$= \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}},$$

Jacques Philippe Marie Binet (1843)

$$\Phi = \frac{1+\sqrt{5}}{2}, \quad -\Phi^{-1} = \frac{1-\sqrt{5}}{2},$$

$$X^2 - X - 1 = (X - \Phi)(X + \Phi^{-1}).$$

# The so–called Binet Formula

Formula of A. De Moivre (1718, 1730), Daniel Bernoulli (1726), L. Euler (1728, 1765), J.P.M. Binet (1843) : for $n \geq 0$,

$$F_n = \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^n .$$

| Abraham de Moivre (1667–1754) | Daniel Bernoulli (1700–1782) | Leonhard Euler (1707–1783) | Jacques P.M. Binet (1786–1856) |
| --- | --- | --- | --- |
|  |  |  |  |

# Generating series

A single series encodes all the Fibonacci sequence :

$$\sum_{n\geq 0} F_n X^n = X + X^2 + 2X^3 + 3X^4 + 5X^5 + \cdots + F_n X^n + \cdots$$

Fact : this series is the Taylor expansion of a rational fraction :

$$\sum_{n\geq 0} F_n X^n = \frac{X}{1 - X - X^2}.$$

Proof : the product

$$(X + X^2 + 2X^3 + 3X^4 + 5X^5 + 8X^6 + \cdots)(1 - X - X^2)$$

is a telescoping series

$$\begin{aligned} X + X^2 + 2X^3 + 3X^4 + 5X^5 + 8X^6 + \cdots \\ -X^2 - X^3 - 2X^4 - 3X^5 - 5X^6 - \cdots \\ -X^3 - X^4 - 2X^5 - 3X^6 - \cdots \end{aligned}$$

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# Generating series of the Fibonacci sequence

`Remark.` The denominator $1 - X - X^2$ in the right hand side of

$$X + X^2 + 2X^3 + 3X^4 + \cdots + F_n X^n + \cdots = \frac{X}{1 - X - X^2}$$

is $X^2 f(X^{-1})$, where $f(X) = X^2 - X - 1$ is the irreducible polynomial of the Golden ratio $\Phi$.

# Fibonacci and powers of matrices

The Fibonacci linear recurrence relation $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$ can be written

$$\begin{pmatrix} F_{n+1} \\ F_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix}.$$

By induction one deduces, for $n \geq 0$,

$$\begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

An equivalent formula is, for $n \geq 1$,

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}.$$

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# Characteristic polynomial

The characteristic polynomial of the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

is

$$\det(XI - A) = \det \begin{pmatrix} X & -1 \\ -1 & X - 1 \end{pmatrix} = X^2 - X - 1,$$

which is the irreducible polynomial of the Golden ratio $\Phi$.

# Fibonacci sequence and the Golden ratio (continued)

For $n \geq 1$, $\Phi^n \in \mathbb{Z}[\Phi] = \mathbb{Z} + \mathbb{Z}\Phi$ is a linear combination of $1$ and $\Phi$ with integer coefficients, namely

$$\Phi^n = F_{n-1} + F_n\Phi$$

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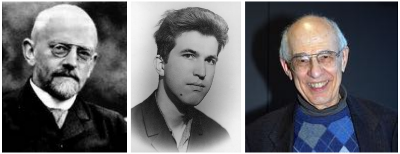
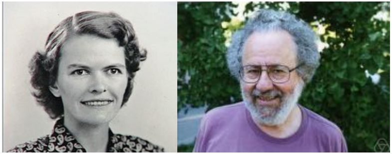
# Fibonacci sequence and Hilbert's 10th problem

Yuri Matiyasevich (1970) showed that there is a polynomial $P$ in $n$, $m$, and a number of other variables $x, y, z, \dots$ having the property that $n = F_{2m}$ iff there exist integers $x, y, z, \dots$ such that $P(n, m, x, y, z, \dots) = 0$.

This completed the proof of the impossibility of the tenth of Hilbert's problems (*does there exist a general method for solving Diophantine equations ?*) thanks to the previous work of Hilary Putnam, Julia Robinson and Martin Davis.





# The Fibonacci Quarterly

The Fibonacci sequence satisfies a lot of very interesting properties. Four times a year, the *Fibonacci Quarterly* publishes an issue with new properties which have been discovered.

The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

TABLE OF CONTENTS




VOLUME 50 AUGUST 2012 NUMBER 3

# Lucas sequence

http://oeis.org/000032

The Lucas sequence $(L_n)_{n\geq 0}$ satisfies the same recurrence relation as the Fibonacci sequence, namely

$$L_{n+2} = L_{n+1} + L_n \qquad (n \geq 0),$$

only the initial values are different :

$$L_0 = 2,\ L_1 = 1.$$

The sequence of Lucas numbers starts with

$$2,\ 1,\ 3,\ 4,\ 7,\ 11,\ 18,\ 29,\ 47,\ 76,\ 123,\ 199,\ 322\ldots$$

A closed form involving the Golden ratio $\Phi$ is

$$L_n = \Phi^n + (-\Phi)^{-n},$$

from which it follows that for $n \geq 2$, $L_n$ is the nearest integer to $\Phi^n$.

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# François Édouard Anatole Lucas (1842 - 1891)

Edouard Lucas is best known for his results in number theory. He studied the Fibonacci sequence and devised the test for Mersenne primes still used today.

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Lucas.html

# Generating series of the Lucas sequence

The generating series of the Lucas sequence

$$\sum_{n\geq 0} L_n X^n = 2 + X + 3X^2 + 4X^3 + \cdots + L_n X^n + \cdots$$

is nothing else than

$$\frac{2 - X}{1 - X - X^2}.$$

# The Lucas sequence and power of matrices

From the linear recurrence relation $L_{n+2} = L_{n+1} + L_n$ one deduces, (as we did for the Fibonacci sequence), for $n \geq 0$,

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Any one of the three sequences $(F_n)_{n\geq 0}$, $(L_n)_{n\geq 0}$, $(\Phi^n)_{n\geq 0}$ can be written as a linear combination of the two others.

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# Perrin sequence http://oeis.org/A001608

The Perrin sequence (also called *skiponacci sequence*) is the linear recurrence sequence defined by

$$P_{n+3} = P_{n+1} + P_n \quad \text{for} \quad n \geq 0,$$

with the initial conditions

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It starts with

$$3,\ 0,\ 2,\ 3,\ 2,\ 5,\ 5,\ 7,\ 10,\ 12,\ 17,\ 22,\ 29,\ 39,\ 51,\ 68\ldots$$

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# Plastic (or silver) constant

https://oeis.org/A060006

The ratio of successive terms in the Perrin sequence approaches the plastic number $\varrho$, which is the minimal Pisot–Vijayaraghavan number, real root of

$$x^3 - x - 1$$

which has a value of approximately $1.324718$.

This constant is equal to

$$\varrho = \frac{\sqrt[3]{108 + 12\sqrt{69}} + \sqrt[3]{108 - 12\sqrt{69}}}{6}.$$

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# Perrin sequence and the plastic constant

Decompose the polynomial $X^3 - X - 1$ into irreducible factors over $\mathbb{C}$

$$X^3 - X - 1 = (X - \varrho)(X - \rho)(X - \overline{\rho}).$$

and over $\mathbb{R}$

$$X^3 - X - 1 = (X - \varrho)(X^2 + \varrho X + \varrho^{-1}).$$

Hence $\rho$ and $\overline{\rho}$ are the roots of $X^2 + \varrho X + \varrho^{-1}$. Then, for $n \geq 0$,

$$P_n = \varrho^n + \rho^n + \overline{\rho}^n,$$

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It follows that, for $n \geq 0$, $P_n$ is the nearest integer to $\varrho^n$

# Perrin sequence and the plastic constant

Decompose the polynomial $X^3 - X - 1$ into irreducible factors over $\mathbb{C}$

$$X^3 - X - 1 = (X - \varrho)(X - \rho)(X - \overline{\rho}).$$

and over $\mathbb{R}$

$$X^3 - X - 1 = (X - \varrho)(X^2 + \varrho X + \varrho^{-1}).$$

Hence $\rho$ and $\overline{\rho}$ are the roots of $X^2 + \varrho X + \varrho^{-1}$. Then, for $n \geq 0$,

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# Generating series of the Perrin sequence

The generating series of the Perrin sequence

$$\sum_{n\geq 0} P_n X^n = 3 + 2X^2 + 3X^3 + 2X^4 + \cdots + P_n X^n + \cdots$$

is nothing else than

$$\frac{3 - X^2}{1 - X^2 - X^3}.$$

The denominator $1 - X^2 - X^3$ is $X^3 f(X^{-1})$ where $f(X) = X^3 - X - 1$ is the irreducible polynomial of $\varrho$.

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# Perrin sequence and power of matrices

From

$$P_{n+3} = P_{n+1} + P_n$$

we deduce

$$\begin{pmatrix} P_{n+1} \\ P_{n+2} \\ P_{n+3} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_n \\ P_{n+1} \\ P_{n+2} \end{pmatrix}$$

Hence

$$\begin{pmatrix} P_n \\ P_{n+1} \\ P_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$$

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# Characteristic polynomial

The characteristic polynomial of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

is

$$\det(XI - A) = \det \begin{pmatrix} X & -1 & 0 \\ 0 & X & -1 \\ -1 & -1 & X \end{pmatrix} = X^3 - X - 1,$$

which is the irreducible polynomial of the plastic constant $\varrho$.

# Perrin pseudoprimes

https://oeis.org/A013998

If $p$ is prime, then $p$ divides $P_p$.

The smallest composite $n$ such that $n$ divides $P_n$ is $521^2$.

For $n$ either $271441 = 521^2$ or $904631 = 7 \times 13 \times 9941$, the number $n$ divides $P_n$.
Jon Grantham has proved that there are infinitely many Perrin pseudoprimes.
The number $c$ of decimal digits of $P_{271441}$ satisfies $10^c = \varrho^{271441}$, hence $c = 271441(\log \varrho)/(\log 10) \sim 33\,150$.

The website www.Perrin088.org maintained by Richard Turk is devoted to Perrin numbers. See OEISA113788.

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# Padovan sequence

https://oeis.org/A000931

The Padovan sequence $p_n$ satisfies the same recurrence

$$p_{n+3} = p_{n+1} + p_n$$

as the Perrin sequence but has different initial values :

$$p_0 = 1, \quad p_1 = p_2 = 0.$$

It starts with

$$1, 0, 0, 1, 0, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65 \dots$$

Richard Padovan

http://mathworld.wolfram.com/LinearRecurrenceEquation.html

# Generating series and power of matrices

$$1 + X^3 + X^5 + \cdots + p_n X^n + \cdots = \frac{1 - X^2}{1 - X^2 - X^3}.$$

For $n \geq 0$,

$$\begin{pmatrix} p_n \\ p_{n+1} \\ p_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

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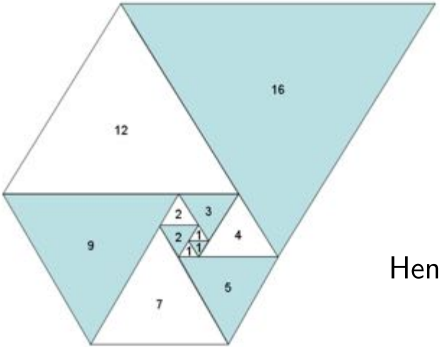
# Padovan triangles




$$p_n = p_{n-2} + p_{n-3}$$

$$p_{n-1} = p_{n-3} + p_{n-4}$$

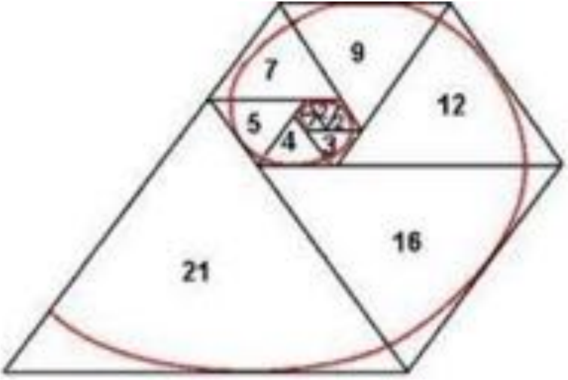
$$p_{n-2} = p_{n-4} + p_{n-5}$$

Hence

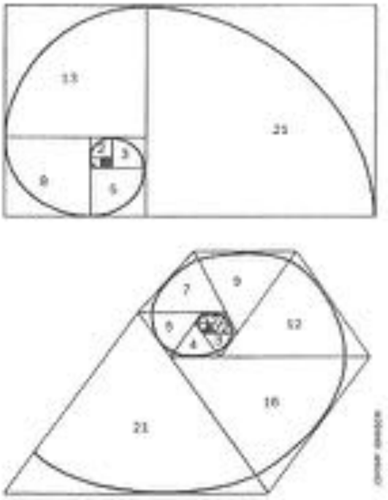
$$p_n - p_{n-1} = p_{n-5}$$

$$p_n = p_{n-1} + p_{n-5}$$

# Padovan triangles



# Padovan triangles vs Fibonacci squares



# Narayana sequence

https://oeis.org/A000930

Narayana sequence is defined by the recurrence relation

$$C_{n+3} = C_{n+2} + C_n$$

with the initial values $C_0 = 2$, $C_1 = 3$, $C_2 = 4$.

It starts with

$$2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, 129, 189, 277, 406, 595, \dots$$

Real root of $x^3 - x^2 - 1$

$$\frac{\sqrt[3]{\frac{29+3\sqrt{93}}{2}} + \sqrt[3]{\frac{29-3\sqrt{93}}{2}} + 1}{3} = 1.465571231876768\dots$$

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$$2 + 3X + 4X^2 + 6X^3 + \cdots + C_nX^n + \cdots = \frac{2 + X + X^2}{1 - X - X^3}.$$

For $n \geq 0$,

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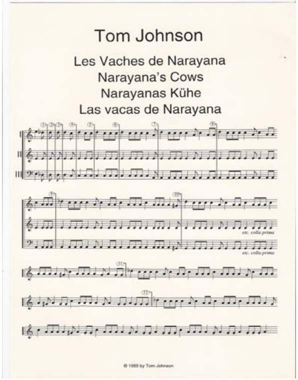
# Narayana's cows

Narayana was an Indian mathematician in the 14th century who proposed the following problem :
*A cow produces one calf every year. Beginning in its fourth year each calf produces one calf at the beginning of each year. How many calves are there altogether after, for example,* $17$ *years ?*

# Music : http://www.pogus.com/21033.html

In working this out, Tom Johnson found a way to translate this into a composition called *Narayana's Cows.*
*Music :* Tom Johnson
*Saxophones :* Daniel Kientzy




# Narayana's cows

http://webusers.imj-prg.fr/~michel.waldschmidt/

| Year | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Original Cow | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| Second generation | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| Third generation | 0 | 0 | 0 | 1 | 3 | 6 | 10 | 15 | 21 | 28 | 36 | 45 | 55 | 66 |
| Fourth generation | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 4 | 10 | 20 | 35 | 56 | 84 | 120 |
| Fifth generation | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 5 | 15 | 35 | 70 |
| Sixth generation | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 6 |
| Total | **2** | **3** | **4** | **6** | **9** | **13** | **19** | **28** | **41** | **60** | **88** | **129** | **189** | **277** |

# Jean-Paul Allouche and Tom Johnson

http://webusers.imj-prg.fr/~jean-paul.allouche/bibliorecente.html
http://www.math.jussieu.fr/~allouche/johnson1.pdf

# Cows, music and morphisms

Jean-Paul Allouche and Tom Johnson

- Narayana's Cows and Delayed Morphisms
In 3èmes Journées d'Informatique Musicale (JIM '96), Ile de Tatihou, Les Cahiers du GREYC (1996 no. 4), pages 2-7, May 1996.
http://kalvos.org/johness1.html

- Finite automata and morphisms in assisted musical composition,
Journal of New Music Research, no. 24 (1995), 97 – 108.
http://www.tandfonline.com/doi/abs/10.1080/09298219508570676
http://web.archive.org/web/19990128092059/www.swets.nl/jnmr/vol24_2.html

# Linear recurrence sequences : definitions

A *linear recurrence sequence* is a sequence of numbers $\mathbf{u} = (u_0, u_1, u_2, \dots)$ for which there exist a positive integer $d$ together with numbers $a_1, \dots, a_d$ with $a_d \neq 0$ such that, for $n \geq 0$,

$$(\star) \qquad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n.$$

Here, a *number* means an element of a field $\mathbb{K}$ of zero characteristic.

Given $\underline{a} = (a_1, \dots, a_d) \in \mathbb{K}^d$, the set $E_{\underline{a}}$ of linear recurrence sequences $\mathbf{u} = (u_n)_{n \geq 0}$ satisfying $(\star)$ is a $\mathbb{K}$–vector subspace of dimension $d$ of the space $\mathbb{K}^{\mathbb{N}}$ of all sequences .

The characteristic (or companion) polynomial of the linear recurrence is

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# Linear recurrence sequence : examples

• Constant sequence : $u_n = u_0$.
Linear recurrence sequence of order $1$ : $u_{n+1} = u_n$.
Characteristic polynomial : $f(X) = X - 1$.
Generating series :

$$\sum_{n\geq 0} X^n = \frac{1}{1-X}.$$

• Geometric progression : $u_n = u_0\gamma^n$.
Linear recurrence sequence of order $1$ : $u_n = \gamma u_{n-1}$.
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Generating series :

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# Linear recurrence sequence : examples

- $u_n = n$. Linear recurrence sequence of order $2$ :

$$n+2 = 2(n+1) - n.$$

Characteristic polynomial

$$f(X) = X^2 - 2X + 1 = (X-1)^2.$$

Generating series

$$\sum_{n\geq 0} nX^n = \frac{1}{1-2X+X^2}.$$

Power of matrices :

$$\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}^n = \begin{pmatrix} -n+1 & n \\ -n & n+1 \end{pmatrix}.$$

# Linear recurrence sequence : examples

• $u_n = f(n)$, $f$ polynomial of degree $d$. Linear recurrence sequence of order $d+1$.

**Proof.** The sequences

$$(f(n))_{n\geq 0}, \quad (f(n+1))_{n\geq 0}, \quad \cdots, \quad (f(n+k))_{n\geq 0}$$

are $\mathbb{K}$–linearly independent in $\mathbb{K}^{\mathbb{N}}$ for $k = d-1$ and linearly dependent for $k = d$ .

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# Linear sequences which are ultimately recurrent

The sequence

$$1, 0, 0, \ldots$$

is not a linear recurrence sequence.

The condition

$$u_{n+1} = u_n$$

is satisfied only for $n \geq 1$.

The relation

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# Order of a linear recurrence sequence

If $\mathbf{u} = (u_n)_{n\geq 0}$ satisfies the linear recurrence, the characteristic polynomial of which is $f$, then, for any monic polynomial $g \in \mathbb{K}[X]$, this sequence $\mathbf{u}$ also satisfies the linear recurrence, the characteristic polynomial of which is $fg$.

`Example` : for $g(X) = X - \gamma$, from

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we deduce

$$u_{n+d+1} - a_1 u_{n+d} - \cdots - a_d u_{n+1} \\ + \gamma(u_{n+d} - a_1 u_{n+d-1} - \cdots - a_d u_n) = 0.$$

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# Polynomial combinations of powers

The sum of any two linear recurrence sequences is a linear recurrence sequence.

The set $\cup_{\underline{a}} E_{\underline{a}}$ of all linear recurrence sequences with coefficients in $\mathbb{K}$ is a sub–$\mathbb{K}$–algebra of $\mathbb{K}^{\mathbb{N}}$.

Given polynomials $p_1, \ldots, p_\ell$ in $\mathbb{K}[X]$ and elements $\gamma_1, \ldots, \gamma_\ell$ in $\mathbb{K}^\times$, the sequence

$$\bigl(p_1(n)\gamma_1^n + \cdots + p_\ell(n)\gamma_\ell^n\bigr)_{n \geq 0}$$

is a linear recurrence sequence.

**Fact :** any linear recurrence sequence is of this form.

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# Linear recurrence sequence and Brahmagupta–Pell–Fermat Equation

Let $d$ be a positive integer, not a square. The solutions $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ of the Brahmagupta–Pell–Fermat Equation

$$x^2 - dy^2 = \pm 1$$

form a sequence $(x_n, y_n)_{n \in \mathbb{Z}}$ defined by

$$x_n + \sqrt{d} y_n = (x_1 + \sqrt{d} y_1)^n.$$

From

$$2x_n = (x_1 + \sqrt{d} y_1)^n + (x_1 - \sqrt{d} y_1)^n$$

we deduce that $(x_n)_{n \geq 0}$ is a linear recurrence sequence. Same for $y_n$, and also for $n \geq 0$.

# Linear recurrence sequence and Brahmagupta–Pell–Fermat Equation

Let $d$ be a positive integer, not a square. The solutions $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ of the Brahmagupta–Pell–Fermat Equation

$$x^2 - dy^2 = \pm 1$$

form a sequence $(x_n, y_n)_{n \in \mathbb{Z}}$ defined by

$$x_n + \sqrt{d} y_n = (x_1 + \sqrt{d} y_1)^n.$$

From

$$2x_n = (x_1 + \sqrt{d} y_1)^n + (x_1 - \sqrt{d} y_1)^n$$

we deduce that $(x_n)_{n \geq 0}$ is a linear recurrence sequence. Same for $y_n$, and also for $n \geq 0$.

# Doubly infinite linear recurrence sequence

A sequence $(u_n)_{n\in\mathbb{Z}}$ indexed by $\mathbb{Z}$ is a linear recurrence sequence if it satisfies

$$(\star) \qquad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n.$$

for all $n \in \mathbb{Z}$.

Recall $a_d \neq 0$.

Such a sequence is determined by $d$ consecutive values.

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# Linear recurrence sequence : simple roots

A basis of $E_{\underline{a}}$ over $\mathbb{K}$ is obtained by attributing to the initial values $u_0, \ldots, u_{d-1}$ the values given by the canonical basis of $\mathbb{K}^d$.

Given $\gamma$ in $\mathbb{K}^\times$, a necessary and sufficient condition for a sequence $(\gamma^n)_{n\geq 0}$ to satisfy $(\star)$ is that $\gamma$ is a root of the characteristic polynomial

$$f(X) = X^d - a_1X^{d-1} - \cdots - a_d.$$

If this polynomial has $d$ distinct roots $\gamma_1, \ldots, \gamma_d$ in $\mathbb{K}$,

$$f(X) = (X - \gamma_1) \cdots (X - \gamma_d), \quad \gamma_i \neq \gamma_j,$$

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The characteristic polynomial of the linear recurrence $u_n = 2\gamma u_{n-1} - \gamma^2 u_{n-2}$ is $X^2 - 2\gamma X + \gamma^2 = (X-\gamma)^2$ with a double root $\gamma$.

The sequence $(n\gamma^n)_{n\geq 0}$ satisfies

$$n\gamma^n = 2\gamma(n-1)n\gamma^{n-1} - \gamma^2(n-2)\gamma^{n-2}.$$

A basis of $E_{\underline{a}}$ for $a_1 = 2\gamma$, $a_2 = -\gamma^2$ is given by the two sequences $(\gamma^n)_{n\geq 0}$, $(n\gamma^n)_{n\geq 0}$.

Given $\gamma \in \mathbb{K}^\times$, a necessary and sufficient condition for the sequence $n\gamma^n$ to satisfy the linear recurrence relation $(\star)$ is that $\gamma$ is a root of multiplicity $\geq 2$ of $f(X)$.

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# Linear recurrence sequence : multiple roots

In general, when the characteristic polynomial splits as

$$X^d - a_1 X^{d-1} - \cdots - a_d = \prod_{i=1}^{\ell} (X - \gamma_i)^{t_i},$$

a basis of $E_{\underline{a}}$ is given by the $d$ sequences

$$(n^k \gamma_i^n)_{n \geq 0}, \qquad 0 \leq k \leq t_i - 1, \quad 1 \leq i \leq \ell.$$

# Generating series of a linear recurrence sequence

Let $\mathbf{u} = (u_n)_{n\geq 0}$ be a linear recurrence sequence

$$(\star) \qquad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n \quad \text{for} \quad n \geq 0$$

with characteristic polynomial

$$f(X) = X^d - a_1 X^{d-1} - \cdots - a_d.$$

Denote by $f^-$ the reciprocal polynomial of $f$ :

$$f^-(X) = X^d f(X^{-1}) = 1 - a_1 X - \cdots - a_0 X^d.$$

Then

$$\sum_{n=0}^{\infty} u_n X^n = \frac{r(X)}{f^-(X)},$$

where $r$ is a polynomial of degree less than $d$ determined by the initial values of $\mathbf{u}$.

# Generating series of a linear recurrence sequence

Assume

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# Taylor coefficients of rational functions

Conversely, the coefficients the Taylor expansion of any rational fraction $a(X)/b(X)$ with $\deg a < \deg b$ satisfies the recurrence relation with characteristic polynomial $f \in K[X]$ given by $f(X) = b^{-}(X)$.

Therefore a sequence $\mathbf{u} = (u_n)_{n \geq 0}$ satisfies the recurrence relation $(\star)$ with characteristic polynomial $f \in K[X]$ if and only if

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# Matrix notation for a linear recurrence sequence

The linear recurrence sequence

$$(\star) \qquad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n \quad \text{for} \quad n \geq 0$$

can be written

$$\begin{pmatrix} u_{n+1} \\ u_{n+2} \\ \vdots \\ u_{n+d} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_d & a_{d-1} & a_{d-2} & \cdots & a_1 \end{pmatrix} \begin{pmatrix} u_n \\ u_{n+1} \\ \vdots \\ u_{n+d-1} \end{pmatrix}.$$

# Matrix notation for a linear recurrence sequence

$$U_{n+1} = AU_n$$

with

$$U_n = \begin{pmatrix} u_n \\ u_{n+1} \\ \vdots \\ u_{n+d-1} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_d & a_{d-1} & a_{d-2} & \cdots & a_1 \end{pmatrix}.$$

The determinant of $I_d X - A$ (the characteristic polynomial of $A$) is nothing but

$$f(X) = X^d - a_1 X^{d-1} - \cdots - a_d,$$

the characteristic polynomial of the linear recurrence sequence. By induction

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# Powers of matrices

Let $A = (a_{ij})_{1 \le i,j \le d} \in \mathrm{GL}_{d \times d}(\mathbb{K})$ be a $d \times d$ matrix with coefficients in $\mathbb{K}$ and nonzero determinant. For $n \ge 0$, define

$$A^n = \big(a_{ij}(n)\big)_{1 \le i,j \le d}.$$

Then each of the $d^2$ sequences $\big(a_{ij}(n)\big)_{n \ge 0}$, $(1 \le i, j \le d)$ is a linear recurrence sequence. The roots of the characteristic polynomial of these linear recurrences are the eigenvalues of $A$.

In particular the sequence $\big(\mathrm{Tr}(A^n)\big)_{n \ge 0}$ satisfies the linear recurrence, the characteristic polynomial of which is the characteristic polynomial of the matrix $A$.

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# Conversely :

Given a linear recurrence sequence $\mathbf{u} \in \mathbb{K}^{\mathbb{N}}$, there exist an integer $d \geq 1$ and a matrix $A \in \mathrm{GL}_d(\mathbb{K})$ such that, for each $n \geq 0$,

$$u_n = a_{11}(n).$$

The characteristic polynomial of $A$ is the characteristic polynomial of the linear recurrence sequence.

Everest G., van der Poorten A., Shparlinski I., Ward T. – *Recurrence sequences,* Mathematical Surveys and Monographs (AMS, 2003), volume 104.

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# Discrete version of linear differential equations

A sequence $\mathbf{u} \in \mathbb{K}^{\mathbb{N}}$ can be viewed as a linear map $\mathbb{N} \longrightarrow \mathbb{K}$. Define the discrete derivative $\mathcal{D}$ by

$$\begin{array}{rccc} \mathcal{D}\mathbf{u} : & \mathbb{N} & \longrightarrow & \mathbb{K} \\ & n & \longmapsto & u_{n+1} - u_n. \end{array}$$

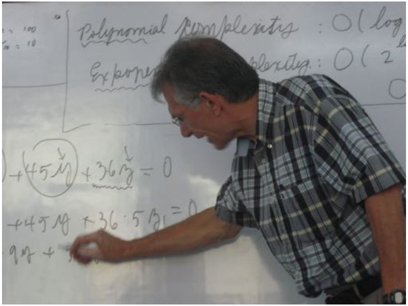
A sequence $\mathbf{u} \in \mathbb{K}^{\mathbb{N}}$ is a linear recurrence sequence if and only if there exists $Q \in \mathbb{K}[T]$ with $Q(1) \neq 1$ such that

$$Q(\mathcal{D})\mathbf{u} = 0.$$

Linear recurrence sequences are a discrete version of linear differential equations with constant coefficients.

The condition $Q(1) \neq 0$ reflects $a_d \neq 0$ – otherwise one gets *ultimately* recurrent sequences.

# Joint work with Claude Levesque



*Linear recurrence sequences and twisted binary forms.*
Proceedings of the International Conference on Pure and Applied Mathematics ICPAM-GOROKA 2014. South Pacific Journal of Pure and Applied Mathematics.

http://webusers.imj-prg.fr/~michel.waldschmidt//articles/pdf/ProcConfPNG2014.pdf

# Families of binary forms

Consider a binary form $F_0(X,Y) \in \mathbb{C}[X,Y]$ which satisfies $F_0(1,0)=1$. We write it as

$$F_0(X,Y) = X^d + a_1X^{d-1}Y + \cdots + a_dY^d = \prod_{i=1}^{d}(X-\alpha_iY).$$

Let $\epsilon_1,\dots,\epsilon_d$ be $d$ nonzero complex numbers not necessarily distinct. Twisting $F_0$ by the powers $\epsilon_1^n,\dots,\epsilon_d^n$ $(n\in\mathbb{Z})$, we obtain the family of binary forms

$$F_n(X,Y) = \prod_{i=1}^{d}(X-\alpha_i\epsilon_i^nY),$$

which we write as

$$X^d - U_1(n)X^{d-1}Y + \cdots + (-1)^dU_d(n)Y^d.$$

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$$U_h(0) = (-1)^h a_h \qquad (1\le h\le d).$$

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# Families of Diophantine equations

With Claude Levesque, we consider some families of diophantine equations

$$F_n(x,y) = m$$

obtained in the same way from a given irreducible form $F(X,Y)$ with coefficients in $\mathbb{Z}$, when $\epsilon_1,\dots,\epsilon_d$ are algebraic units and when the algebraic numbers $\alpha_1\epsilon_1,\dots,\alpha_d\epsilon_d$ are Galois conjugates with $d \geq 3$.

**Theorem**. *Let $\mathbb{K}$ be a number field of degree $d \geq 3$, $S$ a finite set of places of $\mathbb{K}$ containing the places at infinity. Denote by $\mathcal{O}_S$ the ring of $S$–integers of $\mathbb{K}$ and by $\mathcal{O}_S^\times$ the group of $S$–units of $\mathbb{K}$. Assume $\alpha_1,\dots,\alpha_d,\epsilon_1,\dots,\epsilon_d$ belong to $\mathbb{K}^\times$ Then there are only finitely many $(x,y,n)$ in $\mathcal{O}_S \times \mathcal{O}_S \times \mathbb{Z}$ satisfying*

$$F_n(x,y) \in \mathcal{O}_S^\times, \quad xy \neq 0 \quad \textit{and} \quad \mathrm{Card}\{\alpha_1\epsilon_1^n,\dots,\alpha_d\epsilon_d^n\} \geq 3.$$

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is a linear recurrence sequence.

For example, for $n \in \mathbb{Z}$,

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For $1 \le h \le d$, the sequence $\big(U_h(n)\big)_{n\in\mathbb{Z}}$ is a linear combination of the sequences

$$\big((\epsilon_{i_1}\cdots\epsilon_{i_h})^n\big)_{n\in\mathbb{Z}}, \quad (1 \le i_1 < \cdots < i_h \le d).$$

# Units of Bernstein and Hasse

Let $t$ and $s$ be two positive integers, $D$ an integer $\geq 1$, and $c \in \{-1, +1\}$. Let $\omega > 1$ satisfy

$$\omega^{st} = D^{st} + c,$$

where it is assumed that $\mathbb{Q}(\omega)$ is of degree $st$.

Consider

$$\alpha = D - \omega, \quad \epsilon = D^t - \omega^t.$$

L. Bernstein and H. Hasse noticed that $\alpha$ and $\epsilon$ are units of degree $st$ and $s$ respectively, and showed that these units can be obtained from the Jacobi–Perron algorithm. H.-J. Stender proved that for $s = t = 2$, $\{\alpha, \epsilon\}$ is a fundamental system of units of the quartic field $\mathbb{Q}(\omega)$.

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# Helmut Hasse (1898-1979)

$D > 0$, $s \geq 1$, $t \geq 1$,
$c \in \{-1, +1\}$, $\omega > 0$,

$$\omega^{st} = D^{st} + c,$$

$$\alpha = D - \omega,$$

$$\epsilon = D^t - \omega^t.$$

$$(\alpha - D)^{st} = (-1)^{st}(D^{st} + c).$$

# Diophantine equations associated with some units of Bernstein and Hasse

The irreducible polynomial of $\alpha$ is $F_0(X,1)$, with

$$F_0(X,Y) = (X - DY)^{st} - (-1)^{st}(D^{st} + c)Y^{st}.$$

For $n \in \mathbb{Z}$, the binary form $F_n(X,Y)$, obtained by twisting $F_0(X,Y)$ with the powers $\epsilon^n$ of $\epsilon$, is the homogeneous version of the irreducible polynomial $F_n(X,1)$ of $\alpha\epsilon^n$. So $F_n$ depends of the parameters $n$, $D$, $s$, $t$ and $c$.

**Theorem** (with Claude Levesque). *Suppose $st \ge 3$. There exists an effectively computable constant $\kappa$, depending only on $D$, $s$ and $t$, with the following property. Let $m$, $a$, $x$, $y$ be rational integers satisfying $m \ge 2$, $xy \ne 0$, $[\mathbb{Q}(\alpha\epsilon^a) : \mathbb{Q}] = st$ and*

$$F_n(x,y)| \le m.$$

*Then*

$$\max\{\log|x|, \log|y|, |n|\} \le \kappa \log m.$$

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# Linear recurrence sequences, exponential polynomials and Diophantine approximation

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