Fourth lecture: April 15, 2011

Example 20. Let a and b be two positive integers with $a < b$. Define two sequences of integers $(\delta_p)_{p\geq 1}$ and $(P_\ell)_{\ell\geq 1}$ by the induction formulae

$$\delta_p = \delta_{p-a} + \delta_{p-b} \quad \text{for } p \geq b+1$$

with initial conditions

$$\begin{cases} \delta_0 = 1, & \\ \delta_p = 0 & \text{for } 1 \leq p \leq b-1 \text{ if } a \text{ does not divide } p, \\ \delta_p = 1 & \text{for } a \leq p \leq b-1 \text{ if } a \text{ divides } p, \\ \delta_b = 1 & \text{if } a \text{ does not divide } b, \\ \delta_b = 2 & \text{if } a \text{ divides } b \end{cases}$$

and

$$P_\ell = P_{\ell-a} + P_{\ell-b} \quad \text{for } \ell \geq b+1$$

with initial conditions

$$\begin{cases} P_\ell = 0 & \text{for } 1 \leq \ell < b \text{ if } a \text{ does not divides } \ell, \\ P_\ell = a & \text{for } a \leq \ell < b \text{ if } a \text{ divides } \ell, \\ P_b = b & \text{if } a \text{ does not divides } b, \\ P_b = a+b & \text{if } a \text{ divides } b. \end{cases}$$

Under the hypotheses of Lemma 13, the following properties are equivalent:

(i) *The Hilbert–Poincaré series of A is*

$$\mathcal{H}_A(t) = \frac{1}{1-t^a-t^b}.$$

(ii) *For any $p \geq 1$, we have $d_p = \delta_p$.*

(iii) *For any $\ell \geq 1$, we have*

$$\sum_{n|\ell} nN(n) = P_\ell$$

(iv) *For any $k \geq 1$, we have*

$$N(k) = \frac{1}{k}\sum_{\ell|k} \mu(k/\ell)P_\ell.$$

Proof. Condition (ii) means that the sequence $(d_p)_{p\geq 0}$ satisfies

$$(1-t^a-t^b)\sum_{p\geq 0} d_p t^p = 1.$$

The equivalence between (i) and (ii) follows from the definition of d_p in condition (ii): the series

$$\mathcal{H}_A(t) = \sum_{p\geq 0} d_p t^p$$

satisfies

$$(1 - t^a - t^b)\mathcal{H}_A(t) = 1$$

if and only if the sequence $(d_p)_{p\geq 0}$ is the same as $(\delta_p)_{p\geq 0}$. The definition of the sequence $(P_\ell)_{\ell\geq 0}$ can be written

$$(1 - t^a - t^b)\sum_{\ell\geq 1} P_\ell t^{\ell-1} = at^{a-1} + bt^{b-1}.$$

The equivalence between (iii) and (iv) follows from Möbius inversion formula. It remains to check that conditions (i) and (iii) are equivalent. The logarithmic derivative of $1/(1 - t^a - t^b)$ is

$$\frac{at^{a-1} + bt^{b-1}}{1 - t^a - t^b} = \sum_{\ell\geq 1} P_\ell t^{\ell-1},$$

while the logarithmic derivative of $\prod_{k\geq 1} 1/(1 - t^k)^{N(k)}$ is given by (18). This completes the proof. $\square$

A first special case of example 20 is with $a = 1$ and $b = 2$: the sequence $(d_p)_{p\geq 1} = (1, 2, 3, 5\dots)$ is the Fibonacci sequence $(F_n)_{n\geq 0}$ shifted by 1: $d_p = F_{p+1}$ for $p \geq 1$ while the sequence $(P_\ell)_{\ell\geq 1}$ is the sequence of Lucas numbers

$$1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, \dots$$

See [3] `A000045` for the Fibonacci sequence and `A000032` for the Lucas sequence.

For the application to MZV, we are interested with the special case where $a = 2$, $b = 3$ in example 20. In this case the recurrence formula for the sequence $(P_\ell)_{\ell\geq 1}$ is $P_\ell = P_{\ell-2} + P_{\ell-3}$ and the initial conditions are $P_1 = 0$, $P_2 = 2$, $P_3 = 3$, so that, if we set $P_0 = 3$, then the sequence $(P_\ell)_{\ell\geq 0}$ is

$$3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39, 51, 68, 90, 119, \dots$$

This is the so–called *Perrin sequence* or *Ondrej Such sequence* (see [3] `A001608`), defined by

$$P_\ell = P_{\ell-2} + P_{\ell-3} \quad \text{for } \ell \geq 3$$

with the initial conditions

$$P_0 = 3, \quad P_1 = 0, \quad P_2 = 2.$$

The sequence $N(p)$ of the number of Lyndon words of weight p on the alphabet $\{2, 3\}$ satisfies, for $p \geq 1$,

$$N(p) = \frac{1}{p}\sum_{\ell | p} \mu(p/\ell)P_\ell.$$

The generating function of the sequence $(P_\ell)_{\ell\ge 1}$ is

$$\sum_{\ell\ge 1} P_\ell t^{\ell-1} = \frac{3-t^2}{1-t^2-t^3}\cdot$$

For $\ell > 9$, P_ℓ is the nearest integer to r^ℓ, with $r = 1.324\,717\,957\,244\,7\ldots$ the real root of x^3-x-1 (see [3] `A060006`), which has been also called the *silver number*, also the *plastic number*: this is the smallest *Pisot-Vijayaraghavan number*.

Example 21 (Words on the alphabet $\{f_3, f_5, \ldots, f_{2n+1}\ldots\}$)**.** Consider the alphabet $\{f_3, f_5, \ldots, f_{2n+1}\ldots\}$ with countably many letters, one for each odd weight. The free algebra on this alphabet (see §6.2) is the so–called *concatenation algebra* $\mathcal{C} := \mathbf{Q}\langle f_3, f_5, \ldots, f_{2n+1}\ldots\rangle$.

We get a word of weight p by concatenating a word of weight $p-(2k+1)$ with f_{2k+1}; in other terms, starting with a word of weight q having the last letter say f_{2k+1}, the prefix obtained by removing the last letter has weight $q-2k-1$. Hence the number of words with weight p satisfies

$$d_p = d_{p-3} + d_{p-5} + \cdots$$

(a finite sum for each p) with $d_0 = 1$ (the empty word), $d_1 = d_2 = 0$, $d_3 = 1$. The Hilbert–Poincaré series of $\mathcal{C}$

$$\mathcal{H}_{\mathcal{C}}(t) := \sum_{p\ge 0} d_p t^p$$

satisfies

$$\mathcal{H}_{\mathcal{C}}(t) = 1 - t^3\mathcal{H}_{\mathcal{C}}(T) - t^5\mathcal{H}_{\mathcal{C}}(t) - \cdots$$

Since

$$(1-t^2)(1-t^3-t^5-t^7-\cdots) = 1-t^3-t^5-t^7-\cdots-t^2+t^{-5}+t^7+\cdots = 1-t^2-t^3$$

(telescoping series), we deduce

$$\mathcal{H}_{\mathcal{C}}(t) = \frac{1-t^2}{1-t^2-t^3}\cdot$$

Recall (example 15) that the Hilbert–Poincaré series of the commutative polynomial algebra $\mathbf{Q}[f_2]$ with f_2 a single variable of weight 2 is $1/(1-t^2)$. The algebra $\mathcal{C}\otimes_{\mathbf{Q}}\mathbf{Q}[f_2]$, which plays an important role in the theory of mixed Tate motives, can be viewed either as the free algebra on the alphabet $\{f_3, f_5, \ldots, f_{2n+1}\ldots\}$ over the commutative ring $\mathbf{Q}[f_2]$, or as the algebra $\mathcal{C}[f_2]$ of polynomials in the single variable f_2 with coefficients in $\mathcal{C}$. The Hilbert–Poincaré series of this algebra is the product

$$\mathcal{H}_{\mathcal{C}[f_2]}(t) = \frac{1-t^2}{1-t^2-t^3}\cdot\frac{1}{1-t^2} = \frac{1}{1-t^2-t^3},$$

which is conjectured to be also the Hilbert–Poincaré series of the algebra $\mathfrak{Z}$.

5.4 Filtrations

A *filtration* on a A-module E is an increasing or decreasing sequence of sub–A–modules

$$\{0\} = E_0 \subset E_1 \subset \cdots \subset E_n \subset \cdots$$

or

$$E = E_0 \supset E_1 \supset \cdots \supset E_n \supset \cdots$$

Sometimes one writes $\mathcal{F}^n(E)$ in place of E_n. For instance if φ is an endomorphism of a A–module E, the sequence of kernels of the iterates

$$\{0\} \subset \ker\varphi \subset \ker\varphi^2 \subset \cdots \subset \ker\varphi^n \subset \cdots$$

is an increasing filtration on E, while the images of the iterates

$$E \supset \mathrm{Im}\varphi \supset \mathrm{Im}\varphi^2 \subset \cdots \supset \mathrm{Im}\varphi^n \supset \cdots$$

is a decreasing filtration on E.

A *filtration* on a ring A is an increasing or decreasing sequence of (abelian) additive subgoups

$$A = A_0 \supset A_1 \supset \cdots \supset A_n \supset \cdots$$

or

$$\{0\} = A_0 \subset A_1 \subset \cdots \subset A_n \subset \cdots$$

such that $A_nA_m \subset A_{n+m}$. In this case A_0 is a subring of A and each A_n is a A_0–module.

As an example, if $\mathfrak{A}$ is an ideal of A, a filtration on the ring A is given by the powers of $\mathfrak{A}$:

$$A = \mathfrak{A}^0 \supset \mathfrak{A}^1 \supset \cdots \supset \mathfrak{A}^n \supset \cdots$$

The *first graduated ring* associated with this filtration is

$$\bigoplus_{n\ge 0} \mathfrak{A}^n,$$

and the *second graduated ring* is

$$\bigoplus_{n\ge 0} \mathfrak{A}^n/\mathfrak{A}^{n+1}.$$

For instance if $\mathfrak{A}$ is a *proper* ideal (that means distinct from $\{0\}$ and from A) and is principal, the the first graduated ring is isomorphic to the ring of polynomials $A[t]$ and the second to $(A/\mathfrak{A})[t]$.

We come back to MZV. We have seen that the length k defines a filtration on the algebra $\mathfrak{Z}$ of multiple zeta values. Recall that $\mathcal{F}^k\mathfrak{Z}_p$ denotes the $\mathbf{Q}$-vector subspace of $\mathbf{R}$ spanned by the $\zeta(\underline{s})$ with $\underline{s}$ of weight p and length $\le k$ and that $d_{p,k}$ be the dimension of $\mathcal{F}^k\mathfrak{Z}_p/\mathcal{F}^{k-1}\mathfrak{Z}_p$. The next conjecture is proposed by D. Broadhurst.

Conjecture 22.

$$\left(\sum_{p\ge 0}\sum_{k\ge 0} d_{p,k}X^pY^k\right)^{-1} = (1-X^2Y)\left(1-\frac{X^3Y}{1-X^2}+\frac{X^{12}Y^2(1-Y^2)}{(1-X^4)(1-X^6)}\right).$$

That Conjecture 22 implies Zagier's Conjecture 5 is easily seen by substituting $Y=1$ and using (4).

The left hand side in Conjecture 22 can be written as an infinite product: the next Lemma can be proved in the same way as Lemma 13.

Lemma 23. *Let $D(p,k)$ for $p\ge 0$ and $k\ge 1$ be non–negative integers. Then*

$$\prod_{p\ge 0}\prod_{k\ge 1}(1-X^pY^k)^{-D(p,k)} = \sum_{p\ge 0}\sum_{k\ge 1} d_{p,k}X^pY^k,$$

where $d_{p,k}$ is the number of tuples of non–negative integers of the form $\underline{h} = (h_{ij\ell})_{i\ge 0\ j\ge 1,\ 1\le \ell\le D(p,k)}$ satisfying

$$\sum_{i\ge 0}\sum_{j\ge 1}\sum_{n=1}^{D(p,k)} ih_{ij\ell} = p \quad \textit{and} \quad \sum_{i\ge 0}\sum_{j\ge 1}\sum_{n=1}^{D(p,k)} jh_{ij\ell} = k.$$

If one believes Conjecture 12, a transcendence basis $\mathcal{T}$ of the field generated over $\mathbf{Q}$ by $\mathfrak{Z}$ should exist such that

$$D(p,k) = \mathrm{Card}\big(\mathcal{T}\cap \mathcal{F}^k\mathfrak{Z}_p\big)$$

is the number of Lyndon words on the alphabet $\{2,3\}$ with weight p and length k, so that

$$\prod_{p\ge 3}\prod_{k\ge 1}(1-X^pY^k)^{D(p,k)} = 1-\frac{X^3Y}{1-X^2}+\frac{X^{12}Y^2(1-Y^2)}{(1-X^4)(1-X^6)}.$$