Fifth lecture: April 18, 2011

6 Words, non–commutative polynomials: free monoids and free algebras

6.1 Free structures as solution of universal problems

We shall consider the categories of monoids, groups, abelian groups, vector spaces, commutative algebras, algebras, and for each of them we shall consider the following *universal problem* of existence of an *initial object*: *given a non–empty set* X*, does there exists an object* $A(X)$ *in this category and a map* $\iota : X \to A(X)$ *with the following property: for any object* B *in the category and any map* $f : X \to B$*, there exists a unique morphism* $\underline{f} : A(X) \to B$ *in this category for which the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & B \\ & \searrow^{\iota} & \Big\uparrow \underline{f} \\ & & A(X) \end{array}$$

commutes. In each of these categories the answer is yes, therefore the solution $(A(X), i)$ is unique up to a unique isomorphism, meaning that if (A', ι') is another solution to this problem, then there is a unique isomorphism ι in the category between $A(X)$ and A' for which the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ \iota'\ } & A' \\ & \searrow^{\iota} & \Big\uparrow \tilde{\iota} \\ & & A(X) \end{array}$$

commutes.

Our first example is the category of *monoid*, where the objects are pairs $(M, \cdot)$, where M is a non empty set and $\cdot$ a law

$$\begin{array}{ccc} M \times M & \longrightarrow & M \\ (a,b) & \longmapsto & ab \end{array}$$

which is associative with a neutral element e:

$$(ab)c = a(bc) \quad \text{and} \quad ae = ea = a$$

for all a, b and c in M, while the morphisms between two monoids M and M' are the maps $\varphi : M \to M'$ such that $\varphi(e) = e'$ and $\varphi(ab) = \varphi(a)\varphi(b)$.

When X is a non–empty set, we denote by $X^* = X^{(\mathbf{N})}$ the set of finite sequences of elements in X, including the empty sequence e. Write $x_1 \cdots x_p$ with $p \geq 0$ such a sequence (which is called a *word* on the *alphabet* X - the elements x_i in X are the *letters*). This set $X^\star$ is endowed with a monoid structure, the law on $X^\star$ is the *concatenation*:

$$(x_1 \cdots x_p)(x_{p+1} \cdots x_{p+q}) = x_1 \cdots x_{p+q},$$

which produces *the universal free monoid with basis* X. The neutral element is the *empty word* e. This monoid X^* with the canonical map $X \to X^*$, which maps a letter onto the corresponding word with a single letter, is the solution of the above mentioned universal problem in the category of monoids. The simplest case of a free monoid is when the given set X has a single element x, in which case the solution X^* of the universal problem is the monoid $\{e, x, x^2, \dots, x^n \dots\}$ with the law $x^k x^\ell = x^{k+\ell}$ for k and ℓ non–negative integers. If one writes the law additively with the neutral element 0 and if we replace x by 1, this is a construction of the monoid $\mathbf{N} = \{0, 1, 2, \dots\}$ of the natural integers. We shall study in §6.2 the free monoid $X^\star$ on a set $X := \{x_0, x_1\}$ with two elements as well as the free monoid $Y^\star$ on a countable set $Y := \{y_1, y_2, \dots\}$.

One can define a monoid by generators and relations: if X is the set of generators, we consider the free monoid X^* on X, we consider the equivalence relation, compatible with the concatenation, induced on X^* by the set of relations, and we take the quotient. For instance, one gets the solution of the above–mentioned universal problem in the category of *commutative monoids* by taking the quotient of the free monoid on X by the equivalence relation induced by the relations $xy = yx$ for x and y in X. The *free commutative monoid* on a set X is the set $\mathbf{N}^{(X)}$ of maps $f : x \mapsto n_x$ from X to $\mathbf{N}$ with finite *support*: recall that the support of a map $f : X \to \mathbf{N}$ is $\{x \in X\ ;\ f(n) \neq 0\}$.

Another example is the construction of the *free group* on a set X: we consider the free monoid Y^* on the set Y which is the disjoint union of two copies of X. One can take, for instance, $Y = X_1 \cup X_2$ where $X_1 = X \times \{1\}$ and $X_2 = X \times \{2\}$. Next, we take the equivalence relation on $Y^\star$ induced by $(x, 1)(x, 2) = e$ and $(x, 2)(x, 1) = e$ for all $x \in X$. A set of representative is given by the set $R(X)$ of so–called *reduced words*, which are the words $y_1 \cdots y_s$ in $Y^\star$ having no consecutive letters of the form $(x, 1)$, $(x, 2)$ nor of the form $(x, 2)$, $(x, 1)$ for some $x \in X$. One defines in a natural way a surjective map

$$\begin{array}{rccc} r: & Y^\star & \longrightarrow & R(X) \\ & w & \longmapsto & r(w) \end{array}$$

by induction on the number ℓ of letters of the w's in $Y^\star$ as follows: if $w \in R(X)$, then $r(w) = w$; if $w \notin R(X)$, then there are two consecutive letters in the word w which are either of the form $(x, 1)$, $(x, 2)$ or of the form $(x, 2)(x, 1)$; we consider the word w' having $\ell - 2$ letters deduced from w by omitting such two consecutive letters (there may be several choices) and we use the induction hypothesis which allows us to define $r(w) = r(w')$. This maps is well defined (independent of the choices made) and allows one to define a law on the set of reduced words by $ww' = r(ww')$; this endows the set $R(X)$ with a structure of group. This is the *free group on* X.

Similarly, we get the *free abelian group on a set* X, which is just the group $\mathbf{Z}^{(X)}$. If X is finite with n elements, this is $\mathbf{Z}^n$.

The group defined by a set of generators X and a set of relations can be seen as the monoid constructed as follows: we consider the disjoint union Y of two copies of X, say X_1 and X_2 as above, the set of relations on X induces a set of relations on Y to which one adds the relations $(x,1)(x,2) = e$ and $(x,2)(x,1) = e$ for $x \in X$, and we take the quotient of $Y^\star$ by these relations.

In the category of *vector spaces over a field* K, the solution of the universal problem associated with a set X is the *free vector space on* X, obtained by taking a set of variables e_x indexed by $x \in X$ and by considering the K–vector space E with basis $\{e_x\}_{x\in X}$. If X is finite with n elements, the free vector space over X has dimension n and is isomorphic to K^n. In general, the free vector space is $K^{(X)}$, which it the set of maps $X \to K$ with finite support, with the natural structure of K–vector space and with the natural injection $X \to K^{(X)}$ which maps $x \in X$ onto the characteristic function δ_x of $\{x\}$ (*Kronecker's symbol*):

$$\delta_x(y) = \begin{cases} 1 \text{ if } y = x, \\ 0 \text{ if } y \neq x, \end{cases} \qquad \text{for } y \in X.$$

In the category of *commutative algebras over a field* K, the solution of the universal problem is the ring $K[\{T_x\}_{x\in X}]$ of polynomials in a set of variables indexed by X. One could take the elements of X as variables, and we shall often do so, but if X has already some structure, it may be more convenient to introduce new letters (variables). For instance, when $X = \mathbf{N}$, it is better to introduce countably many variables $T_0, T_1, \dots$ in order to avoid confusion. Let us recall the construction of the *commutative polynomial algebra over a set* X, because the construction of the free algebra over X will be similar, only commutativity will not be there. Given the set of variables T_x with $x \in X$, an element in $K[\{T_x\}_{x\in X}]$ is a finite linear combination of *monomials*. Now, a monomial is a finite product

$$\prod_{x\in X} T_x^{n_x}.$$

Hence, to give a monomial is the same as to give a set $\underline{n} := \{n_x\}_{x\in X}$ of non–negative integers, all of which but a finite number are 0. We have seen that the set of such $\underline{n}$ is the underlying set of the free commutative monoid $\mathbf{N}^{(X)}$ on X. Hence, the K–vector space underlying the algebra of commutative polynomials $K[\{T_x\}_{x\in X}]$ is the free vector space on $\mathbf{N}^{(X)}$: to an element $\underline{n}$ in this set $\mathbf{N}^{(X)}$, we associate the coefficient $c_{\underline{n}}$ of the corresponding monomial, and an element in $K[\{T_x\}_{x\in X}]$ can be written in a unique way

$$\sum_{\underline{n}\in\mathbf{N}^{(X)}} c_{\underline{n}} \prod_{x\in X} T_x^{n_x}.$$

In the category of *algebras*, the solution of the universal problem will be denoted by $K\langle X\rangle$. As a K–vector space, it is the free vector space over the free monoid X^*; hence, the elements of $K\langle X\rangle$ are the linear combinations of words with coefficients in K. The set underlying this space is the set $K^{(X^*)}$ of maps $S : X^* \to K$ having a finite support; for such a map, we denote by $(S|w)$ the image of $w \in X^*$ in K, so that the support is $\{w \in X^* \; ; \; (S|w) \neq 0\}$. We write also

$$S = \sum_{w \in X^*} (S|w)w.$$

On $K^{(X^*)}$, define an addition by

$$(S + T|w) = (S|w) + (T|w) \quad \text{for any } w \in X^* \tag{24}$$

and a multiplication [2] by

$$(ST|w) = \sum_{uv=w} (S|u)(T|v) \tag{25}$$

where, for each $w \in X^*$, the sum is over the (finite) set of (u, v) in $X^* \times X^*$ such that $uv = w$. Further, for $\lambda \in K$ and $S \in K^{(X^*)}$, define $\lambda S \in K^{(X^*)}$ by

$$(\lambda S|w) = \lambda(S|w) \quad \text{for any } w \in X^*. \tag{26}$$

With these laws, one checks that the set $K^{(X^*)}$ becomes a K-algebra, solution of the above universal problem, which is denoted by $K\langle X\rangle$ and is called the *free algebra on* X.

This is a graded algebra, when elements of X are given weight 1: the weight of a word $x_1 \cdots x_p$ is p, and for $p \geq 0$ the set $K\langle X\rangle_p$ of $S \in K\langle X\rangle$ for which

$$(S|w) = 0 \quad \text{if } w \in X^* \text{ has weight} \neq p$$

is the K–vector subspace whose basis is the set of words of length p. For $p = 0$, $K\langle X\rangle_0$ is the set Ke of *constant* polynomials λe, with $\lambda \in K$ – this is the K-subspace of dimension 1 spanned by e. For any $S \in K\langle X\rangle_p$ and $T \in K\langle X\rangle_q$, we have

$$ST \in K\langle X\rangle_{p+q}.$$

If X is finite with n elements, then for each $p \geq 0$ there are n^p words of weight p, hence the dimension of $K\langle X\rangle_p$ over K is n^p, and the Hilbert–Poincaré series of the graded algebra $K\langle X\rangle$ is

$$\sum_{p \geq 0} t^p \dim_K K\langle X\rangle_p = \frac{1}{1 - nt}\cdot$$

For $n = 1$, this algebra $K\langle X\rangle$ is simply the (commutative) ring of polynomials $K[X]$ in one variable.

[2]Sometimes called *Cauchy product* - it is the usual multiplication, in opposition to the *Hadamard product* where $(ST|w) = (S|w)(T|w)$.