Ninth lecture: April 28, 2011

11 The Zagier–Broadhurst formula

Theorem 41. *For any $n \geq 1$,*

$$\zeta\big(\{3,1\}_n\big) = \frac{1}{2^{2n}}\zeta\big(\{4\}_n\big).$$

This formula was originally conjectured by D. Zagier and first proved by D. Broadhurst.

The right hand side is known to be

$$\frac{1}{2n+1}\zeta\big(\{2\}_{2n}\big) = 2 \cdot \frac{\pi^{4n}}{(4n+2)!}.$$

These relations can be written

$$\widehat{\zeta}(y_{3,1}^n) = \frac{1}{2^{2n}}\widehat{\zeta}(y_4^n) = \frac{1}{2n+1}\widehat{\zeta}(y_2^{2n}) = 2 \cdot \frac{\pi^{4n}}{(4n+2)!}.$$

11.1 Rational power series

We shall use power series without formal justification; the necessary bases for that will be given in § 11.4 below.

We introduce a new map, denoted again with a star $\star$ in the exponent[3] from the set of series S in $\widehat{\mathfrak{H}}$ which satisfy $(S|e) = 0$ to $\widehat{\mathfrak{H}}$, defined by

$$S^\star = \sum_{n \geq 0} S^n = e + S + S^2 + \cdots \tag{42}$$

The fact that the right hand side of (42) is well defined is a consequence of the assumption $(S|e) = 0$. Notice that $S^\star$ is the unique solution to the equation

$$(1 - S)S^\star = e,$$

and it is also the unique solution to the equation

$$S^\star(1 - S) = e.$$

A *rational* series is a series in $\widehat{\mathfrak{H}}$ which is obtained by starting with a finite number of letters (this is a restriction only in case in case X is infinite) and using only a finite number of *rational operations*, namely addition (24), product (25), multiplication (26) by an element in K and the star (42). The set of rational series over K is a field $\mathrm{Rat}_K(X)$.

[3] There should be no confusion with the notation $X^\star$ for the set of words, nor with the star in the exponent for the dual, nor with the harmonic product.

For instance, for $x \in X$, the series

$$e + x^2 + x^4 + \cdots + x^{2n} + \cdots = x^\star(-x)^\star$$

is rational, and also, for $m \geq 1$, the series

$$\sum_{p\geq 0} \varphi_m(p)x^p = (mx)^\star,$$

when $\varphi_m(p) = m^p$. Notice that $\varphi_m(p)$ is also the number of words of weight p on the alphabet with m letters. Series like

$$\sum_{p\geq 0} x^p/p, \quad \sum_{p\geq 0} x^p/p!, \quad \sum_{p\geq 0} x^{2^p}$$

are not rational: if X has a single element, say x, one can prove that *rational series can be identified with elements in $K(x)$ with no poles at $x = 0$.*

For a series S without constant term, i.e. such that $(S|e) = 0$, one defines

$$\exp(S) = \sum_{n=0}^{\infty} \frac{S^n}{n!}.$$

It is easy to check that if T satisfies $(T|e) = 0$, then the series

$$S = \sum_{n=1}^{\infty} \frac{T^n}{n}$$

is well defined and has

$$\exp(S) = T^\star.$$

11.2 Syntaxic identities, following Hoang Ngoc Minh and Petitot

Here we assume that the next *syntaxic identity* (due to Minh and Petitot) hold. We shall prove it in §11.3. The star has been defined in (42) and we use rational series which belong to the algebra of power series considered in §11.4.2.

Lemma 43. *The following identity holds:*

$$(x_0x_1)^\star ш (-x_0x_1)^\star = (-4x_0^2x_1^2)^\star.$$

Taking this identity for granted, we complete the proof of Theorem 41.

Proof of Theorem 41. We shall introduce generating series; we work in the ring $\mathbf{R}[[t]]$ of formal power series, where t is a variable. The integration will be according to Chen integrals with respect to ω_0 and ω_1 (not with respect to t!).

In the left hand side of the formula of Lemma 43, we replace x_0 by $t\omega_0$, x_1 by $t\omega_1$ and integrate between 0 and 1. We have, for $k \geq 0$,

$$\int_0^1 (\omega_0\omega_1)^k = \widehat{\zeta}(y_2^k) = \zeta\big(\{2\}_k\big), \quad \int_0^1 (\omega_0^2\omega_1^2)^k = \widehat{\zeta}((y_3y_1)^k) = \zeta\big(\{3,1\}_k\big)$$

and

$$\int_0^1 (\omega_0^3\omega_1)^k = \widehat{\zeta}(y_4^k) = \zeta\big(\{4\}_k\big).$$

From Proposition 7

$$\prod_{n\geq 1}\left(1+\frac{t}{n^s}\right) = \sum_{k\geq 0}\zeta\big(\{s\}_k\big)t^k = \sum_{k\geq 0}\widehat{\zeta}\big((y_s)^k\big)t^k$$

with $s=2$ and $s=4$ and replacing t by t^2, $-t^2$ and $-t^4$ respectively, we deduce

$$\sum_{k=0}^{\infty} t^{2k}\zeta\big(\{2\}_k\big) = \prod_{n\geq 1}\left(1+\frac{t^2}{n^2}\right), \quad \sum_{k=0}^{\infty}(-t^2)^k\zeta\big(\{2\}_k\big) = \prod_{n\geq 1}\left(1-\frac{t^2}{n^2}\right)$$

and

$$\sum_{k=0}^{\infty}(-t^4)^k\zeta\big(\{4\}_k\big) = \prod_{n\geq 1}\left(1-\frac{t^4}{n^4}\right).$$

On the other hand if we replace x_0 by $t\omega_0$ and x_1 by $t\omega_1$ in the left hand side of the formula in Lemma 43, we get

$$\left(\sum_{k=0}^{\infty} t^{2k}(\omega_0\omega_1)^k\right) ш \left(\sum_{k=0}^{\infty}(-t^2)^k(\omega_0\omega_1)^k\right) = \sum_{k=0}^{\infty}(-4t^4)^k(\omega_0^3\omega_1)^k.$$

Thanks to the compatibility of the shuffle product with Chen integrals (Lemma 29), one deduces

$$\left(\int_0^1\sum_{k=0}^{\infty} t^{2k}(\omega_0\omega_1)^k\right)\left(\int_0^1\sum_{k=0}^{\infty}(-t^2)^k(\omega_0\omega_1)^k\right) = \int_0^1\sum_{k=0}^{\infty}(-4t^4)^k(\omega_0^2\omega_1^2)^k.$$

Hence Theorem 41 follows from

$$\prod_{n\geq 1}\left(1+\frac{t^2}{n^2}\right)\prod_{n\geq 1}\left(1-\frac{t^2}{n^2}\right) = \prod_{n\geq 1}\left(1-\frac{t^4}{n^4}\right).$$

□

In §11.4.3, we shall use the same arguments to prove another syntaxic identity, where the star in the exponent is the rational map introduced in §11.1, while the operator $\star$ is the harmonic product.

Lemma 44. *The following identity holds:*

$$y_2^\star \star (-y_2)^\star = (-y_4)^\star.$$

11.3 Shuffle product and automata

There is a description of the shuffle product in terms of automata due to Schutzenberger. Here is a sketch of proof of Lemma 43.

Sketch of proof of Lemma 43. To a series $S^\star$ one associates an automaton, with the following property: the sum of paths going out from the entry gate is S. As an example the series associated to

$$\begin{tikzcd} \begin{array}{c}\Longleftarrow\\ \Longrightarrow\end{array} & \boxed{1} \arrow[r, shift right, "x_0"'] & \boxed{2} \arrow[l, shift right, "x_1"'] \end{tikzcd} \tag{45}$$

is

$$S_1 = e + x_0x_1 + (x_0x_1)^2 + \cdots + (x_0x_1)^n + \cdots = (x_0x_1)^\star$$

and similarly the series associated to

$$\begin{tikzcd} \begin{array}{c}\Longleftarrow\\ \Longrightarrow\end{array} & \boxed{A} \arrow[r, shift right, "-x_0"'] & \boxed{B} \arrow[l, shift right, "x_1"'] \end{tikzcd} \tag{46}$$

is

$$S_A = e - x_0x_1 + (x_0x_1)^2 + \cdots + (-x_0x_1)^n + \cdots = (-x_0x_1)^\star.$$

The cartesian product of these two automata is the following:

$$\begin{tikzcd} \begin{array}{c}\Longleftarrow\\ \Longrightarrow\end{array} & \boxed{1A} \arrow[rr, shift right, "x_0"'] \arrow[d, shift right, "-x_0"'] & & \boxed{2A} \arrow[ll, shift right, "x_1"'] \arrow[d, shift right, "-x_0"'] \\ & \boxed{1B} \arrow[rr, shift right, "x_0"'] \arrow[u, shift right, "x_1"'] & & \boxed{2B} \arrow[ll, shift right, "x_1"'] \arrow[u, shift right, "x_1"'] \end{tikzcd} \tag{47}$$

Let S_{1A} be the series associated with this automaton (47). One computes it by solving a system of linear (noncommutative) equations as follows. Define also S_{1B}, S_{2A} and S_{2B} as the series associated with the paths going out from the corresponding vertex. Then

$$\begin{aligned} S_{1A} &= e - x_0S_{1B} + x_0S_{2A},\\ S_{1B} &= x_1S_{1A} + x_0S_{2B},\\ S_{2A} &= x_1S_{1A} - x_0S_{2B},\\ S_{2B} &= x_1S_{1B} + x_1S_{2A}. \end{aligned}$$

The rule is as follows: if Σ is the sum associated with a vertex (also denoted by Σ) with oriented edges $\xi_i : \Sigma \to \Sigma_i \quad (1 \le i \le m)$, then

$$\Sigma = x_1\Sigma_1 + \cdots + x_m\Sigma_m,$$

and $x_i\Sigma_i$ is replaced by e for the entry gate.

In the present situation one deduces

$$S_{1A} = e - x_0(S_{1B} - S_{2A}), \quad S_{1B} - S_{2A} = -2x_0 S_{2B},$$

$$S_{2B} = x_1(S_{1B} + S_{2A}), \quad S_{1B} + S_{2A} = 2x_1 S_{1A}$$

and therefore

$$S_{1A} = e + 4x_0^2 x_1^2 S_{1A},$$

which completes the proof of Lemma 43, since the series associated with the automaton (47) is the shuffle product of the series associated with the automata (45) and (46).

A proof that the cartesian product of two automata recognizes the shuffle of the two languages which are recognized by each factor can be found in [9], pp. 19–20.

□

11.4 Formal power series, rational series, symmetric and quasi-symmetric series

11.4.1 Dual

Let K be a field and X a set. Denote by E_X the free vector space on X (see §6.1). By definition of E_X as a solution of a universal problem, there is a one to one map j between the set of linear maps $E_X \to K$ (which is the *dual* of the vector space E_X which we denote by E^*) and the set K^X of maps $X \to K$. Notice that there is no restriction on the support of these maps (such a condition on the support makes a difference only when X is infinite: for a finite set X, we have $K^X = K^{(X)}$). There is a natural structure of K–vector space on the dual E^* and there is also a natural structure of K–vector space on K^X, and the bijective map j is an isomorphism of K–vector spaces.

Consider next the free commutative algebra $K[X]$ on X (see §6.1). By definition of $K[X]$ as a solution of a universal problem, there is a one to one map ι between the set of morphisms of algebras $K[X] \to K$ (which is also called the *dual* of the algebra $K[X]$) and the set K^X of maps $X \to K$. There is a natural structure of K–algebra on the dual and there is also a natural structure of K–algebra on K^X, and the bijective map ι is an isomorphism of K–algebras.

Consider a non–empty set X. According to the definition of $K\langle X\rangle$ as a solution of a universal problem, for each K-algebra A the map $f \to \underline{f}$ defines a bijection between A^X and the set of morphisms of K-algebras $K\langle X\rangle \to A$. This is one dual of $\mathfrak{H}$, considered as an algebra, but this is not the one we are going to consider: we are interested with the dual $\widehat{\mathfrak{H}}$ of $\mathfrak{H}$ as a K–vector space.

11.4.2 The Algebra $\widehat{\mathfrak{H}} = K\langle\langle X\rangle\rangle$ of formal power series

Let X be a non–empty set. We are interested in the dual of the free K–algebra $K\langle X\rangle$ as a K–vector space, namely the K–vector space $\mathrm{Hom}_K\big(K\langle X\rangle, K\big)$ of K-linear maps $K\langle X\rangle \to K$. We shall see that there is a natural structure of algebra on this dual which is the algebra $\widehat{\mathfrak{H}} = K\langle\langle X\rangle\rangle$ *of formal power series on* X.

The underlying set of the algebra $K\langle\langle X\rangle\rangle$ is the set K^{X^*} of maps $X^* \to K$. Here, there is no restriction on the support. For such a map S write $(S|w)$ the image of $w \in X^*$ in K and write also

$$S = \sum_{w\in X^*} (S|w)w.$$

On this set K^{X^*} the addition is defined by (24) and the multiplication is again Cauchy product (25). Further, for $\lambda \in K$ and $S \in K^{(X^*)}$, define $\lambda S \in K^{X^*}$ by (26). With these laws one checks that the set K^{X^*} becomes a K-algebra which we denote by either $K\langle\langle X\rangle\rangle$ of $\widehat{\mathfrak{H}}$.

To a formal power series S we associate a K-linear map:

$$\begin{array}{ccc} K\langle X\rangle & \longrightarrow & K \\ P & \longmapsto & \displaystyle\sum_{w\in X^*} (S|w)(P|w). \end{array}$$

Notice that the sum is finite since $P \in K\langle X\rangle$ has finite support.

Since X^* is a basis of the K-vector space $K\langle X\rangle$, a linear map $f \in \mathrm{Hom}_K\bigl(K\langle X\rangle, K\bigr)$ is uniquely determines by its values $(f|w)$ on the set X^*. Hence, the map

$$\begin{array}{ccc} \mathrm{Hom}_K\bigl(K\langle X\rangle, K\bigr) & \longrightarrow & \widehat{\mathfrak{H}} \\ f & \longmapsto & \sum_{w\in X^*}(f|w)w \end{array}$$

is an isomorphism of vector spaces between the dual [4] $\mathrm{Hom}_K\bigl(K\langle X\rangle, K\bigr)$ of $\mathfrak{H} = K\langle X\rangle$ and $\widehat{\mathfrak{H}}$.

11.4.3 Symmetric Series, Quasi-Symmetric Series and Harmonic product

Denote by $\underline{t} = (t_1, t_2, \dots)$ a sequence of commutative variables. To $\underline{s} = (s_1, \dots, s_k)$, where each s_j is an integer ≥ 1, associate the series

$$\mathrm{S}_{\underline{s}}(\underline{t}) = \sum_{\substack{n_1\geq 1,\dots,n_k\geq 1\\ n_1,\dots,n_k \text{pairwise distinct}}} t_{n_1}^{s_1}\cdots t_{n_k}^{s_k}.$$

The space of power series spanned by these $\mathrm{S}_{\underline{s}}$ is denoted by Sym and its elements are called *symmetric series*. A basis of Sym is given by the series $\mathrm{S}_{\underline{s}}$ with $s_1 \geq s_2 \geq \cdots \geq s_k$ and $k \geq 0$.

[4]This is the classical dual; there are other notions of dual, in particular the “graduate dual”, which in the present case is isomorphic to $\mathfrak{H}$, and the “restricted dual”, which is the field $\mathrm{Rat}_K(X)$ of series which are “rational” (see § 1.3).

A *quasi-symmetric series* is an element of the algebra QSym spanned by the series

$$\mathrm{QS}_{\underline{s}}(\underline{t}) = \sum_{n_1 > \cdots > n_k \geq 1} t_{n_1}^{s_1} \cdots t_{n_k}^{s_k},$$

where $\underline{s}$ ranges over the set of tuples $(s_1, \ldots, s_k)$ with $k \geq 0$ and $s_j \geq 1$ for $1 \leq j \leq k$. Notice that, for $\underline{s} = (s_1, \ldots, s_k)$ of length k,

$$\mathrm{S}_{\underline{s}} = \sum_{\tau \in \mathfrak{S}_k} \mathrm{QS}_{\underline{s}^\tau},$$

where $\mathfrak{S}_k$ is the symmetric group on k elements and $\underline{s}^\tau = (s_{\tau(1)}, \ldots, s_{\tau(k)})$. Hence any symmetric series is also quasi-symmetric. Therefore Sym is a subalgebra of QSym.

Notice that these algebras are commutative.

Proposition 48. *The K-linear map $\phi : \mathfrak{H}^1 \to \mathrm{QSym}$ defined by $y_{\underline{s}} \mapsto \mathrm{QS}_{\underline{s}}$ is an isomorphism of K-algebras from the harmonic algebra $\mathfrak{H}^1_\star$ to* QSym.

In other terms, we can write (31) as follows:

$$\mathrm{QS}_{\underline{s}}(\underline{t})\, \mathrm{QS}_{\underline{s}'}(\underline{t}) = \sum_{\underline{s}''} \mathrm{QS}_{\underline{s}''}(\underline{t}),$$

which means

$$\sum_{n_1 > \cdots > n_k \geq 1} t_{n_1}^{s_1} \cdots t_{n_k}^{s_k} \sum_{n'_1 > \cdots > n'_k \geq 1} t_{n'_1}^{s'_1} \cdots t_{n'_k}^{s'_k} = \sum_{\underline{s}''} \sum_{n''_1 > \cdots > n''_k \geq 1} t_{n''_1}^{s''_1} \cdots t_{n''_k}^{s''_k}$$

where $\underline{s}''$ is the same as for the definition of the harmonic product. The star (stuffle) law gives an explicit way of writing the product of two quasi-symmetric series as a sum of quasi-symmetric series.

Let QSym^0 be the subspace of QSym spanned by the $\mathrm{QS}_{\underline{s}}(\underline{t})$ for which $s_1 \geq 2$. The restriction of ϕ to $\mathfrak{H}^0$ gives an isomorphism of K-algebra from $\mathfrak{H}^0$ to QSym^0. The specialization $t_n \to 1/n$ for $n \geq 1$ restricted QSym^0 maps $\mathrm{QS}_{\underline{s}}$ onto $\zeta(\underline{s})$. Hence we have a commutative diagram:

$$\begin{array}{ccc}
\mathfrak{H} & & \\
\bigcup & & \\
\mathfrak{H}^1 & \xrightarrow[\sim]{\phi} & \mathrm{QSym} \\
\bigcup & & \bigcup \\
\mathfrak{H}^0 & \xrightarrow{\sim} & \mathrm{QSym}^0 \\
{\scriptstyle \hat{\zeta}}\Big\downarrow & \swarrow & \\
\mathbf{R} & &
\end{array}
\qquad\qquad
\begin{array}{ccc}
y_{\underline{s}} & \longmapsto & \mathrm{QS}_{\underline{s}}(\underline{t}) \\
\downarrow & \swarrow & \\
\zeta(\underline{s}) & &
\end{array}$$

Proof of Lemma 44. From the definition of ϕ in Proposition 48, we have

$$\phi(y_2^\star) = \sum_{k=0}^{\infty} \sum_{n_1>\cdots>n_k\geq 1} t_{n_1}^2 \cdots t_{n_k}^2,$$

$$\phi\big((-y_2)^\star\big) = \sum_{k=0}^{\infty} (-1)^k \sum_{n_1>\cdots>n_k\geq 1} t_{n_1}^2 \cdots t_{n_k}^2$$

and

$$\phi\big((-y_4)^\star\big) = (-1)^k \sum_{n_1>\cdots>n_k\geq 1} t_{n_1}^4 \cdots t_{n_k}^4.$$

Hence from the identity

$$\prod_{n=1}^{\infty}(1+t_n t) = \sum_{k=0}^{\infty} t^k \sum_{n_1>\cdots>n_k\geq 1} t_{n_1}\cdots t_{n_k}$$

one deduces

$$\phi(y_2^\star) = \prod_{n=1}^{\infty}(1+t_n^2), \quad \phi\big((-y_2)^\star\big) = \prod_{n=1}^{\infty}(1-t_n^2) \quad \text{and} \quad \phi\big((-y_4)^\star\big) = \prod_{n=1}^{\infty}(1-t_n^4),$$

which implies Lemma 44.

□

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