On the Ramanujan street puzzle of Prasanta Chandra Mahalanobis

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An interesting street number

The puzzle itself was about a street in the town of Louvain in Belgium, where houses are numbered consecutively. One of the house numbers had the peculiar property that the total of the numbers lower than it was exactly equal to the total of the numbers above it. Furthermore, the mysterious house number was greater than 50 but less than 500.

Prasanta Chandra Mahalanobis
1893 – 1972

Srinivasa Ramanujan
1887 – 1920

http://mathshistory.st-andrews.ac.uk/Biographies/Mahalanobis.html
https://www.math.auckland.ac.nz/~butcher/miniature/miniature2.pdf

Street number : examples

Examples :

- House number 6 in a street with 8 houses :

$$1+2+3+4+5=15, \quad 7+8=15.$$

- House number 35 in a street with 49 houses. To compute

$$S := 1+2+3+\cdots+32+33+34$$

write

$$S = 34+33+32+\cdots+3+2+1$$

so that $2S = 34 \times 35$:

$$1+2+3+\cdots+34 = \frac{34 \times 35}{2} = 595.$$

On the other side of the house,

$$36+37+\cdots+49 = \frac{49 \times 50}{2} - \frac{35 \times 36}{2} = 1225 - 630 = 595.$$

Other solutions to the puzzle

- House number 1 in a street with 1 house.
- House number 0 in a street with 0 house.

Ramanujan : *if no banana is distributed to no student, will each student get a banana ?*

The puzzle requests the house number between 50 and 500.

Street number

Let m be the house number and n the number of houses :

$$1 + 2 + 3 + \cdots + (m-1) = (m+1) + (m+2) + \cdots + n.$$

$$\frac{m(m-1)}{2} = \frac{n(n+1)}{2} - \frac{m(m+1)}{2}.$$

This is $2m^2 = n(n+1)$. Complete the square on the right :

$$8m^2 = (2n+1)^2 - 1.$$

Set $x = 2n+1$, $y = 2m$. Then

$$x^2 - 2y^2 = 1.$$

Mahalanobis puzzle $x^2 - 2y^2 = 1$, $x = 2n+1,\ y = 2m$

Fundamental solution : $(x_1, y_1) = (3, 2)$.
Other solutions (x_ν, y_ν) with

$$x_\nu + y_\nu\sqrt{2} = (3 + 2\sqrt{2})^\nu.$$

- $\nu = 0$, trivial solution : $x = 1$, $y = 0$, $m = n = 0$.
- $\nu = 1$, $x_1 = 3$, $y_1 = 2$, $m = n = 1$.
- $\nu = 2$, $x_2 = 17$, $y_2 = 12$, $n = 8$, $m = 6$,

$$x_2 + y_2\sqrt{2} = (3 + 2\sqrt{2})^2 = 17 + 12\sqrt{2}.$$

- $\nu = 3$, $x_3 = 99$, $y_3 = 70$, $n = 49$, $m = 35$,

$$x_3 + y_3\sqrt{2} = (3 + 2\sqrt{2})^3 = 99 + 70\sqrt{2}.$$

Remark : $\frac{99}{70} = \frac{297}{210}$: $A4$ format.

Brahmagupta (628)

Brahmasphutasiddhanta :
Solve in integers the equation

$$x^2 - 92y^2 = 1$$

Brahmagupta

Answer : $(x, y) = (1151, 120)$

The continued fraction expansion of $\sqrt{92}$ is

$$\sqrt{92} = [9, \overline{1,\ 1,\ 2,\ 4,\ 2,\ 1,\ 1,\ 18}].$$

Compute

$$[9,\ 1,\ 1,\ 2,\ 4,\ 2,\ 1,\ 1] = \frac{1151}{120}.$$

Indeed $1151^2 - 92 \cdot 120^2 = 1\,324\,801 - 1\,324\,800 = 1.$

Bhaskara II (12th Century)

Lilavati
(*Bijaganita*, 1150)
$x^2 - 61y^2 = 1$
A solution is :
$x = 1\,766\,319\,049$,
$y = 226\,153\,980$.

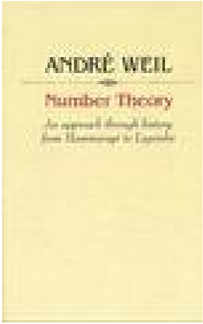
Cyclic method (Chakravala) of Brahmagupta.

$$\sqrt{61} = [7, \overline{1,\ 4,\ 3,\ 1,\ 2,\ 2,\ 1,\ 3,\ 4,\ 1,\ 14}]$$

$$[7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4, 3, 1, 2, 2, 1, 3, 5] = \frac{1\,766\,319\,049}{226\,153\,980}$$

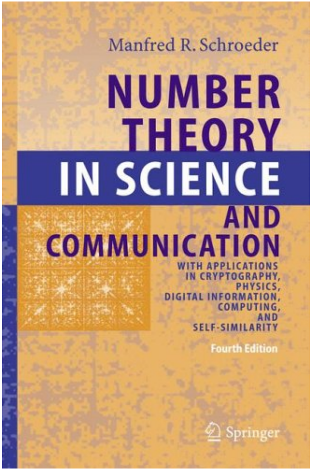
Reference to Indian mathematics

André Weil
Number theory :
An approach through history. From Hammurapi to Legendre.
Birkhäuser Boston, Inc., Boston, Mass., (1984) 375 pp.
MR 85c:01004



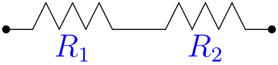
Number Theory in Science and communication

M.R. Schroeder.
Number theory in science and communication :
with applications in cryptography, physics, digital information, computing and self similarity
Springer series in information sciences **7** 1986.
4th ed. (2006) 367 p.



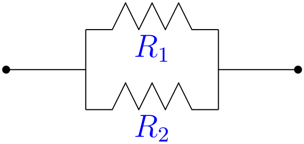
Electric networks

- The resistance of a network in series



is the sum $R_1 + R_2$.

- The resistance R of a network in parallel

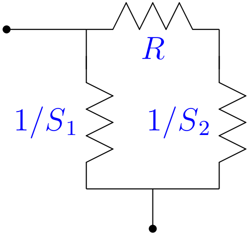


satisfies

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

Electric networks and continued fractions

The resistance U of the circuit

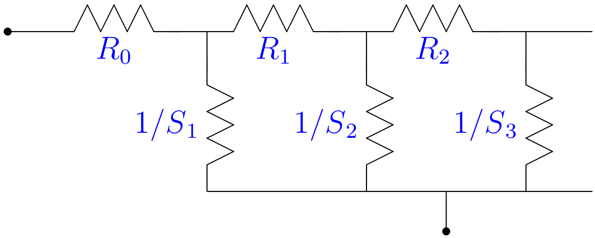


is given by

$$U = \cfrac{1}{S_1 + \cfrac{1}{R + \cfrac{1}{S_2}}} := [0, S_1, R, S_2].$$

A circuit for a continued fraction expansion

- For the network



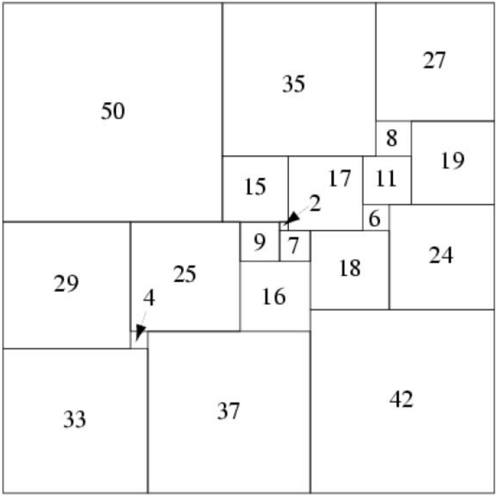
the resistance is given by a continued fraction expansion

$$R_0 + \cfrac{1}{S_1 + \cfrac{1}{R_1 + \cfrac{1}{S_2 + \cfrac{1}{\ddots}}}} := [R_0, S_1, R_1, S_2, R_2, \dots]$$

Decomposition of a square in squares

Electric networks and continued fractions have been used to find the first solution to the problem of decomposing an integer square into a disjoint union of integer squares, all of which are distinct.

Squaring the square



21-*square perfect square*

There is a unique simple perfect square of order 21 (the lowest possible order), discovered in 1978 by A. J. W. Duijvestijn (Bouwkamp and Duijvestijn 1992). It is composed of 21 squares with total side length 112, and is illustrated above.

Continued fraction of $\sqrt{2}$, partial quotients $\frac{x_\nu}{y_\nu}$

Trivial solution of $x^2 - 2y^2 = 1$: $x_0 = 1$, $y_0 = 0$.
First non trivial solution : $x_1 = 3$, $y_1 = 2$. We have

$$\frac{x_1}{y_1} = \frac{3}{2} = 1 + \frac{1}{2} = [1, 2].$$

Second solution : $x_2 = 17$, $y_2 = 12$

$$\frac{x_2}{y_2} = \frac{17}{12} = 1 + \frac{5}{12}, \quad \frac{12}{5} = 2 + \frac{2}{5}, \quad \frac{5}{2} = 2 + \frac{1}{2},$$

hence

$$\frac{17}{12} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}} = [1, 2, 2, 2].$$

Third solution of $x^2 - 2y^2 = 1$

$(3+2\sqrt{2})^3 = x_3 + y_3\sqrt{2}$
$x_3 = 99,\ y_3 = 70$.

$$\frac{x_3}{y_3} = \frac{99}{70} = 1 + \frac{29}{70}, \quad \frac{70}{29} = 2 + \frac{12}{29}, \quad \frac{29}{12} = 2 + \frac{5}{12}$$

with

$$\frac{12}{5} = 2 + \frac{2}{5}, \quad \frac{5}{2} = 2 + \frac{1}{2}$$

hence

$$\frac{99}{70} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \frac{1}{2}}}}} = [1, 2, 2, 2, 2, 2].$$

Fourth solution of $x^2 - 2y^2 = 1$

$(3+2\sqrt{2})^4 = x_4 + y_4\sqrt{2}$

$$[1,2,2,2,2,2,2,2] = 1 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\frac{1}{2}}}}}}} = \frac{577}{408}.$$

$577^2 - 2 \times 408^2 = 1, \qquad 577 = 2 \times 288 + 1, \quad 408 = 2 \times 204.$

Hence the solution to the puzzle is : *the house number is* 204 *in a street with* 288 *houses :*

$$1+2+3+4+5+\cdots+203 = \frac{203 \times 204}{2} = 20706,$$

$$205+206+\cdots+288 = \frac{288 \times 289}{2} - \frac{204 \times 205}{2} = 20706.$$

Pell's equation $x^2 - 2y^2 = 1$ and Euclid

Euclid of Alexandria about 325 BC - about 265 BC , Elements, II $\S$ 10

$$17^2 - 2 \cdot 12^2 = 289 - 2 \cdot 144 = 1.$$

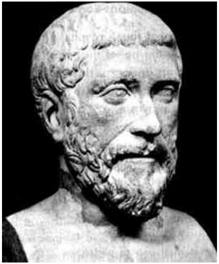
$$99^2 - 2 \cdot 70^2 = 9\,801 - 2 \cdot 4900 = 1.$$

$$577^2 - 2 \cdot 408^2 = 332\,929 - 2 \cdot 166\,464 = 1.$$

Pythagorean triples

Pythagoras of Samos

about 569 BC - about 475 BC



Which are the right angle triangles with integer sides such that the two sides of the right angle are consecutive integers?

$$x^2 + y^2 = z^2, \qquad y = x + 1.$$

$$2x^2 + 2x + 1 = z^2$$

$$(2x+1)^2 - 2z^2 = -1$$

$$X^2 - 2Y^2 = -1$$

$$(X, Y) = \quad (1,1), \quad (7,5), \quad (41,29) \ldots$$

$x^2 - 2y^2 = \pm 1$

$$\sqrt{2} = 1,4142135623730950488016887242\ \dots$$

satisfies

$$\sqrt{2} = 1 + \frac{1}{\sqrt{2}+1}\cdot$$

Hence the continued fraction expansion is periodic with period length 1 :

$$\sqrt{2} = [1,\ 2,\ 2,\ 2,\ 2,\ 2,\ \dots] = [1,\ \overline{2}],$$

The fundamental solution of $x^2 - 2y^2 = -1$ is $x_1 = 1$, $y_1 = 1$

$$1^2 - 2 \cdot 1^2 = -1,$$

the continued fraction expansion of x_1/y_1 is $[1]$.

Pell's equation $x^2 - 2y^2 = -1$

The fundamental solution of

$$x^2 - 2y^2 = -1$$

is $x = y = 1$. The norm of $1 + \sqrt{2}$ is

$$(1 + \sqrt{2})(1 - \sqrt{2}) = -1$$

and the norm of $(1 + \sqrt{2})^\nu$ is $(-1)^\nu$, hence is -1 for ν odd.

The sequence (x_j, y_j) of solutions of $x^2 - 2y^2 = -1$ is given by the continued fractions

$$\frac{x_1}{y_1} = 1, \quad \frac{x_2}{y_2} = [1, 2, 2], \quad \frac{x_3}{y_3} = [1, 2, 2, 2, 2], \ldots$$

with an even number of 2's.

$x^2 - 2y^2 = \pm 1$

$$(1+\sqrt{2})^n = x_n + y_n\sqrt{2}$$

$$(1-\sqrt{2})^n = x_n - y_n\sqrt{2}$$

$$x_n = \frac{1}{2}(1+\sqrt{2})^n + \frac{1}{2}(1-\sqrt{2})^n$$

$$y_n = \frac{1}{2\sqrt{2}}(1+\sqrt{2})^n - \frac{1}{2\sqrt{2}}(1-\sqrt{2})^n.$$

$$(x_0, y_0) = (1, 0), \quad (x_1, y_1) = (1, 1).$$

Linear recurrence sequence :

$$u_{n+2} = 2u_{n+1} + u_n$$

The space of solutions of the recurrence

The set of sequences $(u_n)_{n\geq 0}$ of complex numbers satisfying the recurrence $u_{n+2} = 2u_{n+1} + u_n$ is a vector space of dimension 2, a basis is given by the two sequences with initial conditions $(0,1)$ and $(1,0)$.

Another basis is given by the two sequences $(\gamma_1^n)_{n\geq 0}$ and $(\gamma_2^n)_{n\geq 0}$ where

$$X^2 - 2X - 1 = (X - \gamma_1)(X - \gamma_2),$$

hence $\gamma_1 = 1 + \sqrt{2}$, $\gamma_2 = 1 - \sqrt{2}$.

The matrix associated with the recurrence

The recurrence relation

$$u_{n+2} = 2u_{n+1} + u_n$$

can be written in a matrix form $\mathbf{u}_{n+1} = A\mathbf{u}_n$ with

$$\mathbf{u}_n = \begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

Hence

$$\mathbf{u}_n = A^n \mathbf{u}_0.$$

The characteristic polynomial of the matrix A is

$$\det \begin{pmatrix} -X & 1 \\ 1 & 2-X \end{pmatrix} = X^2 - 2X - 1.$$

Computing A^n : linear algebra

Diagonalize the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$: $AP = PD$ with

$$P = \begin{pmatrix} 1 & 1 \\ \gamma_1 & \gamma_2 \end{pmatrix}, \quad D = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix},$$

$\gamma_1 = 1 + \sqrt{2}$, $\gamma_2 = 1 - \sqrt{2}$,

$$P^{-1} = \frac{1}{\gamma_2 - \gamma_1} \begin{pmatrix} \gamma_2 & -1 \\ -\gamma_1 & 1 \end{pmatrix}, \qquad \gamma_2 - \gamma_1 = \frac{-\sqrt{2}}{4},$$

so that

$$A^n = PD^nP^{-1} = \frac{\sqrt{2}}{4} \begin{pmatrix} \gamma_1^{n-1} - \gamma_2^{n-1} & \gamma_1^n - \gamma_2^n \\ \gamma_1^n - \gamma_2^n & \gamma_1^{n+1} - \gamma_2^{n+1} \end{pmatrix}.$$

Theorem of Cayley – Hamilton

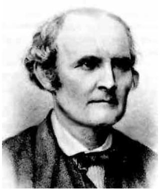
Let A be a square $d \times d$ matrix with characteristic polynomial

$$X^d - a_1 X^{d-1} - \cdots - a_{d-1} X - a_d.$$

Then

$$A^d = a_1 A^{d-1} + \cdots + a_{d-1} A + a_d I_d$$

where I_d is the identity $d \times d$ matrix.



Arthur Cayley
1821 – 1895

Sir William Rowan Hamilton
1805 – 1865

Computing A^n : linear recurrence

For $n \geq 0$, multiply the Cayley – Hamilton relation

$$A^d = a_1 A^{d-1} + \cdots + a_{d-1} A + a_d I_d$$

by A^n :

$$A^{n+d} = a_1 A^{n+d-1} + \cdots + a_{d-1} A^{n+1} + a_d A^n.$$

It follows that each entry $a_{ij}(n)$, $1 \leq i, j \leq d$ of A^n satisfies the linear recurrence relation

$$u_{n+d} = a_1 u_{n+d-1} + \cdots + a_{d-1} u_{n+1} + a_d u_n.$$

Computing A^n for $A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$

The characteristic polynomial is $X^2 - 2X - 1$, the linear recurrence is $u_{n+2} = 2u_{n+1} + u_n$, the initial conditions are

$$A^0 = I_d = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A^1 = A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix},$$

which yields

$$A^n = \begin{pmatrix} a_{n-1} & a_n \\ a_n & a_{n+1} \end{pmatrix},$$

with $a_0 = 0$, $a_1 = 1$. Hence

$$a_n = \frac{\sqrt{2}}{4}(\gamma_1^n - \gamma_2^n).$$

The generating series of the recurrence

Assume

$$u_{n+2} = 2u_{n+1} + u_n$$

Consider the formal power series

$$f(t) = \sum_{n \geq 0} u_n t^n.$$

Then (telescoping series)

$$(1 + 2t - t^2) f(t) = u_0 + u_1 t.$$

Therefore $f(t)$ is a rational fraction, the denominator is the reciprocal of the characteristic polynomial of the recurrence.

The exponential generating series of the recurrence

Assume

$$u_{n+2} = 2u_{n+1} + u_n$$

The formal power series

$$y(x) = \sum_{n \geq 0} u_n \frac{x^n}{n!}$$

satisfies the differential equation

$$y'' - 2y' - y = 0.$$

A basis of the space of solutions of the differential equation $y'' - 2y' - y = 0$ is given by $e^{\gamma_1 x}$ and $e^{\gamma_2 x}$ where γ_1 and γ_2 are the roots of the characteristic polynomial $X^2 - 2X - 1$.

The binary recurrence sequence $u_{n+2} = 2u_{n+1} + u_n$

Exercise (Pierre Arnoux) :
$u_{n+2} = 2u_{n+1} + u_n$,
$u_0 = 1$, $u_1 = 1 - \sqrt{2}$.
Use a calculator to estimate u_{100}.

Pierre Arnoux

http://www.imj-prg.fr/~michel.waldschmidt/articles/pdf/Ariane5VI.pdf
PARI GP : https://pari.math.u-bordeaux.fr/

$$u_{n+2} = 2u_{n+1} + u_n,\ n = 0, 1, \ldots, 22$$

0	1
1	-0,4142135624
2	0,1715728753
3	-0,0710678119
4	0,0294372515
5	-0,0121933088
6	0,0050506339
7	-0,0020920411
8	0,0008665518
9	-0,0003589375
10	0,0001486768
11	-0,0000615839
12	0,0000255089
13	-0,0000105661
14	0,0000043766
15	-0,0000018129
16	0,0000007509
17	-0,0000003111
18	0,0000001286
19	-0,0000000540
20	0,0000000206
21	-0,0000000129
22	-0,0000000052

$$u_{n+2} = 2u_{n+1} + u_n,\ n = 22, \ldots, 100$$

20	0,0000000206	47	-33,4362152981	74	-723037158346,313000
21	-0,0000000129	48	-80,7221644471	75	-1745566113779,370000
22	-0,0000000052	49	-194,8805441923	76	-4214169385905,060000
23	-0,0000000233	50	-470,4832528317	77	-10173904885589,500000
24	-0,0000000519	51	-1135,8470498557	78	-24561979157084,000000
25	-0,0000001271	52	-2742,1773525431	79	-59297863199757,600000
26	-0,0000003060	53	-6620,2017549419	80	-143157705556599,000000
27	-0,0000007391	54	-15982,5808624269	81	-345613274312956,000000
28	-0,0000017843	55	-38585,3634797957	82	-834384254182511,000000
29	-0,0000043078	56	-93153,3078220182	83	-2014381782677980,000000
30	-0,0000103999	57	-224891,9791238320	84	-4863147819538470,000000
31	-0,0000251077	58	-542937,2660696820	85	-11740677421754900,000000
32	-0,0000606152	59	-1310766,5112632000	86	-28344502663048300,000000
33	-0,0001463381	60	-3164470,2885960800	87	-68429682747851500,000000
34	-0,0003532915	61	-7639707,0884553500	88	-165203868158751000,000000
35	-0,0008529212	62	-18443884,4655068000	89	-398837419065354000,000000
36	-0,0020591338	63	-44527476,0194689000	90	-962878706289460000,000000
37	-0,0049711888	64	-107498836,5044450000	91	-2324594831644270000,000000
38	-0,0120015115	65	-259525149,0283580000	92	-5612068369578010000,000000
39	-0,0289742118	66	-626549134,5611610000	93	-13548731570800300000,000000
40	-0,0699499351	67	-1512623418,1506800000	94	-32709531511178600000,000000
41	-0,1688740819	68	-3651795970,8625200000	95	-78967794593157400000,000000
42	-0,4076980989	69	-8816215359,8757200000	96	-190645120697493000000,000000
43	-0,9842702797	70	-21284226690,6140000000	97	-460258035988144000000,000000
44	-2,3762386583	71	-51384668741,1036000000	98	-1111161192673780000000,000000
45	-5,7367475963	72	-124053564172,8210000000	99	-2682580421335710000000,000000
46	-13,8497338509	73	-299491797086,7460000000	100	-6476322035345200000000,000000

The linear recurrence sequence $u_{n+2} = 2u_{n+1} + u_n$

A basis of the space of solutions is $\big((\gamma_1^n)_{n\ge 0},\quad (\gamma_2^n)_{n\ge 0}\big)$ with

$$\gamma_1 = 1+\sqrt{2},\quad \gamma_2 = 1-\sqrt{2}.$$

The general solution is

$$u_n = a\gamma_1^n + b\gamma_2^n.$$

Notice that $\gamma_1 > 1 > |\gamma_2|$.
If $a \neq 0$, then $u_n \sim a\gamma_1^n \to \infty$ as $n \to \infty$.
If $a = 0$, then $u_n = b\gamma_2^n \to 0$.

If two consecutive terms are of the same sign, then $a \neq 0$, all the next ones have the same sign and $|u_n|$ tends to infinity.

$u_{n+2} = 2u_{n+1} + u_n$ with $u_0 = 1$, $u_1 = 1 - \sqrt{2}$

The value of u_{100} is $(1-\sqrt{2})^{100}$.

$1-\sqrt{2} = -0.4142135624\dots \qquad \log(\sqrt{2}-1) = -0.881373587\dots.$

$$(1-\sqrt{2})^{100} = \mathrm{e}^{-88.1373587\dots} = 5.277 \cdot 10^{-39}.$$

For $u_0 = 1$ and $u_1 = 1 - \sqrt{2} + \eta$, we have

$$u_n = \frac{1}{4}\eta\sqrt{2}(1+\sqrt{2})^n + \left(1 - \frac{1}{4}\eta\sqrt{2}\right)(1-\sqrt{2})^n;$$

hence $u_n \to 0$ if and only if $\eta = 0$.

Infinitely many solutions to the puzzle

Ramanujan said he has infinitely many solutions (but a single one between 50 and 500).
Sequence of balancing numbers (number of the house)
https://oeis.org/A001109

$0, 1, 6, 35, \mathbf{204}, 1189, 6930, 40391, 235416, 1372105, 7997214\dots$

This is a linear recurrence sequence $u_{n+2} = 6u_{n+1} - u_n$ with the initial conditions $u_0 = 0$, $u_1 = 1$.

The number of houses is https://oeis.org/A001108

$$0, 1, 8, 49, \mathbf{288}, 1681, 9800, 57121, 332928, 1940449, \dots$$

OEIS

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http://oeis.org/A001109 http://oeis.org/A001108

The puzzle is attributed to David Gales in the Puzzles Column of the Emissary MSRI
https://www.msri.org/attachments/media/news/emissary/EmissaryFall2005.pdf

Balancing numbers

A balancing number is an integer $B \geq 0$ such that there exists C with

$$1 + 2 + 3 + \cdots + (B-1) = (B+1) + (B+2) + \cdots + C.$$

Same as $B^2 = C(C+1)/2$: a balancing number is an integer B such that B^2 is a triangular number (and a square !).
Sequence of balancing numbers : https://oeis.org/A001109

$$0, 1, 6, 35, 204, 1189, 6930, 40391, 235416, 1372105, 7997214 \ldots$$

This is a linear recurrence sequence

$$B_{n+1} = 6B_n - B_{n-1}$$

with the initial conditions $B_0 = 0$, $B_1 = 1$.

Balancing numbers and the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 6 \end{pmatrix}$

$$\begin{pmatrix} B_{n+1} \\ B_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 6 \end{pmatrix} \begin{pmatrix} B_n \\ B_{n+1} \end{pmatrix} \quad (n \geq 0).$$

Powers of A :

$$\begin{pmatrix} 0 & 1 \\ -1 & 6 \end{pmatrix}^n = \begin{pmatrix} -B_n & B_{n+1} \\ -B_{n+1} & B_{n+2} \end{pmatrix} \quad (n \geq 0).$$

Characteristic polynomial :

$$\det(XI - A) = \det \begin{pmatrix} X & -1 \\ 1 & X-6 \end{pmatrix} = X^2 - 6X + 1.$$

The sequence of balancing numbers

Characteristic polynomial :
$f(X) = X^2 - 6X + 1 = (X - 3 - 2\sqrt{2})(X - 3 + 2\sqrt{2})$.
Closed formula :

$$B_n = \frac{1}{4\sqrt{2}}\big((3 + 2\sqrt{2})^n - (3 - 2\sqrt{2})^n\big).$$

Generating series :

$$\varphi(t) = \sum_{n \geq 0} B_n t^n = t + 6t^2 + 35t^3 + \cdots = \frac{t}{1 - 6t + t^2}.$$

`Exercise` :

$$t^2\varphi' = (1 - t^2)\varphi^2.$$

Takao Komatsu & Prasanta Kumar Ray. *Higher-order identities for balancing numbers*. `arXiv:1608.05925 [math.NT]`

Exponential generating series of the sequence of balancing numbers

$$y(x) = \sum_{n \geq 0} B_n \frac{x^n}{n!}$$
$$= x + 3x^2 + \frac{35}{6}x^3 + \cdots$$
$$= \frac{1}{4\sqrt{2}} \left(e^{(3+2\sqrt{2})x} - e^{(3-2\sqrt{2})x} \right).$$

This is a solution of the homogeneous linear differential equation of order 2

$$y'' = 6y' - y$$

with the initial conditions $y(0) = 0$, $y'(0) = 1$.

The sequence $(C_n)_{n\geq 0}$

$$2B_n^2 = C_n(C_n+1)$$

The corresponding sequence $(C_n)_{n\geq 0}$ is
https://oeis.org/A001108

$$0, 1, 8, 49, 288, 1681, 9800, 57121, 332928, 1940449, \ldots$$

The solutions of $x^2 - 2y^2 = 1$ are given by

$$x_n = 2B_n, \quad y_n = 2C_n + 1.$$

Both sequences $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ satisfy

$$u_{n+2} = 6u_{n+1} - u_n.$$

with $x_0 = 0$, $x_1 = 2$, $y_0 = 1$, $y_1 = 3$.
Hence

$$C_{n+1} = 6C_n - C_{n-1} + 2.$$

On the Ramanujan street puzzle of Prasanta Chandra Mahalanobis

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