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Criteria for linear independence and transcendence, following Yuri Nesterenko, Stéphane Fischler, Wadim Zudilin and Amarisa Chantanasiri

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Abstract

Most irrationality proofs rest on the following criterion :

A real number x is irrational if and only if, for any $\epsilon > 0$, there exist two rational integers p and q with $q > 0$, such that

$$0 < |qx - p| < \epsilon.$$

We survey generalisations of this criterion to linear independence, transcendence and algebraic independence.

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Numbers : algebraic, transcendental

Algebraic number : a complex number which is root of a non-zero polynomial with rational coefficients.

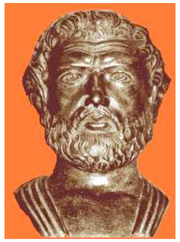
Examples :
rational numbers : a/b, root of $bX - a$.
$\sqrt{2}$, root of $X^2 - 2$.
i, root of $X^2 + 1$.

The sum and the product of algebraic numbers are algebraic numbers. The set of complex algebraic numbers is a field, the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$.

A transcendental number is a complex number which is not algebraic.

Irrationality of $\sqrt{2}$

Pythagoreas school



Hippasus of Metapontum (around 500 BC).

Sulba Sutras, Vedic civilization in India, $\sim$800-500 BC.

Irrationality criteria

A real number is rational if and only if its binary (or decimal, or in any basis $b \geq 2$) expansion is *ultimately periodic*.

Also a real number is rational if and only if its continued fraction expansion is finite.

Consequence : it should not be so difficult to decide whether a given number is rational or not.

To prove that certain numbers (occurring as constants in analysis) are irrational is most often an impossible challenge. However to construct irrational (even transcendental) numbers is easy.

First decimals of $\sqrt{2}$

http://wims.unice.fr/wims/wims.cgi

1.4142135623730950488016887242096980785696718753769480731766797379907324784621070388503875343276415727350138462309122970249248360558507372126441214970999358314132226659275055927557999505011527820605714701095599716059702745345968620147285174186408891986095523292304843087143214508397626036279952514079896872533965463318088296406206152583523950547457502877599617298355752203375318570113543746034084988471603868999706990048150305440277903164542478230684929369186215805784631115966687130130156185689872372352885092648612494977154218334204285686060146824720771435854874155657069677653720226485447015858801620758474922657226002085584466521458398893944370926591800311388246468157082630100594858704003186480342194897278290641045072636881313739855256117322040245091227700226941127573627280495738108967504018369868368450725799364729060762996941380475654823728997180326802474420629269124859052181004459842150591120249441341728531478105803603371077309182869314710171111683916581726889419758716582152128229518488472089694633862891562882765952635140542267653239694617511291602408715510135150455381287560052631468017127402653969470240300517495318862925631385188163478001569369176881852378684052287837629389214300655869568685964595155501644724509836896036887323114389415576651040883914292338113206052433629485317049915771756228549741438999188021762430965206564211827316726257539594717255934637238632261482742622208671155839599926521176252698917540988159348640083457085181472231814204070426509056532333398436457865796796519267292399875366617215982578860263363617827495994219403777753681426217738799194551397231274066898329989895386728822856378697749662519966583525776198939322845344735694794962952168891485492538904755828834526096524096542889394538646625744927492221447817 . . .

First binary digits of $\sqrt{2}$

http://wims.unice.fr/wims/wims.cgi

1.0110101000001001111001100110011111110011101111001100100100001000101100101111101100010011011001101110101010010101011111010011111000111010110111101100000101110101000100100111011101010000100110011101101000101111010110010000101100000110011001110011001000101010100101011111100100000110000010000111010101110001010001011000011101010001011000111111110011011111101110010000011110110110011100100001111011101001010100001011110010000111001110001111011010010100111100000000100100001110011011000111101111110100010011101101000110100100010000000101110100001110100001010101111000111111010011100101001100000101100111000110000000010001101111000011001101111011110010101011000110111100100100010001011010001000010001011000101001000110000010101011110001110010001011110111111000100111000110011110001101101010110101000101000111000101110110111111101001110111001100101100101010011000110100001100110001111110011111001000010011011111101010010111100010010000011111000001101101110010110000010111011101010101001001010000010001001100100000010000000110010100100101010000000100111001010010100000 . . .

Euler–Mascheroni constant

Euler's Constant is

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right)$$
$$= 0.577\,215\,664\,901\,532\,860\,606\,512\,090\,082\ldots$$

Is–it a rational number ?

$$\gamma = \sum_{k=1}^{\infty}\left(\frac{1}{k}-\log\left(1+\frac{1}{k}\right)\right) = \int_1^{\infty}\left(\frac{1}{[x]}-\frac{1}{x}\right)dx$$
$$= -\int_0^1\int_0^1 \frac{(1-x)dxdy}{(1-xy)\log(xy)}.$$

Riemann zeta function

The function

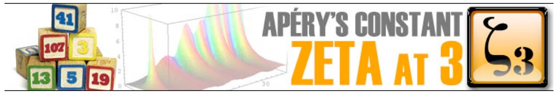
$\zeta(s) = \sum_{n\geq 1}\frac{1}{n^s}$
was studied by Euler (1707– 1783)
for integer values of s
and by Riemann (1859) for complex values of s.

Euler : for any even integer value of $s \geq 2$, the number $\zeta(s)$ is a rational multiple of π^s.

Examples : $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, $\zeta(6) = \pi^6/945$, $\zeta(8) = \pi^8/9450\cdots$

Coefficients : Bernoulli numbers.

Riemann zeta function



The number

$$\zeta(3) = \sum_{n\geq 1}\frac{1}{n^3} = 1,202\,056\,903\,159\,594\,285\,399\,738\,161\,511\ \ldots$$

is irrational *(Apéry 1978)*.

Recall that $\zeta(s)/\pi^s$ is rational for any even value of $s \geq 2$.

Open question : Is the number $\zeta(3)/\pi^3$ irrational ?

Riemann zeta function

Is the number

$$\zeta(5) = \sum_{n\geq 1}\frac{1}{n^5} = 1.036\,927\,755\,143\,369\,926\,331\,365\,486\,457\ldots$$

irrational ?

T. Rivoal (2000) : infinitely many $\zeta(2n+1)$ are irrational.

Motivations

- Squaring the circle
- Dynamical systems
- Solving Diophantine equations
- Theoretical computer sciences : rounding values
- Main goal : to understand the underlying theory.

Known results

Irrationality of the number π :

Āryabhaṭa, b. 476 AD : $\pi \sim 3.1416$.

Nīlakaṇṭha Somayājī, b. 1444 AD : *Why then has an approximate value been mentioned here leaving behind the actual value ? Because it (exact value) cannot be expressed.*

K. Ramasubramanian, *The Notion of Proof in Indian Science*, 13th World Sanskrit Conference, 2006.

Irrationality of π

Johann Heinrich Lambert (1728 - 1777)
Mémoire sur quelques propriétés remarquables des quantités transcendantes circulaires et logarithmiques,
Mémoires de l'Académie des Sciences de Berlin, **17** (1761), p. 265-322 ;
read in 1767 ; Math. Werke, t. II.

$\tan(v)$ is irrational for any rational value of $v \neq 0$ and $\tan(\pi/4) = 1$.

Lambert and Frederick II, King of Prussia

— Que savez vous, Lambert ?
— Tout, Sire.
— Et de qui le tenez–vous ?
— De moi-même !

Leonhard Euler (1707 – 1783)

1748 : Irrationality of the number $e = 2.718\,281\,828\,459\,0\ldots$

The number

$$e = \sum_{n\geq 0} \frac{1}{n!}$$

is irrational

Continued fractions expansion.

http://www-history.mcs.st-andrews.ac.uk/

Joseph Fourier (1768 – 1830)

Proof of Euler's 1748 result on the irrationality of the number e by truncating the series

$$e = \sum_{n\geq 0} \frac{1}{n!}.$$

Course of analysis at the École Polytechnique Paris, 1815.

Irrationality of e, following J. Fourier

$$e = \sum_{n=0}^{N} \frac{1}{n!} + \sum_{m\geq N+1} \frac{1}{m!}.$$

Multiply by $N!$:

$$N!e = \sum_{n=0}^{N} \frac{N!}{n!} + \sum_{m\geq N+1} \frac{N!}{m!}.$$

Set

$$B_N = N!, \qquad A_N = \sum_{n=0}^{N} \frac{N!}{n!}, \quad R_N = \sum_{m\geq N+1} \frac{N!}{m!},$$

so that

$$B_N e = A_N + R_N.$$

Irrationality of e, following J. Fourier

Then A_N and B_N are in $\mathbf{Z}$ and

$$0 < R_N = \frac{1}{N+1} + \frac{1}{(N+1)(N+2)} + \cdots < \frac{e}{N+1}.$$

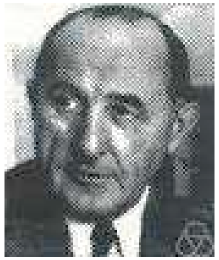
In the formula

$$B_N e - A_N = R_N,$$

the numbers A_N and $B_N = N!$ are integers, while the right hand side is > 0 and tends to 0 when N tends to infinity. Hence $N!\,e$ is not an integer, therefore e is irrational.

C.L Siegel (1949) : irrationality of e^{-1}

$$N!e^{-1} = \sum_{n=0}^{N} \frac{(-1)^n N!}{n!} + \sum_{m\geq N+1} \frac{(-1)^m N!}{m!}.$$



C.L. Siegel (1896 – 1981)

Take for N a large odd integer and set

$$A_N = \sum_{n=0}^{N} \frac{(-1)^n N!}{n!}.$$

Then $A_N \in \mathbf{Z}$ and

$$A_N < N!e^{-1} < A_N + \frac{1}{N+1}.$$

Hence e^{-1} is irrational.

e is not a quadratic irrationality (Liouville, 1840)

Write the quadratic equation as $ae + b + ce^{-1} = 0$.

$$bN! + \sum_{n=0}^{N}(a + (-1)^n c)\frac{N!}{n!}$$
$$= -\sum_{k\geq 0}\left(a + (-1)^{N+1+k}c\right) \cdot \frac{N!}{(N+1+k)!}.$$

Using Fourier's argument, we deduce that the LHS and RHS are 0 for any sufficiently large N.

Irrationality proof

Let $\vartheta \in \mathbf{Q}$, say $\vartheta = a/b$. Then for any $p/q \in \mathbf{Q}$ with $p/q \neq \vartheta$ we have

$$|q\vartheta - p| \geq \frac{1}{b}.$$

Proof : $|qa - pb| \geq 1$.

Consequence. *Let $\vartheta \in \mathbf{R}$. Assume that for any $\epsilon > 0$, there exists $p/q \in \mathbf{Q}$ with*

$$0 < |q\vartheta - p| < \epsilon.$$

Then ϑ is irrational.

Criterion : necessary and sufficient condition

We saw that any $\vartheta \in \mathbf{R}$ for which there exists a sequence $(p_n/q_n)_{n\geq 0}$ of rational numbers with

$$0 < |q_n\vartheta - p_n| < \epsilon_n \quad \text{with} \quad \epsilon_n \to 0$$

is irrational.

Conversely, given $\vartheta \in \mathbf{R} \setminus \mathbf{Q}$, there exists a sequence $(p_n/q_n)_{n\geq 0}$ with

$$0 < |q_n\vartheta - p_n| < \epsilon_n \quad \text{and} \quad \epsilon_n \to 0.$$

More precisely, given $\vartheta \in \mathbf{R}$, for each real number $Q > 1$, there exists $p/q \in \mathbf{Q}$ with

$$|q\vartheta - p| \leq \frac{1}{Q} \quad \text{and} \quad 0 < q < Q.$$

Hence, for $\vartheta \notin \mathbf{Q}$, there exists a sequence $(p_n/q_n)_{n\geq 0}$ with

$$0 < |q_n\vartheta - p_n| < \frac{1}{q_n} \quad \text{and} \quad q_n \to \infty.$$

Gustave Lejeune–Dirichlet (1805 – 1859)

G. Dirichlet

1842 : Box (pigeonhole) principle

A map $f : E \to F$ *with* $\mathrm{Card}E > \mathrm{Card}F$ *is not injective.*

A map $f : E \to F$ *with* $\mathrm{Card}E < \mathrm{Card}F$ *is not surjective.*

Pigeonhole Principle

More holes than pigeons

More pigeons than holes

Existence of rational approximations

For any $\vartheta \in \mathbf{R}$ *and any real number* $Q > 1$, *there exists* $p/q \in \mathbf{Q}$ *with*

$$|q\vartheta - p| \le \frac{1}{Q}$$

and $0 < q < Q$.

Proof. For simplicity assume $Q \in \mathbf{Z}$. Take

$$E = \{0,\ \{\vartheta\},\ \{2\vartheta\}, \dots, \{(Q-1)\vartheta\},\ 1\} \subset [0,1],$$

where $\{x\}$ denotes the fractional part of x, F is the partition

$$\left[0, \frac{1}{Q}\right), \left[\frac{1}{Q}, \frac{2}{Q}\right), \dots, \left[\frac{Q-2}{Q}, \frac{Q-1}{Q}\right), \left[\frac{Q-1}{Q}, 1\right],$$

of $[0,1]$, so that

$$\mathrm{Card}E = Q+1 > Q = \mathrm{Card}F,$$

and $f : E \to F$ maps $x \in E$ to $I \in F$ with $I \ni x$.

Hermann Minkowski (1864 – 1909)

H. Minkowski

1896 : Geometry of numbers.
The set

$$\mathcal{C} = \{(u,v) \in \mathbf{R}^2\,;\, |v| \le Q,\ |v\vartheta - u| \le 1/Q\}$$

is convex, symmetric, compact, with volume 4.
Hence $\mathcal{C} \cap \mathbf{Z}^2 \ne \{(0,0)\}$.

Adolf Hurwitz (1859 – 1919)

A. Hurwitz

1891
For any $\vartheta \in \mathbf{R} \setminus \mathbf{Q}$, there exists a sequence $(p_n/q_n)_{n\geq 0}$ of rational numbers with

$$0 < |q_n\vartheta - p_n| < \frac{1}{\sqrt{5}q_n}$$

and $q_n \to \infty$.
Methods : Continued fractions, Farey sections.

Best possible for the Golden ratio

$$\frac{1+\sqrt{5}}{2} = 1.618\,033\,988\,749\,9\ldots$$

Irrationality criterion

Let ϑ be a real number. The following conditions are equivalent.
(i) *ϑ is irrational.*
(ii) *For any $\epsilon > 0$, there exists $p/q \in \mathbf{Q}$ such that*

$$0 < \left|\vartheta - \frac{p}{q}\right| < \frac{\epsilon}{q}.$$

(iii) *For any real number $Q > 1$, there exists an integer q in the interval $1 \leq q < Q$ and there exists an integer p such that*

$$0 < \left|\vartheta - \frac{p}{q}\right| < \frac{1}{qQ}.$$

(iv) *There exist infinitely many $p/q \in \mathbf{Q}$ satisfying*

$$\left|\vartheta - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}.$$

Irrationality criterion (continued)

Let ϑ be a real number. The following conditions are equivalent.
(i) *ϑ is irrational.*
(ii)' *For any $\epsilon > 0$, there exist two linearly independent linear forms*

$$L_0(X_0, X_1) = a_0X_0 + b_0X_1 \quad \text{and} \quad L_1(X_0, X_1) = a_1X_0 + b_1X_1,$$

with rational integer coefficients, such that

$$\max\left\{|L_0(1,\vartheta)|\,,\,|L_1(1,\vartheta)|\right\} < \epsilon.$$

Proof of (ii) $\iff$ (ii)'

(ii) *For any $\epsilon > 0$, there exists $p/q \in \mathbf{Q}$ such that*

$$0 < \left|\vartheta - \frac{p}{q}\right| < \frac{\epsilon}{q}.$$

(ii)' *For any $\epsilon > 0$, there exist two linearly independent linear forms L_0, L_1 in $\mathbf{Z}X_0 + \mathbf{Z}X_1$ such that*

$$\max\left\{|L_0(1,\vartheta)|\,,\,|L_1(1,\vartheta)|\right\} < \epsilon.$$

Proof of (ii)' $\Longrightarrow$ (ii)
Since L_0, L_1 are linearly independent, one at least of them does not vanish at $(1,\vartheta)$. Write it $pX_0 - qX_1$.
Proof of (ii) $\Longrightarrow$ (ii')
Using (ii), set $L_0(X_0, X_1) = pX_0 - qX_1$, and use (ii) again with ϵ replaced by $|q\vartheta - p|$.

Irrationality of at least one number

Let $\vartheta_1, \dots, \vartheta_m$ be real numbers. The following conditions are equivalent

(i) *One at least of $\vartheta_1, \dots, \vartheta_m$ is irrational.*

(ii) *For any $\epsilon > 0$, there exist $p_1, \dots, p_m, q$ in $\mathbf{Z}$ with $q > 0$ such that*

$$0 < \max_{1 \le i \le m} \left| \vartheta_i - \frac{p_i}{q} \right| < \frac{\epsilon}{q}.$$

(iii) *For any $\epsilon > 0$, there exist $m+1$ linearly independent linear forms $L_0, \dots, L_m$ with coefficients in $\mathbf{Z}$ in $m+1$ variables $X_0, \dots, X_m$, such that*

$$\max_{0 \le k \le m} |L_k(1, \vartheta_1, \dots, \vartheta_m)| < \epsilon.$$

(iv) *For any real number $Q > 1$, there exists $(p_1, \dots, p_m, q)$ in $\mathbf{Z}^{m+1}$ such that $1 \le q \le Q$ and*

$$0 < \max_{1 \le i \le m} \left| \vartheta_i - \frac{p_i}{q} \right| \le \frac{1}{qQ^{1/m}}.$$

Linear independence

Irrationality of ϑ : means that $1, \vartheta$ are linearly independent over $\mathbf{Q}$.

Irrationality of at least one of $\vartheta_1, \dots, \vartheta_m$: means $(\vartheta_1, \dots, \vartheta_m) \notin \mathbf{Q}^m$. Also : means that the dimension of the $\mathbf{Q}$–vector space spanned by $1, \vartheta_1, \dots, \vartheta_m$ is ≥ 2.

Linear independence of $1, \vartheta_1, \dots, \vartheta_m$ over $\mathbf{Q}$: means that for any hyperplane $H : a_0 z_0 + \cdots + a_m z_m = 0$ of $\mathbf{R}^{m+1}$ rational over $\mathbf{Q}$ (i.e. $a_i \in \mathbf{Q}$), the point $(1, \vartheta_1, \dots, \vartheta_m)$ does not belong to H.

Transcendence of ϑ : means that $1, \vartheta, \vartheta^2, \dots, \vartheta^n \dots$ are linearly independent over $\mathbf{Q}$.

Charles Hermite (1822 – 1901)

Charles Hermite

1873 : Hermite's method for proving linear independence.
Let $\vartheta_1, \dots, \vartheta_m$ be real numbers and $a_0, a_1, \dots, a_m$ rational integers, not all of which are 0. The goal is to prove that the number

$$L = a_0 + a_1\vartheta_1 + \cdots + a_m\vartheta_m$$

is not 0.

Hermite's idea is to approximate simultaneously $\vartheta_1, \dots, \vartheta_m$ by rational numbers $p_1/q, \dots, p_m/q$ with the same denominator $q > 0$.

$L = a_0 + a_1\vartheta_1 + \cdots + a_m\vartheta_m$

Let $q, p_1, \dots, p_m$ be rational integers with $q > 0$. For $1 \le k \le m$, set

$$\epsilon_k = q\vartheta_k - p_k.$$

Then $qL = M + R$ with

$$M = a_0 q + a_1 p_1 + \cdots + a_m p_m \in \mathbf{Z}$$

and

$$R = a_1\epsilon_1 + \cdots + a_m\epsilon_m \in \mathbf{R}.$$

If $M \ne 0$ and $|R| < 1$ we deduce $L \ne 0$.

Zero estimate

Main difficulty : to check $M \neq 0$.

We wish to find a simultaneous rational approximation $(q, p_1, \ldots, p_m)$ to $(\vartheta_1, \ldots, \vartheta_m)$ outside the hyperplane $a_0 z_0 + a_1 z_1 + \cdots + a_m z_m = 0$ of $\mathbf{Q}^{m+1}$.

This needs to be checked for all hyperplanes.

Solution : to construct not only one tuple $\mathbf{u} = (q, p_1, \ldots, p_m)$ in $\mathbf{Z}^{m+1} \setminus \{0\}$, but $m+1$ such tuples which are linearly independent.

This yields $m+1$ pairs (M_k, R_k), $k = 0, \ldots, m$ in place of a single pair (M, R), and from $(a_0, \ldots, a_m) \neq 0$ one deduces that one at least of $M_0, \ldots, M_m$ is not 0.

Rational approximations (following Michel Laurent)

Let $(\vartheta_1, \ldots, \vartheta_m) \in \mathbf{R}^m$.
Then the following conditions are equivalent.
(i) *The numbers* $1, \vartheta_1, \ldots, \vartheta_m$ *are linearly independent over* $\mathbf{Q}$.
(ii) *For any* $\epsilon > 0$, *there exist* $m+1$ *linearly independent elements* $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_m$ *in* $\mathbf{Z}^{m+1}$, *say*

$$\mathbf{u}_i = (q_i, p_{1i}, \ldots, p_{mi}) \quad (0 \leq i \leq m)$$

with $q_i > 0$, *such that*

$$\max_{1 \leq k \leq m} \left| \vartheta_k - \frac{p_{ki}}{q_i} \right| \leq \frac{\epsilon}{q_i} \quad (0 \leq i \leq m).$$

Hermite – Lindemann Theorem

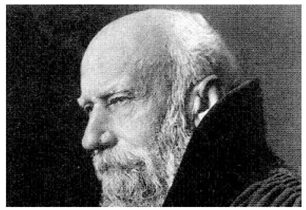
Hermite (1873) :
transcendence of e.

Lindemann (1882) :
transcendence of π.

Hermite – Lindemann Theorem
For any non–zero complex number z, *at least one of the two numbers* z, e^z *is transcendental.*

Corollaries : transcendence of $\log \alpha$ and e^β for α and β non–zero algebraic numbers with $\log \alpha \neq 0$.

Lindemann – Weierstraß Theorem (1888)

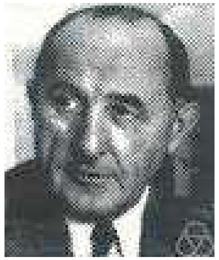


Let $\beta_1, \ldots, \beta_n$ *be algebraic numbers which are linearly independent over* $\mathbf{Q}$. *Then the numbers* $e^{\beta_1}, \ldots, e^{\beta_n}$ *are algebraically independent over* $\mathbf{Q}$.

Equivalent to :
Let $\alpha_1, \ldots, \alpha_m$ *be distinct algebraic numbers. Then the numbers* $e^{\alpha_1}, \ldots, e^{\alpha_m}$ *are linearly independent over* $\mathbf{Q}$.

Carl Ludwig Siegel (1896 – 1981)

Siegel's method for proving linear independence.
Let $\vartheta_1, \dots, \vartheta_m$ be complex numbers.



C.L. Siegel

1929 :
Assume that, for any $\epsilon > 0$, there exists $m+1$ linearly independent linear forms $L_0, \dots, L_m$, with coefficients in $\mathbf{Z}$, such that

$$\max_{0 \le k \le m} |L_k(1, \vartheta_1, \dots, \vartheta_m)| < \frac{\epsilon}{H^{m-1}}$$

where $H = \max_{0 \le k \le m} H(L_k)$.

Then $1, \vartheta_1, \dots, \vartheta_m$ are linearly independent over $\mathbf{Q}$.

Linear independence, following Siegel (1929)

Height of a linear form : $H(L) = \max |\text{coefficients of } L|$.

Example : $m = 1$ (irrationality criterion). *A real number ϑ is irrational if and only, for any $\epsilon > 0$, if there exists two linearly independent linear forms $L_0(X_0, X_1)$ and $L_1(X_0, X_1)$ in $\mathbf{Z}X_0 + \mathbf{Z}X_1$ such that $|L_i(1, \vartheta)| < \epsilon$.*

Sketch of proof of Siegel's criterion. Assume $1, \vartheta_1, \dots, \vartheta_m$ are linearly dependent over $\mathbf{Q}$. Let $L \in \mathbf{Z}X_0 + \cdots + \mathbf{Z}X_m$ be a non–zero linear form vanishing at $(1, \vartheta_1, \dots, \vartheta_m)$. Among $L_0, \dots, L_m$, select m linear forms, say $L_1, \dots, L_m$, which constitute with L a complete system of linearly independent forms in $m+1$ variables. The determinant Δ of $L, L_1, \dots, L_m$ is a non–zero integer, hence its absolute value is ≥ 1. Inverting the matrix, write Δ as a linear combination with integer coefficients of the $L_i(1, \vartheta_1, \dots, \vartheta_m)$ $(1 \le i \le m)$ and estimate the coefficients.

Criterion of Yu. V. Nesterenko

Let $\vartheta_1, \dots, \vartheta_m$ be complex numbers.

Yu.V.Nesterenko (1985)

Let m be a positive integer and α a positive real number satisfying $\alpha > m - 1$. Assume there is a sequence $(L_n)_{n \ge 0}$ of linear forms in $\mathbf{Z}X_0 + \mathbf{Z}X_1 + \ldots + \mathbf{Z}X_m$ of height $\le e^n$ such that

$$|L_n(1, \vartheta_1, \dots, \vartheta_m)| = e^{-\alpha n + o(n)}.$$

Then $1, \vartheta_1, \dots, \vartheta_m$ are linearly independent over $\mathbf{Q}$.

Example : $m = 1$ – irrationality criterion.

Simplified proof of Nesterenko's Theorem

Francesco Amoroso

Pierre Colmez

Refinements : Raffaele Marcovecchio, Pierre Bel (2008).

Irrationality measure for $\log 2$: history

$$\left|\log 2 - \frac{p}{q}\right| > \frac{1}{q^\mu}$$

Hermite–Lindemann, Mahler, Baker, Gel'fond, Feldman,. . . : transcendence measures

G. Rhin 1987 $\mu(\log 2) < 4.07$

E.A. Rukhadze 1987 $\mu(\log 2) < 3.89$

R. Marcovecchio 2008 $\mu(\log 2) < 3.57$

Recent developments

Stéphane Fischler and Wadim Zudilin, *A refinement of Nesterenko's linear independence criterion with applications to zeta values.*
Math. Annalen, to appear. Preprint MPIM 2009-35.

Criteria for transcendence and algebraic independence

A complex number ϑ is *transcendental* if and only if $1, \vartheta, \vartheta^2, \ldots, \vartheta^n \ldots$ are linearly independent (over $\mathbf{Q}$).

Complex numbers $\vartheta_1, \ldots, \vartheta_m$ are *algebraically independent* if and only if the numbers $\vartheta_1^{i_1} \cdots \vartheta_m^{i_m}$, $((i_1, \ldots, i_m) \in \mathbf{Z}_{\geq 0}^m$ are linearly independent.

Hence, criteria for linear independence yield criteria for transcendence and for algebraic independence.

Furthermore, criteria for transcendence are special case $(m = 1)$ of criteria for algebraic independence.

Amarisa Chantanasiri

Criteria for linear independence, transcendence and algebraic independence

Université P. et M. Curie (Paris VI), Ph.D. 2011 ?

New criterion for algebraic independence

Let $\vartheta_1, \dots, \vartheta_m$ be real numbers and $(\tau_d)_{d\geq 1}$, $(\eta_d)_{d\geq 1}$ two sequences of positive real numbers satisfying

$$\frac{\tau_d}{d^{m-1}(1+\eta_d)} \longrightarrow +\infty.$$

Assume that for all sufficiently large d, there is a sequence $(P_n)_{n\geq n_0(d)}$ of polynomials in $\mathbf{Z}[X_1, \dots, X_m]$, where P_n has degree $\leq d$ and height $\leq e^n$, such that

$$e^{-(\tau_d+\eta_d)n} \leq |P_n(\vartheta_1, \dots, \vartheta_m)| \leq e^{-\tau_d n}.$$

Then $\vartheta_1, \dots, \vartheta_m$ are algebraically independent.

Mahidol University, Bangkok October 29-31, 2009
Franco-Thai Seminar in Pure and Applied Mathematics,
http://www.sc.mahidol.ac.th/cem/franco_thai/

Criteria for linear independence and transcendence, following Yuri Nesterenko, Stéphane Fischler, Wadim Zudilin and Amarisa Chantanasiri

Michel Waldschmidt

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Lecture given on October 31, 2009.