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Zeros of linear recurrence sequences

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Abstract. Let $f(x) = P_0(x)\alpha_0^x + \cdots + P_k(x)\alpha_k^x$ be an exponential polynomial over a field of zero characteristic. Assume that for each pair i, j with $i \neq j$, α_i/α_j is not a root of unity. Define $\Delta = \sum_{j=0}^{k}(\deg P_j + 1)$. We introduce a partition of $\{\alpha_0, \ldots, \alpha_k\}$ into subsets $\{\alpha_{i0}, \ldots, \alpha_{ik_i}\}$ $(1 \leq i \leq m)$, which induces a decomposition of f into $f = f_1 + \cdots + f_m$, so that, for $1 \leq i \leq m$, $(\alpha_{i0} : \cdots : \alpha_{ik_i}) \in \mathbb{P}_{k_i}(\overline{\mathbb{Q}})$, while for $1 \leq i \neq u \leq m$, the number α_{i0}/α_{u0} either is transcendental or else is algebraic with not too small a height. Then we show that for all but at most $\exp\big(\Delta(5\Delta)^{5\Delta}\big)$ solutions $x \in \mathbb{Z}$ of $f(x) = 0$, we have

$$f_1(x) = \cdots = f_m(x) = 0.$$

1. Introduction

Let $\mathbb{K}$ be a field of zero characteristic, $\alpha_1, \ldots, \alpha_k$ be non-zero elements of $\mathbb{K}$ and $P_1, \ldots, P_k$ non-zero polynomials with coefficients in $\mathbb{K}$. Consider an exponential polynomial

$$f(x) = \sum_{j=0}^{k} P_j(x)\alpha_j^x.$$

We study the equation

$$f(x) = 0 \tag{1.1}$$

in $x \in \mathbb{Z}$. We suppose that for each pair i, j with $i \neq j$,

$$\alpha_i/\alpha_j \quad \text{is not a root of unity.} \tag{1.2}$$

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We set

$$\Delta(f) = \sum_{j=0}^{k} (\deg P_j + 1).$$

It is well known that $\big(f(0), f(1), \dots\big)$ is a linear recurrence sequence of order $\Delta(f)$, which is "non-degenerate". Vice versa, any non-degenerate linear recurrence sequence $\big(u_0, u_1, \dots\big)$ of elements of $\mathbb{K}$ of order q has some representation $u_n = f(n)$, where f is an exponential polynomial as above satisfying (1.2) and with $\Delta(f) = q$. For more details, cf. e.g. [8]. So in studying (1.1) we study the zeros of linear recurrence sequences. An old conjecture says that the number of solutions $x \in \mathbb{Z}$ of equation (1.1) is bounded above by a function that depends only upon $\Delta(f)$. Let us briefly review what is known so far in this context[1].

For $q = 1$, equation (1.1) reduces to

$$a_0 \alpha_0^x = 0$$

and clearly there is no solution x at all.

For $q = 2$, we have one of the following two equations

$$(a_0 + a_1 x)\alpha_0^x = 0 \quad \text{or} \quad a_0\alpha_0^x + a_1\alpha_1^x = 0.$$

In either case, in view of our assumption (1.2) on non-degeneracy, we clearly do not have more than one solution x.

The first non-trivial case is $q = 3$. Here, Schlickewei [4] proved the conjecture to be true. His bound has been improved by Beukers and Schlickewei [1]. They showed that for $q = 3$ equation (1.1) does not have more than 61 solutions.

Now suppose $q \geq 4$. In a recent paper [3], Evertse, Schlickewei and Schmidt proved the following: *Suppose that in (1.1) the polynomials f_i for $i = 0, \dots, k$ are constant. Then equation (1.1) does not have more than* $\exp\big((7k)^{3k}\big)$ *solutions.* As in this situation $q = k + 1$, we see that when the polynomials f_i in (1.1) are all constant, the conjecture is true.

There remains the case when $q \geq 4$ and when not all f_i's are constant. Now obviously in (1.1) we may suppose without loss of generality that $\alpha_0 = 1$. With this normalization, Schlickewei [5] proved the following: *Suppose that $\alpha_1, \dots, \alpha_k$ are algebraic and that $[\mathbb{Q}(\alpha_1, \dots, \alpha_k) : \mathbb{Q}] \leq d$. Then the number of solutions of equation (1.1) is bounded in terms of q and d only.* (A bound was given explicitly). Schlickewei and Schmidt [6] later on established the bound $(2q)^{35q^3} d^{6q^2}$.

[1] After the present paper was written, the second author [7] settled this conjecture.

We denote by $\overline{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$ in $\mathbb{K}$ (this is the field of algebraic elements in $\mathbb{K}$). We define an equivalence relation on the set $\mathbb{K}^\times$ of non-zero elements of $\mathbb{K}$ by the condition

$$z_1 \sim z_2 \quad \Longleftrightarrow \quad z_1/z_2 \quad \text{is algebraic.}$$

This relation induces a partition of $\{\alpha_0, \dots, \alpha_k\}$:

$$\{\alpha_0, \dots, \alpha_k\} = \bigcup_{i=1}^{m} \{\alpha_{i0}, \dots, \alpha_{ik_i}\},$$

where, for $1 \le i \le m$,

$$(\alpha_{i0} : \dots : \alpha_{ik_i}) \in \mathbb{P}_{k_i}(\overline{\mathbb{Q}}),$$

while for $1 \le i \ne u \le m$, the number α_{i0}/α_{u0} is transcendental. Accordingly, f is decomposed into

$$f = f_1 + \dots + f_m, \tag{1.3}$$

with

$$f_i(x) = P_{i0}(x)\alpha_{i0}^x + \dots + P_{ik_i}(x)\alpha_{ik_i}^x \qquad (1 \le i \le m).$$

We prove

Theorem 1.1. *Suppose we have (1.2). Define $\Delta = \Delta(f)$ and*

$$F(\Delta) = \exp\big(\Delta(5\Delta)^{5\Delta}\big).$$

Then for all but at most $F(\Delta)$ solutions $x \in \mathbb{Z}$ of (1.1), we have

$$f_1(x) = \dots = f_m(x) = 0. \tag{1.4}$$

Our result, in other words, says that the only case when the conjecture possibly could fail to be true arises from the algebraic case, i.e. when $\alpha_0, \dots, \alpha_k$ are in $\overline{\mathbb{Q}}$. Moreover we shall see that the conjecture would follow from the special case where $\alpha_0, \dots, \alpha_k$ are algebraic and each α_i/α_j has a small height. Actually our method of proof gives a result of the type stated in the Theorem also under the assumption that the quotients α_i/α_u are not transcendental but have logarithmic height bounded away from zero (for more details, see the final remark in Sect. 6).

We mention that our proof was inspired by a similar result for $q = 3$ by Beukers and Tijdeman [2]. They showed:

Let α and β be non-zero elements of $\mathbb{K}$. Suppose that α, β and α/β are not roots of unity. Let a and b be non-zero elements of $\mathbb{K}$. Suppose that the equation

$$a\alpha^x + b\beta^x = 1$$

has at least 4 solutions $x \in \mathbb{Z}$. Then α and β are algebraic.

Our proof uses a recent result of Schlickewei and Schmidt [6] on polynomial exponential equations.

2. Heights

Let K be a number field of degree d. Write $M(K)$ for the set of places of K. For $v \in M(K)$, let $|\ \ |_v$ be the valuation which extends either the standard absolute value of $\mathbb{Q}$, or if $v|p$ for a rational prime p, let $|\ \ |_v$ be the valuation with $|p|_v = p^{-1}$. Write d_v for the local degree $[K_v : \mathbb{Q}_p]$ and define the absolute value $\|\ \ \|_v$ by

$$\|\ \ \|_v = |\ \ |_v^{d_v/d}.$$

Let $n \geq 1$ and let $\underline{\alpha} = (\alpha_0, \dots, \alpha_n) \neq (0, \dots, 0)$ be a point in K^{n+1}. We then put

$$\|\underline{\alpha}\|_v = \max\bigl\{\|\alpha_0\|_v, \dots, \|\alpha_n\|_v\bigr\}$$

and we define the homogeneous height as

$$\mathrm{H}(\underline{\alpha}) = \prod_{v \in M(K)} \|\underline{\alpha}\|_v.$$

Since it depends only on the class $\underline{\underline{\alpha}} = (\alpha_0 : \cdots : \alpha_n)$ of $\underline{\alpha}$ in $\mathbb{P}_n(\overline{\mathbb{Q}})$, we also denote it by $\mathrm{H}(\underline{\underline{\alpha}})$. Let

$$\mathrm{h}(\underline{\underline{\alpha}}) = \mathrm{h}(\alpha_0 : \cdots : \alpha_n) = \log \mathrm{H}(\underline{\underline{\alpha}})$$

be the homogeneous logarithmic absolute height of $\underline{\underline{\alpha}} \in \mathbb{P}_n(\overline{\mathbb{Q}})$ We shall also need the inhomogeneous absolute heights

$$\mathrm{H}_{\mathrm{in}}(\underline{x}) = \mathrm{H}(1 : x_1 : \cdots : x_n)$$

and

$$\mathrm{h}_{\mathrm{in}}(\underline{x}) = \mathrm{h}(1 : x_1 : \cdots : x_n) = \log \mathrm{H}_{\mathrm{in}}(\underline{x})$$

of $\underline{x} = (x_1, \dots, x_n) \in \overline{\mathbb{Q}}^n$. Further, for $x \in \overline{\mathbb{Q}}$, we set

$$\mathrm{H}_{\mathrm{in}}(x) = \mathrm{H}(1 : x) \quad \text{and} \quad \mathrm{h}_{\mathrm{in}}(x) = \mathrm{h}(1 : x) = \log \mathrm{H}_{\mathrm{in}}(x).$$

Given $D \in \mathbb{N}$ and $\hbar > 0$, we will use the fact that the set of elements $\alpha \in \overline{\mathbb{Q}}^{\times}$ with

$$\deg \alpha \leq D \quad \text{and} \quad \mathrm{h}_{\mathrm{in}}(\alpha) \leq \hbar$$

is finite.

3. Algebraic linear recurrence sequences

The results in this section are consequences of the Subspace Theorem.

Lemma 3.1. *Let* $m \geq 1$ *and* Γ *be a finitely generated subgroup of* $(\overline{\mathbb{Q}}^{\times})^m$ *of rank* $r \geq 0$*. Then the solutions* $\underline{z} = \underline{x} * \underline{y} = (x_1 y_1, \dots, x_m y_m)$ *of*

$$z_1 + \cdots + z_m = 1 \tag{3.1}$$

with $\underline{z} \in \Gamma$, $\underline{y} \in \mathbb{Q}^m$ *and*

$$\mathrm{h}_{\mathrm{in}}(\underline{y}) \leq \frac{1}{4m^2} \mathrm{h}_{\mathrm{in}}(\underline{x}) \tag{3.2}$$

are contained in the union of at most

$$\exp\big((4m)^{4m}(r+1)\big)$$

proper subspaces of $\overline{\mathbb{Q}}^m$.

Proof. This is a variation on Proposition A of [6]. In that proposition there was a distinction between three kinds of solutions:

i) Solutions where some $y_i = 0$, i.e., some $z_i = 0$. These clearly lie in m subspaces.
ii) Solutions where each $y_i \neq 0$ and where $\mathrm{h}_{\mathrm{in}}(\underline{x}) > 2m \log m$. These were called *large solutions* in [6] and it was shown in (10.4) of that paper that they lie in the union of fewer than

$$2^{30m^2}(21m^2)^r$$

proper subspaces.
iii) Solutions where each $y_i \neq 0$ and where $\mathrm{h}_{\mathrm{in}}(\underline{x}) \leq 2m \log m$. These were called *small solutions* in [6]. Here we argue as follows. We have $\mathrm{h}_{\mathrm{in}}(\underline{y}) \leq (2m \log m)/(4m^2) < \log 2$ by (3.2). Then each component has $\mathrm{h}_{\mathrm{in}}(y_i) < \log 2$, which is $\mathrm{H}_{\mathrm{in}}(y_i) < 2$. Since $y_i \in \mathbb{Q}^{\times}$, we have $y_i = \pm 1$. The equation (3.1) now becomes

$$\pm x_1 \pm x_2 \pm \cdots \pm x_m = 1. \tag{3.3}$$

The group Γ' generated by Γ and the vectors $(\pm 1, \cdots, \pm 1)$ contains no more than r multiplicatively independent elements. By Proposition 2.1 of [6], the solutions of (3.3) lie in the union of not more than

$$\exp\big((4m)^{3m} \cdot 2(r+1)\big)$$

proper subspaces of $\overline{\mathbb{Q}}^m$.

Combining our estimates we obtain

$$m + 2^{30m^2}(21m^2)^r + \exp\big((4m)^{3m} \cdot 2(r+1)\big) < \exp\big((4m)^{4m}(r+1)\big). \qquad \square$$

Corollary. *Let $q > 1$ and let Γ be a finitely generated subgroup of $(\overline{\mathbb{Q}}^\times)^q$ of rank $r \geq 0$. Then the solutions of*

$$z_1 + \cdots + z_q = 0 \tag{3.4}$$

where $\underline{z} = \underline{x} * \underline{y}$ with $\underline{x} \in \Gamma$, $\underline{y} \in \mathbb{Q}^q$ and

$$\mathrm{h}(\underline{y}) \leq \frac{1}{4q^2}\mathrm{h}(\underline{x})$$

are contained in the union of fewer than

$$\exp\big((4q)^{4q}(r+1)\big) \tag{3.5}$$

proper subspaces of the space given by (3.4).

Proof. This is just the homogeneous version of Lemma 3.1. We apply Lemma 3.1 with $m = q - 1$. One needs also to consider the possible solutions with $z_q = 0$. But they lie in one subspace, and 1 is absorbed in (3.5) since $q > m$. $\square$

Lemma 3.2. *Let $\alpha \in \overline{\mathbb{Q}}^\times$ be given with $\mathrm{h_{in}}(\alpha) > 0$. Let $a \in \overline{\mathbb{Q}}^\times$. Then there is a $u \in \mathbb{Z}$ such that*

$$\mathrm{h_{in}}(a\alpha^{x-u}) \geq \frac{1}{4}\mathrm{h_{in}}(\alpha)|x|$$

for $x \in \mathbb{Z}$.

Proof. This is the case $r = n = 1$ of Lemma 15.1 in [6]. $\square$

Agreement. We define the degree of the zero polynomial as -1.

Lemma 3.3. *Consider an equation*

$$P_0(x)\alpha_0^x + \cdots + P_k(x)\alpha_k^x = 0 \tag{3.6}$$

where $(\alpha_0, \ldots, \alpha_k) \in (\overline{\mathbb{Q}}^\times)^{k+1}$ and, for $0 \leq j \leq k$, P_j is a non-zero polynomial of degree $d_j \geq 0$ with algebraic coefficients. Write

$$\Delta = \sum_{j=0}^{k}(d_j + 1), \qquad D = \max_{0 \leq j \leq k} d_j.$$

Suppose that $\Delta \geq 3$,

$$\max_{0 \leq i,j \leq k} \mathrm{h}(\alpha_i : \alpha_j) \geq \not{h}$$

where $0 < \hbar \leq 1$ *and set*

$$E = 16\Delta^2 D/\hbar, \qquad t = \exp\big((5\Delta)^{4\Delta}\big) + 5E \log E.$$

Then there are tuples

$$\big(P_0^{(\ell)}, \ldots, P_k^{(\ell)}\big) \neq (0, \ldots, 0) \qquad (1 \leq \ell \leq t)$$

of polynomials where $\deg P_j^{(\ell)} \leq d_j$ *(*$0 \leq j < k, 1 \leq \ell \leq t$*) and* $\deg P_k^{(\ell)} < d_k$ *for* $\ell = 1, \ldots, t$*, such that every solution* $x \in \mathbb{Z}$ *of (3.6) satisfies*

$$P_0^{(\ell)}(x)\alpha_0^x + \cdots + P_k^{(\ell)}(x)\alpha_k^x = 0 \tag{3.7}$$

for some ℓ.

Proof. Suppose $u \in \mathbb{Z}$ and set $y = x + u$. Then (3.6) may be rewritten as

$$P_0(y-u)\alpha_0^{-u}\alpha_0^y + \cdots + P_k(y-u)\alpha_k^{-u}\alpha_k^y = 0,$$

which is the same as

$$\widetilde{P}_0(y)\alpha_0^y + \cdots + \widetilde{P}_k(y)\alpha_k^y = 0, \tag{3.8}$$

with

$$\widetilde{P}_j(Y) = P_j(Y-u)\alpha_j^{-u} \qquad (0 \leq j \leq k).$$

Suppose our assertion is true for (3.8), with polynomials $\widetilde{P}_0^{(\ell)}, \ldots, \widetilde{P}_k^{(\ell)}$ $(1 \leq \ell \leq t)$. Thus every solution of (3.8) satisfies

$$\widetilde{P}_0^{(\ell)}(y)\alpha_0^y + \cdots + \widetilde{P}_k^{(\ell)}(y)\alpha_k^y = 0$$

for some ℓ. But then $x = y - u$ satisfies (3.7) with

$$P_j^{(\ell)}(X) = \widetilde{P}_j^{(\ell)}(X+u)\alpha_j^u \qquad (0 \leq j \leq k).$$

We therefore may make a change of variables $x \mapsto y = x + u$.

We may suppose that $\mathrm{h}(\alpha_0 : \alpha_\iota) \geq \hbar$ for some ι in the range $1 \leq \iota \leq k$. Write $P_j(X) = a_{j0} + a_{j1}X + \cdots + a_{j,d_j}X^{d_j}$. Pick u according to Lemma 3.2 with $\mathrm{h}(a_{0,d_0}\alpha_0^{y-u} : a_{\iota,d_\iota}\alpha_\iota^{y-u}) \geq \frac{1}{4}\hbar|y|$. Writing $\widetilde{P}_j(Y) = P_j(Y-u)\alpha_j^{-u} = b_{j0} + b_{j1}Y + \cdots + b_{j,d_j}Y^{d_j}$, we have $b_{0,d_0} = a_{0,d_0}\alpha_0^{-u}$, $b_{\iota,d_\iota} = a_{\iota,d_\iota}\alpha_\iota^{-u}$, so that

$$\mathrm{h}(b_{0,d_0}\alpha_0^y : b_{\iota,d_\iota}\alpha_\iota^y) \geq \frac{1}{4}\hbar|y|. \tag{3.9}$$

The equation (3.8) may be written as

$$\big(b_{00} + b_{01}y + \cdots + b_{0,d_0}y^{d_0}\big)\alpha_0^y + \cdots + \big(b_{k0} + b_{k1}y + \cdots + b_{k,d_k}y^{d_k}\big)\alpha_k^y = 0.$$

Some coefficients may be zero; omitting the zero coefficients, we rewrite this as

$$\bigl(b'_{00}y^{v_{00}} + \cdots + b_{0,d_0}y^{d_0}\bigr)\alpha_0^y + \cdots + \bigl(b'_{k0}y^{v_{k0}} + \cdots + b_{k,d_k}y^{d_k}\bigr)\alpha_k^y = 0.$$

Let q be the total number of (non-zero) coefficients here, and consider the following vectors in q-space:

$$\begin{aligned} \underline{x} &= \bigl(b'_{00}\alpha_0^y, \ldots, b_{0,d_0}\alpha_0^y, \ldots, b'_{k0}\alpha_k^y, \ldots, b_{k,d_k}\alpha_k^y\bigr) \\ \underline{w} &= \bigl(y^{v_{00}}, \ldots, y^{d_0}, \ldots, y^{v_{k0}}, \ldots, y^{d_k}\bigr). \end{aligned}$$

Our equation becomes

$$z_1 + \cdots + z_q = 0 \tag{3.10}$$

with $\underline{z} = \underline{x} * \underline{w} = (x_1w_1, \ldots, x_qw_q)$. Hence $\underline{x}$ lies in the group Γ of rank $r \le 2$ generated by $(\alpha_0, \ldots, \alpha_0, \ldots, \alpha_k, \ldots, \alpha_k)$ and $(b'_{00}, \ldots, b_{0,d_0}, \ldots, b'_{k0}, \ldots, b_{k,d_k})$. Further

$$\mathrm{h}(\underline{x}) \ge \mathrm{h}(b_{0,d_0}\alpha_0^y : b_{\iota,d_\iota}\alpha_\iota^y) \ge \frac{1}{4}\hslash|y|$$

by (3.9). On the other hand, $\mathrm{h}(\underline{w}) \le D\log|y|$. Therefore when

$$|y| \ge 2E\log E, \tag{3.11}$$

so that $|y| \ge (32q^2D/\hslash)\log(16q^2D/\hslash)$ by $q \le \Delta$, then

$$|y| > (16q^2D/\hslash)\log|y|,$$

and

$$\mathrm{h}(\underline{w}) \le D\log|y| < \frac{\hslash}{16q^2}|y| = \frac{1}{4q^2}\frac{1}{4}\hslash|y| \le \frac{1}{4q^2}\mathrm{h}(\underline{x}).$$

By the corollary, for such y, we have $\underline{z}$ contained in the union of

$$\exp\bigl((4q)^{4q}\cdot 3\bigr) < \exp\bigl((5\Delta)^{4\Delta}\bigr)$$

proper subspaces of the space (3.10). Consider such a subspace $c_1z_1 + \cdots + c_qz_q = 0$ (where $(c_1, \ldots, c_q)$ is not proportional to $(1, \ldots, 1)$). Taking a linear combination of this and (3.10) we obtain a non-trivial relation $c'_1z_1 + \cdots + c'_{q-1}z_{q-1} = 0$. But this means exactly that y satisfies a non-trivial equation

$$\widetilde{Q}_0(y)\alpha_0^y + \cdots + \widetilde{Q}_k(y)\alpha_k^y = 0, \tag{3.12}$$

where $\deg \widetilde{Q}_j \le d_j$ $(0 \le j < k)$ and $\deg \widetilde{Q}_k < d_k$.

There are not more than $5E\log E$ values of y where (3.11) is violated. For fixed y, and since $\Delta = \sum(d_j + 1) \ge 3$, there will certainly be polynomials $\widetilde{Q}_0, \ldots, \widetilde{Q}_k$ as above ($\deg \widetilde{Q}_j \le d_j$ $(0 \le j < k)$ and $\deg \widetilde{Q}_k < d_k$) with (3.12). □

Remark. When $\hslash \ge \exp\bigl(-(5\Delta)^{4\Delta}\bigr)$ we have $t \le \exp\bigl((5\Delta)^{5\Delta}\bigr)$.

4. A specialization-type argument

Lemma 4.1. *Let K be a number field, D, N, M, $L_1, \dots, L_M$ non-negative integers, $A_1, \dots, A_N$ homogeneous polynomials in $K[X_0, \dots, X_k]$, each of degree $\le D$ and $B_{\lambda\mu}$ $(1 \le \lambda \le L_\mu, 1 \le \mu \le M)$ homogeneous polynomials in $\overline{\mathbb{Q}}[X_0, \dots, X_k]$. Assume that there exists $\underline{\underline{\alpha}} \in \mathbb{P}_k(\mathbb{K})$ such that*

(i) $A_1(\underline{\underline{\alpha}}) = \dots = A_N(\underline{\underline{\alpha}}) = 0$ *and*
(ii) *for each $\mu = 1, \dots, M$, there exists $\lambda \in \{1, \dots, L_\mu\}$ with $B_{\lambda\mu}(\underline{\underline{\alpha}}) \neq 0$.*
Then there exist elements $\widetilde{\alpha}_0, \dots, \widetilde{\alpha}_k$ in $\mathbb{K}$, algebraic over $\mathbb{Q}$, not all of which are zero, which generate an extension $\widetilde{K} = K(\widetilde{\alpha}_0, \dots, \widetilde{\alpha}_k)$ of K of degree $\left[\widetilde{K} : K\right] \le D^k$ and such that the point $\underline{\underline{\widetilde{\alpha}}} = (\widetilde{\alpha}_0 : \dots : \widetilde{\alpha}_k) \in \mathbb{P}_k(\widetilde{K})$ satisfies
(i)$_a$ $A_1(\underline{\underline{\widetilde{\alpha}}}) = \dots = A_N(\underline{\underline{\widetilde{\alpha}}}) = 0$ *and*
(ii)$_a$ *for each $\mu = 1, \dots, M$, there exists $\lambda \in \{1, \dots, L_\mu\}$ with $B_{\lambda\mu}(\underline{\underline{\widetilde{\alpha}}}) \neq 0$.*

Proof. Given homogeneous polynomials $Q_1, \dots, Q_N$ in $\mathbb{K}[X_0, \dots, X_k]$, we write

$$Z(Q_1, \dots, Q_N) \subset \mathbb{P}_k(\mathbb{K})$$

for the set of zeros in $\mathbb{P}_k(\mathbb{K})$ of the ideal $(Q_1, \dots, Q_N)$ in $\mathbb{K}[X_0, \dots, X_k]$ generated by $Q_1, \dots, Q_N$.

Let Y be an absolutely irreducible component of $Z(A_1, \dots, A_N) \subset \mathbb{P}_k(\mathbb{K})$ containing $\underline{\underline{\alpha}}$. Consider the Zariski closed subset

$$F = \bigcap_{\mu=1}^{M} Z(B_{1\mu}, \dots, B_{L_\mu,\mu})$$

of $\mathbb{P}_k(\mathbb{K})$. By assumption $\underline{\underline{\alpha}}$ is not in F. Hence Lemma 4.1 is a consequence of the following statement:

> *Let $A_1, \dots, A_N$ be homogeneous polynomials in $K[X_0, \dots, X_k]$, each of degree $\le D$. Let Y be an irreducible component of dimension δ of*
>
> $$Z(A_1, \dots, A_N)$$
>
> *and F a Zariski closed subset of $\mathbb{P}_k(\mathbb{K})$ such that $Y \setminus F$ is not empty. Then there exists an element $\underline{\underline{\widetilde{\alpha}}} = (\widetilde{\alpha}_0 : \dots : \widetilde{\alpha}_k)$ in $Y \setminus F$ whose components $\widetilde{\alpha}_0, \dots, \widetilde{\alpha}_k$ are algebraic over $\mathbb{Q}$ and such that we have*
>
> $$\left[K(\widetilde{\alpha}_0, \dots, \widetilde{\alpha}_k) : K\right] \le D^{k-\delta}.$$

Since Y is absolutely irreducible and not contained in F, we have $\dim(Y \cap F) \le \delta - 1$. Pick linear forms $L_1, \ldots, L_\delta$ with coefficients in K and in sufficiently general position such that

$$Z(L_1) \cap \cdots \cap Z(L_\delta) \cap F \cap Y = \emptyset$$

and such that moreover

$$Z(L_1) \cap \cdots \cap Z(L_\delta) \cap Y$$

is a non-empty finite set which does not contain more than $D^{k-\delta}$ points. Let $\underline{\underline{\gamma}} = (\gamma_0 : \cdots : \gamma_k)$ be one of its elements. One at least among $\gamma_0, \ldots, \gamma_k$ is non-zero, say γ_0. Put $\widetilde{\alpha}_i = \gamma_i/\gamma_0$. Then our construction implies that $\underline{\underline{\widetilde{\alpha}}} = (1 : \widetilde{\alpha}_1 : \cdots : \widetilde{\alpha}_k) = \underline{\underline{\gamma}}$ lies in $Y \setminus F$. Since our linear forms L_i as well as the polynomials $A_1, \ldots, A_N$ have coefficients in K, it follows that for any K-embedding σ of $K(\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_k)$ in $\mathbb{K}$ we have

$$(1 : \sigma\widetilde{\alpha}_1 : \cdots : \sigma\widetilde{\alpha}_k) \in Z(L_1) \cap \cdots \cap Z(L_\delta) \cap Y.$$

Since moreover the right hand side has cardinality $\le D^{k-\delta}$, we may conclude that in fact $\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_k$ are algebraic over K and that

$$[K(\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_k) : K] \le D^{k-\delta}. \qquad \square$$

Here is a consequence of Lemma 4.1.

Lemma 4.2. *Let k be a non-negative integer, $p, S, T, d_1, \ldots, d_S$ positive integers and $\hslash$ a positive real number. For $1 \le s \le S$, let $\underline{C}_s = (C_{1s}, \ldots, C_{ps})$ be a p-tuple of homogeneous polynomials in $\overline{\mathbb{Q}}[X_0, \ldots, X_k]$, each of degree d_s. For $1 \le t \le T$, let $\underline{D}_t = (D_{1t}, \ldots, D_{pt})$ be a p-tuple of homogeneous polynomials in $\overline{\mathbb{Q}}[X_0, \ldots, X_k]$, with $\deg D_{1t} = \cdots = \deg D_{pt}$. Let $\alpha_0, \ldots, \alpha_k$ be non-zero elements of $\mathbb{K}$ and $\underline{\alpha} = (\alpha_0, \ldots, \alpha_k) \in \mathbb{K}^{k+1}$. Denote by V the subspace of $\mathbb{K}^p$ spanned by $\underline{C}_1(\underline{\alpha}), \ldots, \underline{C}_S(\underline{\alpha})$. Assume that for each $t = 1, \ldots, T$, we have $\underline{D}_t(\underline{\alpha}) \notin V$.*
Then there exist non-zero algebraic elements $\widetilde{\alpha}_0, \ldots, \widetilde{\alpha}_k$ in $\mathbb{K}$ such that

$$\underline{\widetilde{\alpha}} = (\widetilde{\alpha}_0, \ldots, \widetilde{\alpha}_k) \in \overline{\mathbb{Q}}^{k+1}$$

has the following properties. The subspace $\widetilde{V}$ of $\mathbb{K}^p$ spanned by $\underline{C}_1(\underline{\widetilde{\alpha}}), \ldots, \underline{C}_S(\underline{\widetilde{\alpha}})$ has $\dim \widetilde{V} = \dim V$. Further, for each $t = 1, \ldots, T$, we have $\underline{D}_t(\underline{\widetilde{\alpha}}) \notin \widetilde{V}$. Furthermore, for $0 \le i, j \le k$, we have

$$\begin{cases} \widetilde{\alpha}_i/\widetilde{\alpha}_j = \alpha_i/\alpha_j & \text{if } \alpha_i/\alpha_j \text{ is algebraic,} \\ \mathrm{h}(\widetilde{\alpha}_i : \widetilde{\alpha}_j) \ge \hslash & \text{if } \alpha_i/\alpha_j \text{ is transcendental.} \end{cases}$$

Proof. Let K denote a number field containing all coefficients of C_{is} ($1 \leq i \leq p$, $1 \leq s \leq S$) and all algebraic elements of $\mathbb{K}$ which belong to the set $\left\{\alpha_i/\alpha_j\,;\ 0 \leq i,\, j \leq k\right\}$. We shall prove the existence of $\underline{\widetilde{\alpha}} = (\widetilde{\alpha}_0, \ldots, \widetilde{\alpha}_k) \in \mathbb{K}^{k+1}$ satisfying the desired properties together with an upper bound for the degree of the number field $\widetilde{K} = K(\widetilde{\alpha}_0, \ldots, \widetilde{\alpha}_k)$, namely

$$\left[\widetilde{K} : K\right] \leq D^k \quad \text{with} \quad D = p \max_{1 \leq s \leq S} d_s.$$

Define $r = \dim V$. Since $\underline{D}_t(\underline{\alpha})$ is not in V, we have $V \neq \mathbb{K}^p$, hence $r < p$. Denote by $\left\{A_1, \ldots, A_J\right\}$ the set of $(r+1) \times (r+1)$ minors of the $p \times S$ matrix

$$\left(\underline{C}_1, \ldots, \underline{C}_S\right).$$

Each of these polynomials $A_1, \ldots, A_J$ is homogeneous of degree

$$\leq (r+1) \max_{1 \leq s \leq S} d_s \leq D.$$

Also, for $1 \leq t \leq T$, denote by $\left\{B_{1t}, \ldots, B_{Lt}\right\}$ the set of $(r+1) \times (r+1)$ minors of the $p \times (S+1)$ matrix

$$\left(\underline{C}_1, \ldots, \underline{C}_S,\ \underline{D}_t\right).$$

Further, let $\left\{A_{J+1}, \ldots, A_N\right\}$ denote the set of polynomials $\alpha_i X_j - \alpha_j X_i$ where (i, j) runs over the set of pairs with $0 \leq i, j \leq k$ for which α_i/α_j is algebraic. Furthermore, denote by $\left\{B_{T+1}, \ldots, B_M\right\}$ the set of polynomials $X_0, \ldots, X_k$, and $\beta X_i - X_j$, where (i, j) runs over the set of pairs with $0 \leq i, j \leq k$ for which α_i/α_j is transcendental, while β runs over the (finite) set of algebraic elements of $\mathbb{K}$ for which

$$[K(\beta) : K] \leq D^k \quad \text{and} \quad \mathrm{h}_{\mathrm{in}}(\beta) \leq \hslash.$$

By assumption the point $\underline{\alpha} \in \mathbb{K}^{k+1}$ satisfies

$$A_1(\underline{\alpha}) = \cdots = A_N(\underline{\alpha}) = 0,$$

$$B_\mu(\underline{\alpha}) \neq 0 \quad \text{for} \quad T+1 \leq \mu \leq M,$$

and for each $\mu = 1, \ldots, T$, there exists $\lambda \in \{1, \ldots, L\}$ such that $B_{\lambda\mu}(\underline{\alpha}) \neq 0$.

From Lemma 4.1 we deduce that there exists $\underline{\widetilde{\alpha}} \in \overline{\mathbb{Q}}^{k+1}$ such that

$$\left[K(\widetilde{\alpha}_0, \ldots, \widetilde{\alpha}_k) : K\right] \leq D^k,$$

$$A_1(\underline{\widetilde{\alpha}}) = \cdots = A_N(\underline{\widetilde{\alpha}}) = 0,$$

$$B_\mu(\underline{\widetilde{\alpha}}) \neq 0 \quad \text{for} \quad T+1 \leq \mu \leq M,$$

and for each $\mu = 1, \ldots, T$, there exists $\lambda \in \{1, \ldots, L\}$ such that $B_{\lambda\mu}(\underline{\widetilde{\alpha}}) \neq 0$. This $\underline{\widetilde{\alpha}}$ then satisfies all desired properties. □

We apply Lemma 4.2 to exponential polynomials.

Lemma 4.3. *Let $k \geq 1$ be an integer, $\hbar$ a positive real number, $d_0, \dots, d_k$ non-negative integers and $\alpha_0, \dots, \alpha_k$ non-zero elements of $\mathbb{K}$ satisfying (1.2). For $0 \leq j \leq k$, let*

$$P_j(X) = \sum_{i=0}^{d_j} a_{ij} X^i$$

be a non-zero polynomial in $\mathbb{K}[X]$ of degree d_j. Define

$$f(x) = \sum_{j=0}^{k} P_j(x)\alpha_j^x$$

and denote by $\mathcal{N}$ the set of solutions $x \in \mathbb{Z}$ of the equation $f(x) = 0$. Let $\mathcal{E}$ be a finite subset of $\mathbb{Z}$. Assume that for each $x \in \mathcal{E}$ we are given a subset $I(x)$ of $\{(i, j)\,;\ 0 \leq i \leq d_j,\ 0 \leq j \leq k\}$ for which

$$\sum_{(i,j)\in I(x)} a_{ij} x^i \alpha_j^x \neq 0. \tag{4.1}$$

Then there exist non-zero algebraic elements $\widetilde{\alpha}_0, \dots, \widetilde{\alpha}_k$ of $\mathbb{K}$ and there exist polynomials $\widetilde{P}_0, \dots, \widetilde{P}_k$ which are not all zero,

$$\widetilde{P}_j(X) = \sum_{i=0}^{d_j} \widetilde{a}_{ij} X^i \qquad (0 \leq j \leq k),$$

with algebraic coefficients $\widetilde{a}_{ij}$, and with the following properties:

$$\deg \widetilde{P}_j \leq d_j \qquad (0 \leq j \leq k) \tag{4.2}$$

$$\sum_{j=0}^{k} \widetilde{P}_j(x)\widetilde{\alpha}_j^x = 0 \quad \textit{for all } x \in \mathcal{N}, \tag{4.3}$$

$$\sum_{(i,j)\in I(x)} \widetilde{a}_{ij} x^i \widetilde{\alpha}_j^x \neq 0 \quad \textit{for each} \quad x \in \mathcal{E}, \tag{4.4}$$

and, for $0 \leq i, j \leq k$,

$$\begin{cases} \widetilde{\alpha}_i/\widetilde{\alpha}_j = \alpha_i/\alpha_j & \textit{if } \alpha_i/\alpha_j \textit{ is algebraic}, \\ \mathrm{h}(\widetilde{\alpha}_i : \widetilde{\alpha}_j) \geq \hbar & \textit{if } \alpha_i/\alpha_j \textit{ is transcendental}. \end{cases} \tag{4.5}$$

Proof. We fix an ordering of the set $I = \{(i, j)\,;\ 0 \le i \le d_j,\ 0 \le j \le k,\ a_{ij} \ne 0\}$ and we denote by p the number of elements in this set. Also we write $\mathcal{N} = \{n_1, \dots, n_S\}$ (recall that $\mathcal{N}$ is finite) and $\mathcal{E} = \{x_1, \dots, x_T\}$. For $1 \le s \le S$, we define $\underline{C}_s$ as the p-tuple composed of the polynomials $n_s^i X_j^{n_s}$ for $(i, j) \in I$. For $1 \le t \le T$, let $\underline{D}_t$ be the p-tuple composed of the polynomials

$$\begin{cases} x_t^i X_j^{x_t} & \text{for } (i, j) \in I \cap I(x_t) \\ 0 & \text{for } (i, j) \in I \setminus I(x_t). \end{cases}$$

From the definition of $\mathcal{N}$ we deduce that the dimension r of the vector space V spanned by $\underline{C}_1(\underline{\alpha}), \dots, \underline{C}_S(\underline{\alpha})$ satisfies $r < p$. According to (4.1), for each $t = 1, \dots, T$ we have $\underline{D}_t(\underline{\alpha}) \notin V$. Therefore Lemma 4.3 follows from Lemma 4.2. □

Remark. Let K denote the field generated over $\mathbb{Q}$ by all algebraic elements which belong to the set $\{\alpha_i/\alpha_j\,;\ 0 \le i, j \le k\}$. The proof of Lemma 4.3 also yields an upper bound for the degree of the number field $\widetilde{K} = K(\widetilde{\alpha}_0, \dots, \widetilde{\alpha}_k)$, namely

$$\left[\widetilde{K} : K\right] \le \left(\Delta \max_{x \in \mathcal{N}} |x|\right)^k$$

with $\Delta = d_1 + \cdots + d_k + k + 1$. One may prove a variant of Lemma 4.3 where (4.3) holds only for some subset $\mathcal{N}'$ of $\mathcal{N}$ with $\operatorname{Card} \mathcal{N}' / \operatorname{Card} \mathcal{N} \ge 1/(k+1)$ but with the estimate

$$\left[\widetilde{K} : K\right] \le \left(\Delta \min_{x \in \mathcal{N}'} |x|\right)^k.$$

5. Dividing exponential polynomials

Let $\alpha_0, \dots, \alpha_k$ be given non-zero elements of $\mathbb{K}$ satisfying (1.2) and $P_0, \dots, P_k$ be polynomials with coefficients in $\mathbb{K}$, possibly zero. Consider the exponential polynomial

$$f(x) = \sum_{j=0}^{k} P_j(x) \alpha_j^x.$$

We set

$$\Delta(f) = \sum_{\substack{j=0 \\ P_j \ne 0}}^{k} (\deg P_j + 1).$$

Thus $\Delta(f) = 0$ precisely when $P_0 = \cdots = P_k = 0$. When

$$g(x) = \sum_{j=0}^{k} Q_j(x) \alpha_j^x$$

is another exponential polynomial with the same frequencies $(\alpha_0, \cdots, \alpha_k)$, we write $g \prec f$ if $\deg Q_j \leq \deg P_j$ for $0 \leq j \leq k$. We write $g \ll f$ if $g \prec f$ and $\Delta(g) < \Delta(f)$.

Lemma 5.1. *Suppose $g \prec f$ and $g \neq 0$. Then there is an exponential polynomial*

$$r(x) = R_0(x)\alpha_0^x + \cdots + R_k(x)\alpha_k^x$$

with $r \ll f$ such that

$$f(x) = r(x) + cx^n g(x)$$

for some c in $\mathbb{K}^\times$ and some $n \geq 0$.

Proof. With f and g written as above, set

$$n = \min_{\substack{0 \leq j \leq k \\ Q_j \neq 0}} (\deg P_j - \deg Q_j).$$

We may suppose $n = \deg P_0 - \deg Q_0$. When

$$P_0 = c_a X^a + c_{a-1} X^{a-1} + \cdots, \qquad Q_0 = d_b X^b + d_{b-1} X^{b-1} + \cdots,$$

where now $a = b + n$, set $c = c_a / d_b$ and

$$r(x) = f(x) - cx^n g(x).$$

If again $r(x) = R_0(x)\alpha_0^x + \cdots + R_k(x)\alpha_k^x$, we have

$$R_0(X) = P_0(X) - (c_a/d_b)x^n Q_0(X),$$

so that $\deg R_0 < \deg P_0$. Also $\deg R_j \leq \max(\deg P_j, n + \deg Q_j) \leq \deg P_j$, so that $r \ll f$. □

Consider an exponential polynomial

$$f(x) = \sum_{j=0}^{k} P_j(x)\alpha_j^x$$

where $\alpha_0, \ldots, \alpha_k$ are non-zero algebraic elements in $\mathbb{K}$ satisfying (1.2). Assume

$$\{\alpha_0, \ldots, \alpha_k\} = \bigcup_{i=1}^{m} \{\alpha_{i0} : \cdots : \alpha_{ik_i}\}$$

is a partition of $\{\alpha_0, \ldots, \alpha_k\}$ and define

$$f_i(x) = P_{i0}(x)\alpha_{i0}^x + \cdots + P_{ik_i}(x)\alpha_{ik_i}^x \qquad (1 \leq i \leq m)$$

so that

$$f(x) = f_1(x) + \cdots + f_m(x).$$

Suppose further, for $1 \le i \ne u \le m$, $0 \le j \le k_i$ and $0 \le v \le k_u$,

$$\mathrm{h}_{\mathrm{in}}(\alpha_{ij}/\alpha_{uv}) \ge 1. \tag{5.1}$$

From (1.2) we deduce

$$\Delta(f) = \Delta(f_1) + \cdots + \Delta(f_m).$$

Set

$$\Delta = \Delta(f)$$

Lemma 5.2. *Define*

$$F(\Delta) = \exp\bigl(\Delta(5\Delta)^{5\Delta}\bigr).$$

Then for all but at most $F(\Delta)$ solutions $x \in \mathbb{Z}$ of $f(x) = 0$, we have

$$f_1(x) = \cdots = f_m(x) = 0. \tag{5.2}$$

Proof. The lemma is non-trivial only when $m \ge 2$ and at least two of $f_1, \dots, f_m$ are non-zero, so that $\Delta \ge 2$. We now proceed by induction on Δ. When $\Delta = 2$ and $m \ge 2$, we have in fact $f(x) = a\alpha_{10}^x + b\alpha_{20}^x$ with $ab \ne 0$ and $\mathrm{h}_{\mathrm{in}}(\alpha_{10}/\alpha_{20}) \ge 1$, so that α_{10}/α_{20} is not a root of 1. There can be at most one zero x of f, for if $f(x) = f(y) = 0$, then $(\alpha_{10}/\alpha_{20})^x = (\alpha_{10}/\alpha_{20})^y = -b/a$, so that $(\alpha_{10}/\alpha_{20})^{x-y} = 1$ hence $x = y$ since α_{10}/α_{20} is not a root of 1.

Now assume $\Delta \ge 3$. In the induction step we apply Lemma 3.3 with $\not{h} = 1$. The condition $\max_{0 \le i,j \le k} \mathrm{h}(\alpha_i : \alpha_j) \ge 1$ is satisfied because $m \ge 2$. Any $x \in \mathbb{Z}$ with $f(x) = 0$ satisfies a relation

$$f^{(\ell)}(x) = 0$$

for some ℓ in the range $1 \le \ell \le t$ where $t = \exp\bigl((5\Delta)^{5\Delta}\bigr)$ and each $f^{(\ell)} \ne 0$ has $f^{(\ell)} \ll f$. By Lemma 5.1 we have, for $1 \le \ell \le t$

$$f(x) = r^{(\ell)}(x) + c^{(\ell)} x^{n^{(\ell)}} f^{(\ell)}(x)$$

with $r^{(\ell)} \ll f$. Write out

$$\begin{aligned} f^{(\ell)}(x) &= f_1^{(\ell)}(x) + \cdots + f_m^{(\ell)}(x), \\ r^{(\ell)}(x) &= r_1^{(\ell)}(x) + \cdots + r_m^{(\ell)}(x) \end{aligned}$$

with

$$\begin{aligned} f_i^{(\ell)}(x) &= P_{i0}^{(\ell)}(x)\alpha_{i0}^x + \cdots + P_{ik_i}^{(\ell)}(x)\alpha_{ik_i}^x, \\ r_i^{(\ell)}(x) &= R_{i0}^{(\ell)}(x)\alpha_{i0}^x + \cdots + R_{ik_i}^{(\ell)}(x)\alpha_{ik_i}^x \end{aligned}$$

and

$$f_i(x) = r_i^{(\ell)}(x) + c^{(\ell)} x^{n^{(\ell)}} f_i^{(\ell)}(x). \tag{5.3}$$

By induction, and since $f^{(\ell)} \ll f$ and $r^{(\ell)} \ll f$, hence $\Delta(f^{(\ell)}) < \Delta(f)$, $\Delta(r^{(\ell)}) < \Delta(f)$, we see that all but at most $F(\Delta-1)$ solutions of $f^{(\ell)}(x) = 0$ have

$$f_1^{(\ell)}(x) = \cdots = f_m^{(\ell)}(x) = 0, \tag{5.4}$$

and similarly all but at most $F(\Delta-1)$ solutions of $r^{(\ell)}(x) = 0$ have

$$r_1^{(\ell)}(x) = \cdots = r_m^{(\ell)}(x) = 0. \tag{5.5}$$

But (5.3), (5.4) and (5.5) imply (5.2). Taking the sum over ℓ in $1 \le \ell \le t$, we see that all but at most

$$2tF(\Delta-1) \le \exp\bigl(1 + (5\Delta)^{5\Delta} + (\Delta-1)(5\Delta)^{5\Delta-5}\bigr) \le F(\Delta)$$

solutions of $f(x) = 0$ have (5.2). □

6. Proof of Theorem 1.1

Assume that the assumptions of Theorem 1.1 are satisfied. Let $\mathcal{E}$ be a set of more than $F(\Delta)$ solutions of (1.1). Assume that for each x in $\mathcal{E}$ there is an index $i = i(x)$ in the range $1 \le i \le m$ such that $f_{i(x)}(x) \neq 0$.

We apply Lemma 4.3 with $\hbar = 1$. We produce algebraic elements $\widetilde{\alpha}_0, \dots, \widetilde{\alpha}_k$ and polynomials with algebraic coefficients $\widetilde{P}_0, \dots, \widetilde{P}_k$ satisfying (4.2), (4.3), (4.4) and (4.5). The exponential polynomial

$$\widetilde{f}(x) = \sum_{j=0}^{k} \widetilde{P}_j(x)\widetilde{\alpha}_j^x$$

can be written

$$\widetilde{f}(x) = \widetilde{f}_1(x) + \cdots + \widetilde{f}_m(x)$$

where, for $1 \le i \le m$,

$$\widetilde{f}_i(x) = \sum_{j=0}^{k_i} \widetilde{P}_{ij}(x)\widetilde{\alpha}_{ij}^x$$

and, for $1 \le i \neq u \le m$, $0 \le j \le k_i$ and $0 \le v \le k_u$,

$$\mathrm{h}_{\mathrm{in}}(\widetilde{\alpha}_{ij}/\widetilde{\alpha}_{uv}) \ge 1.$$

We apply Lemma 5.2 and deduce that one at least of x in $\mathcal{E}$ satisfies $\widetilde{f}_{i(x)}(x) = 0$, which is a contradiction with (4.4). □

Final remark. The proof of Theorem 1.1 yields a stronger result. Fix $\hbar$ with $0 < \hbar \le 1$. If we replace the assumption that α_{i0}/α_{u0} is transcendental by the assumption that either it is transcendental, or else has height $\ge \hbar$, then we get the same conclusion but with $F(\Delta)$ replaced by a function of Δ and $\hbar$, which is equal to $F(\Delta)$ when $\hbar = 1$.

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