Oregon State University, College of Science, Department of Mathematics
Algebra & Number Theory Seminar
On line

On Markoff Numbers

Michel Waldschmidt

Professeur Émérite, Sorbonne Université,
Institut de Mathématiques de Jussieu, Paris
http://www.imj-prg.fr/~michel.waldschmidt/

Abstract

It is easy to check that the equation

$$x^2 + y^2 + z^2 = 3xyz,$$

where the three unknowns x, y, z are positive integers, has infinitely many solutions. There is a simple algorithm which produces all of them. However, this does not answer to all questions on this equation : in particular, Frobenius asked whether it is true that for each integer $z > 0$, there is at most one pair (x, y) such that $x < y < z$ and (x, y, z) is a solution. This question is an active research topic nowadays.

Abstract (continued)

Markoff's equation occurred initially in the study of minima of quadratic forms at the end of the XIX–th century and the beginning of the XX–th century. It was investigated by many a mathematician, including Lagrange, Hermite, Korkine, Zolotarev, Markoff, Frobenius, Hurwitz, Cassels. The solutions are related with the *Lagrange-Markoff spectrum*, which consists of those quadratic numbers which are badly approximable by rational numbers. It occurs also in other parts of mathematics, in particular free groups, Fuchsian groups and hyperbolic Riemann surfaces (Ford, Lehner, Cohn, Rankin, Conway, Coxeter, Hirzebruch and Zagier...).

We discuss some aspects of this topic without trying to cover all of them.

The sequence of Markoff numbers

A *Markoff number* is a positive integer z such that there exist two positive integers x and y satisfying

$$x^2 + y^2 + z^2 = 3xyz.$$

For instance 1 is a Markoff number, since $(x, y, z) = (1, 1, 1)$ is a solution.

Andrei Andreyevich Markoff
1856 – 1922

Photos :
http://www-history.mcs.st-andrews.ac.uk/history/

The On-Line Encyclopedia of Integer Sequences

1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610, 985, 1325, 1597, 2897, 4181, 5741, 6466, 7561, 9077, 10946, 14701, 28657, 33461, 37666, 43261, 51641, 62210, 75025, 96557, 135137, 195025, 196418, 294685, . . .

The sequence of Markoff numbers is available on the web
The On-Line Encyclopedia of Integer Sequences

Neil J. A. Sloane

http://oeis.org/A002559
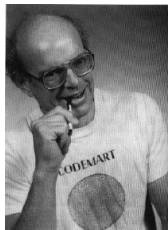
Integer points on a surface

Given a Markoff number z, there exist infinitely many pairs of positive integers x and y satisfying

$$x^2 + y^2 + z^2 = 3xyz.$$

This is a cubic equation in the 3 variables (x, y, z), of which we know a solution $(1, 1, 1)$.

There is an algorithm producing all integer solutions.

Markoff's cubic variety

The surface defined by Markoff's equation

$$x^2 + y^2 + z^2 = 3xyz.$$

is an algebraic variety with many automorphisms : permutations of the variables, changes of signs and

$$(x, y, z) \mapsto (3yz - x, y, z).$$

A.A. Markoff

1856–1922

Algorithm producing all solutions

Let (m, m_1, m_2) be a solution of Markoff's equation :

$$m^2 + m_1^2 + m_2^2 = 3mm_1m_2.$$

Fix two coordinates of this solution, say m_1 and m_2. We get a quadratic equation in the third coordinate m, of which we know a solution, hence, the equation

$$x^2 + m_1^2 + m_2^2 = 3xm_1m_2.$$

has two solutions, $x = m$ and, say, $x = m'$, with $m + m' = 3m_1m_2$ and $mm' = m_1^2 + m_2^2$. This is the *cord and tangent process.*

Hence, *another* solution is (m', m_1, m_2) with $m' = 3m_1m_2 - m$.

Three solutions derived from one

Starting with one solution (m, m_1, m_2), we derive (most often) three *new* solutions :

$$(m', m_1, m_2), \quad (m, m_1', m_2), \quad (m, m_1, m_2').$$

If the solution we start with is $(1, 1, 1)$, we produce only one *new* solution, $(2, 1, 1)$ (up to permutation).

If we start from $(2, 1, 1)$, we produce only two *new* solutions, $(1, 1, 1)$ and $(5, 2, 1)$ (up to permutation).

A *new* solution means *distinct from the one we start with.*

New solutions

We shall see that any solution different from $(1, 1, 1)$ and from $(2, 1, 1)$ yields three new different solutions – and we shall see also that, in each other solution, the three numbers m, m_1 and m_2 are pairwise distinct.

Two solutions are called *neighbors* if they share two components.

For instance

- $(1, 1, 1)$ has a single neighbor, namely $(2, 1, 1)$,
- $(2, 1, 1)$ has two neighbors : $(1, 1, 1)$ et $(5, 2, 1)$,
- any other solution has exactly three neighbors.

A new solution from an old one

Let (m, m_1, m_2) be a solution of Markoff's equation

$$m^2 + m_1^2 + m_2^2 = 3mm_1m_2.$$

Denote by m' the other root of the quadratic polynomial

$$X^2 - 3m_1m_2X + m_1^2 + m_2^2.$$

Then (m', m_1, m_2) is again a solution.
From

$$X^2 - 3m_1m_2X + m_1^2 + m_2^2 = (X - m)(X - m')$$

we deduce

$$m + m' = 3m_1m_2, \quad mm' = m_1^2 + m_2^2.$$

$m_1 \neq m_2$

Let us check that *if $m_1 = m_2$, then $m_1 = m_2 = 1$:*
this holds only for the two exceptional solutions $(1, 1, 1)$,
$(2, 1, 1)$.

Assume $m_1 = m_2$. We have

$$m^2 + 2m_1^2 = 3mm_1^2 \quad \text{hence} \quad m^2 = (3m - 2)m_1^2.$$

Therefore m_1 divides m. Let $m = km_1$. We have
$k^2 = 3km_1 - 2$, hence k divise 2.
For $k = 1$ we get $m = m_1 = 1$.
For $k = 2$ we get $6m_1 = 6$, $m_1 = 1$, $m = 2$.

Consider now a solution distinct from $(1, 1, 1)$ or $(2, 1, 1)$:
hence $m_1 \neq m_2$.

Two larger, one smaller

Assume $m_2 > m_1$.

Consider the number $a = (m_2 - m)(m_2 - m')$.

Since $m + m' = 3m_1 m_2$, and $mm' = m_1^2 + m_2^2$, we have

$$\begin{aligned}
a &= m_2^2 - m_2(m + m') + mm' \\
&= 2m_2^2 + m_1^2 - 3m_1 m_2^2 \\
&= (2m_2^2 - 2m_1 m_2^2) + (m_1^2 - m_1 m_2^2).
\end{aligned}$$

However $2m_2^2 < 2m_1 m_2^2$ and $m_1^2 < m_1 m_2^2$, hence $a < 0$.

This means that $\boxed{m_2 \text{ is in the interval defined by } m \text{ and } m'.}$

Order of the solutions

We order the solution according to the largest coordinate.

If $m < m_2$, we have $m_2 < m'$ and the new solution (m', m_1, m_2) is larger than the initial solution (m, m_1, m_2).

If $m > m_2$, we have $m_2 > m'$ and the new solution (m', m_1, m_2) is smaller than the initial solution (m, m_1, m_2).

$$m < m_1 < m_2 \implies m_1 < m_2 < m' \qquad \text{larger solution}$$
$$m_1 < m < m_2 \implies m_1 < m_2 < m' \qquad \text{larger solution}$$
$$m_1 < m_2 < m \implies m' < m_2 \qquad \text{smaller solution.}$$

Markoff's tree

If we start with (m, m_1, m_2) satisfying $m > m_2 > m_1$, the three new solutions

$$(m', m_1, m_2), \quad (m, m_1', m_2), \quad (m, m_1, m_2').$$

have

$$m_1' > m_2' > m > m'.$$

Then two of the neighbors of (m, m_1, m_2) are larger than the initial solution, the third one is smaller.

Hence, if we start from $(1, 1, 1)$, we produce infinitely many solutions, which we organize in a tree : this is *Markoff's tree*.

This algorithm yields all the solutions

Conversely, starting from any solution other than $(1, 1, 1)$, the algorithm produces a *smaller* solution.

Hence, by induction, we get a sequence of smaller and smaller solutions, until we reach $(1, 1, 1)$.

Therefore the solution we started from was in Markoff's tree.
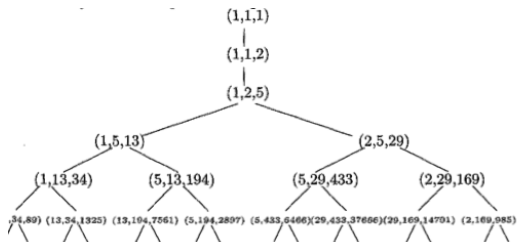
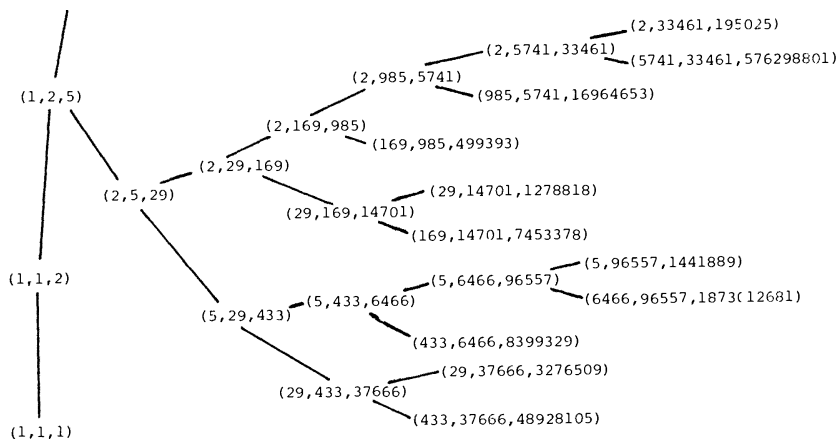# First branches of Markoff's tree

Figure 10. The Tree of Markoff Solutions.

Richard Guy *Unsolved problems in number theory.* Problem books in mathematics, Springer Verlag 1994. Chap. D12 : Markoff numbers.

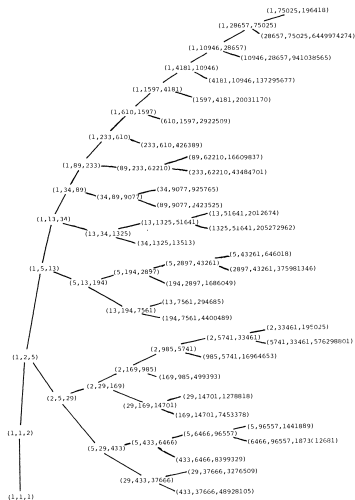# Markoff's tree starting from $(2, 5, 29)$

Markoff's tree up to $100\,000$

FIGURE 2
Markoff triples (p, q, r) with $\max(p, q) \leqslant 100000$

Don Zagier,
On the number of Markoff numbers below a given bound.
Mathematics of Computation,
39 160 (1982), 709–723.

Don Zagier

Continued fractions and the Markoff tree

$$(1,1,1) - (1,1,2) - (2,1,5) \begin{cases} (5,1,13) \begin{cases} (13,1,34) < \begin{array}{l} (34,1,89) < \vdots \\ (13,34,1325) < \vdots \end{array} \\ (5,13,194) < \begin{array}{l} (194,13,7561) < \vdots \\ (5,194,2897) < \vdots \end{array} \end{cases} \\ (2,5,29) \begin{cases} (29,5,433) < \begin{array}{l} (433,5,6466) < \vdots \\ (29,433,37666) < \vdots \end{array} \\ (2,29,169) < \begin{array}{l} (169,29,14701) < \vdots \\ (2,169,985) < \vdots \end{array} \end{cases} \end{cases}$$
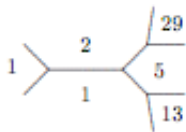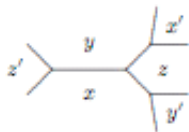
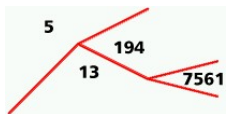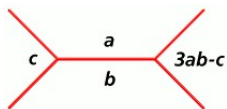E. Bombieri,

Continued fractions and the Markoff tree,

Expo. Math. **25** (2007),

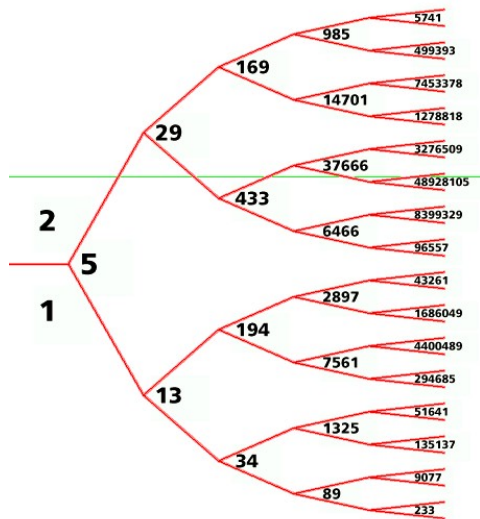Enrico Bombieri   no. 3, 187–213.

$$a^2 + b^2 + c^2 = 3abc$$

$$X^2 - 3abX + a^2 + b^2 =$$
$$(X - c)(X - 3ab + c)$$

$$x' = 3yz - x \qquad y' = 3xz - y \qquad z' = 3xy - z$$

Markoff's tree

Ying Zhang, Congruence and Uniqueness of Certain Markoff Numbers,
Acta Arithmetica **128** 3 (2007), 295-301.

Growth of Markoff's sequence

To identify primitive words in a free group with two generators, H. Cohn (1978) used *Markoff* forms.

Harvey Cohn
1923 – 2014

Order of magnitude of m, m_1 and m_2 for $m^2 + m_1^2 + m_2^2 = 3mm_1m_2$ with $m_1 < m_2 < m$,

$$3mm_1m_2 > m^2 > m_2^2 > m_1^2, \qquad m \sim 3m_1m_2,$$
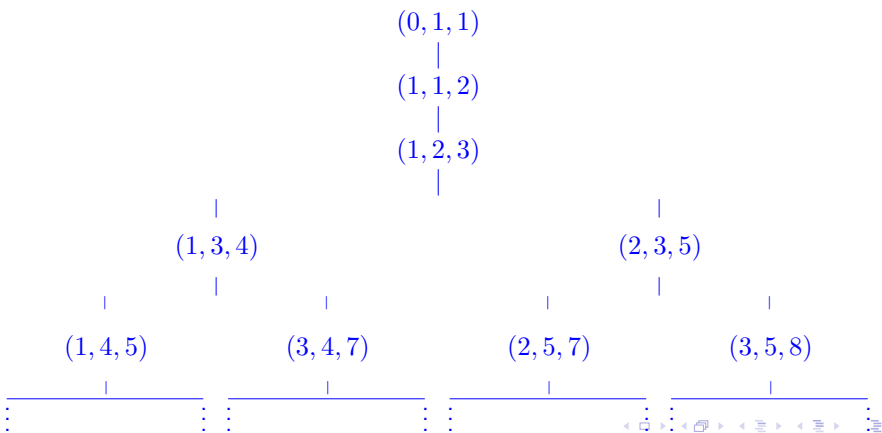
$$\log(3m_1) + \log(3m_2) = \log(3m) + o(1).$$

$$x \mapsto \log(3x) : (m_1, m_2, m) \mapsto (a, b, c) \text{ with } a + b \sim c.$$

Euclidean tree

Start with $(0, 1, 1)$. From a triple (a, b, c) satisfying $a + b = c$ and $a \le b \le c$, one produces two larger such triples $(a, c, a + c)$ and $(b, c, b + c)$ and a smaller one $(a, b - a, b)$ or $(b - a, a, b)$.

The Markoff tree and the Euclide tree

$(1, 13, 34)$ $(5, 13, 194)$ $(2, 29, 169)$ $(5, 29, 433)$ $(1, 4, 5)$ $(3, 4, 7)$ $(2, 5, 7)$ $(3, 5, 8)$

$(1, 15, 13)$ $(2, 5, 29)$ $(1, 3, 4)$ $(2, 3, 5)$

$\xrightarrow{\psi}$

$(1, 2, 5)$ $(1, 2, 3)$
$|$ $|$
$(1, 1, 2)$ $(1, 1, 2)$
$|$ $|$
$(1, 1, 1)$ $(0, 1, 1)$

M E
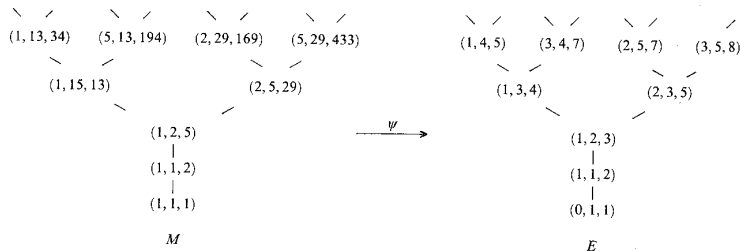
FIGURE 2. THE MARKOFF TREE AND THE EUCLID TREE

Tom Cusik &
Mary Flahive,
*The Markoff— and
Lagrange spectra*,
Math. Surveys and
Monographs **30**, AMS
(1989).

Thomas Cusik

Mary Flahive

Growth of Markoff's sequence

Don Zagier (1982) :
estimating the number of the
Markoff triples bounded by x :

$$C(\log x)^2 + O\big(\log x (\log\log x)^2\big),$$

$$C = 0.180\,717\,0\ldots$$

Don Zagier

Conjecture : the n-th Markoff number m_n is

$$m_n \sim A^{\sqrt{n}} \quad \text{with} \quad A = 10.510\,150\,4\cdots$$

Remark : $C = 1/(\log A)^2$.

Zagier's constant

THEOREM. *The number of Markoff triples (p, q, r) with $p \leqslant q \leqslant r \leqslant x$ is given by*

$$(2) \qquad \mathsf{M}(x) = C(\log x)^2 + O\big(\log x (\log \log x)^2\big) \qquad (x \to \infty),$$

where

$$(3) \qquad C = \frac{3}{\pi^2} \sum_{(p,q,r) \in \mathfrak{M}}^{*} \frac{f(p) + f(q) - f(r)}{f(p)f(q)f(r)} \approx 0.18071704711507;$$

here $\sum^*$ means that the two Markoff triples $(1, 1, 1)$ and $(1, 1, 2)$ with $p = q$ are to be counted with multiplicity $\frac{1}{2}$, and $f(x)$ is the function

$$(4) \qquad f(x) = \log \frac{3x + \sqrt{9x^2 - 4}}{2} = \operatorname{arc\,cosh} \frac{3x}{2} \qquad \left(x \geqslant \frac{2}{3}\right).$$

The Fibonacci sequence and the Markoff equation

The smallest Markoff number is 1. When we impose $z = 1$ in the Markoff equation $x^2 + y^2 + z^2 = 3xyz$, we obtain the equation

$$x^2 + y^2 + 1 = 3xy.$$

Going along Markoff's tree starting from $(1, 1, 1)$, we obtain the subsequence of Markoff numbers

$$1, \ 2, \ 5, \ 13, \ 34, \ 89, \ 233, \ 610, \ 1597, 4181, \ 10946, \ 28657, \ldots$$

which is the sequence of Fibonacci numbers with odd indices

$$F_1 = 1, \ F_3 = 2, \ F_5 = 5, \ F_7 = 13, \ F_9 = 34, \ F_{11} = 89, \ \ldots$$

Leonardo Pisano (Fibonacci)

The Fibonacci sequence
$(F_n)_{n \geq 0}$:

$0, \; 1, \; 1, \; 2, \; 3, \; 5, \; 8, \; 13, \; 21,$

$34, \; 55, \; 89, \; 144, \; 233 \ldots$

is defined by

$$F_0 = 0, \; F_1 = 1,$$

$$F_n = F_{n-1} + F_{n-2} \quad (n \geq 2).$$

Leonardo Pisano (Fibonacci)
$1170 - 1250$

Encyclopedia of integer sequences (again)

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, 514229, 832040, 1346269, 2178309, 3524578, 5702887, 9227465, . . .

The Fibonacci sequence is available online
The On-Line Encyclopedia of Integer Sequences

Neil J. A. Sloane

http://oeis.org/A000045

Encyclopedia of integer sequences A000045

D. E. Knuth writes: "Before Fibonacci wrote his work, the sequence F_{n} had already been discussed by Indian scholars, who had long been interested in rhythmic patterns that are formed from one-beat and two-beat notes. The number of such rhythms having n beats altogether is F_{n+1}; therefore both Gopāla (before 1135) and Hemachandra (c. 1150) mentioned the numbers 1, 2, 3, 5, 8, 13, 21, ... explicitly." (TAOCP Vol. 1, 2nd ed.) – Peter Luschny, Jan 11 2015

In keeping with historical accounts (see the references by P. Singh and S. Kak), the generalized Fibonacci sequence a, b, a + b, a + 2b, 2a + 3b, 3a + 5b, ... can also be described as the Gopala-Hemachandra numbers H(n) = H(n−1) + H(n−2), with F(n) = H(n) for a = b = 1, and Lucas sequence L(n) = H(n) for a = 2, b = 1. – Lekraj Beedassy, Jan 11 2015

Susantha Goonatilake writes: "[T]his sequence was well known in South Asia and used in the metrical sciences. Its development is attributed in part to Pingala (200 BC), later being associated with Virahanka (circa 700 AD), Gopala (circa 1135), and Hemachandra (circa 1150)—all of whom lived and worked prior to Fibonacci." (Toward a Global Science: Mining Civilizational Knowledge, p. 126) – Russ Cox, Sep 08 2021

Also sometimes called Hemachandra numbers.

Fibonacci numbers with odd indices

Fibonacci numbers with odd indices are Markoff's numbers.

(Cassini identity 1680)

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n.$$

Proof :

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.$$

Giovanni Domenico Cassini
1625 – 1712

Replace n by $2n$:

$$F_{2n}^2 = F_{2n-1}F_{2n+1} - 1,$$

$$(F_{2n-1} - F_{2n+1})^2 = F_{2n-1}F_{2n+1} - 1,$$

$$1 + F_{2n-1}^2 + F_{2n+1}^2 = 3F_{2n-1}F_{2n+1}.$$

Prime factors

Let m be a Markoff number with

$$m^2 + m_1^2 + m_2^2 = 3mm_1m_2.$$

1. The GCD of m, m_1 and m_2 is 1 : indeed, if p divides m_1, m_2 and m, then p divides the *new* solutions which are produced by the preceding process – going down in the tree shows that p would divide 1.

2. The odd prime factors of m are all congruent to 1 modulo 4 (since they divide a sum of two relatively prime squares).

3. One can prove that if m is even, then the numbers

$$\frac{m}{2}, \quad \frac{3m-2}{4}, \quad \frac{3m+2}{8},$$

are odd integers.

Prime Markoff numbers

- https://oeis.org/A178444 *Markov numbers that are prime.*

Triples of prime Markoff numbers appear to be very rare.
For Markoff numbers less than 10^{1000}, only five are known :
$(2, 5, 29)$, $(5, 29, 433)$, $(5, 2897, 43261)$, $(2, 5741, 33461)$, and

$(89, 6017226864647074440629, 1606577036114427599277221)$.

- It is conjectured that infinitely many Markoff numbers are prime.

- https://oeis.org/A256395 *Composite Markoff numbers.*
Almost all Markoff numbers are composite (Bourgain, Gamburd, and Sarnak).

Markoff's Conjecture

We have an algorithm which produces the sequence of Markoff numbers. Each Markoff number occurs infinitely often in the tree as one of the components of the solution.

According to the definition, for a Markoff number $m > 2$, there exist a pair (m_1, m_2) of positive integers with $m > m_1 > m_2$ such that $m^2 + m_1^2 + m_2^2 = 3mm_1m_2$.

Question : *Given m, is such a pair (m_1, m_2) unique?*

The answer is yes, as long as $m \leq 10^{105}$.

Frobenius's work

Markoff's Conjecture does not occur in Markoff's 1879 and 1880 papers but in Frobenius's one in 1913.

Ferdinand Georg Frobenius
1849 – 1917

E. Bombieri, *Continued fractions and the Markoff tree,* Expo. Math. **25** (2007), no. 3, 187–213. MR2345177

Special cases

The Conjecture has been proved for certain classes of Markoff numbers m like

$$p^n, \qquad \frac{p^n \pm 2}{3}$$

for p prime.

Arthur Baragar

A. Baragar (1996), $m = p$ and $m = 2p$.

P. Schmutz (1996), $m = p^n$ and $m = 2p^n$
J.O. Button (1998),
M.L. Lang, S.P. Tan (2005),
Ying Zhang (2007) : $4p^n$, $8p^n$.

Powers of a prime number

Anitha Srinivasan, 2007
A really simple proof of the
Markoff conjecture for prime
powers

Number Theory Web
Created and maintained by
Keith Matthews, Brisbane,
Australia

Anitha Srinivasan

Anitha Srinivasan
Markoff numbers and ambiguous classes
Journal de Théorie des Nombres de Bordeaux **21** (2009), 757–770
Numdam

Square Markoff numbers   Richard Guy

In a 2008-02-07 email, Bryan Orman asks if $169 = 13^2$ is the only example of one Markoff number being the square of another. Gary Walsh notes that if the uniqueness conjecture is not true for a fixed z, then there is a divisor u of z with $1 < u \leq \sqrt{z}$ and integers v, w for which (u^2, v, w) is a Markov triple.

```
De : rkg@cpsc.ucalgary.ca
Objet : Rép : Square Markov numbers
Date : 14 février 2008 04:27:57 HNEC
À : NMBRTHRY@LISTSERV.NODAK.EDU
```

Markoff Equation and Nilpotent Matrices
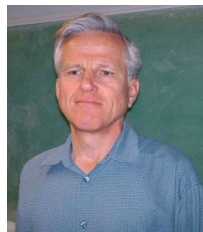# Norbert Riedel (2007) `Arxiv 0709.1499`

A triple (a, b, c) of positive integers is called a Markoff triple iff it satisfies the diophantine equation $a^2 + b^2 + c^2 = abc$. Recasting the Markoff tree, whose vertices are Markoff triples, in the framework of integral upper triangular 3×3 matrices,

it will be shown that the largest member of such a triple determines the other two uniquely. This answers a question which has been open for almost 100 years.

Norbert Riedel

Markoff Equation and Nilpotent Matrices

From : Norbert Riedel
Submission history
[v1] Mon, 10 Sep 2007 22 :11 :39 GMT (11kb)
[v2] Thu, 13 Sep 2007 18 :45 :29 GMT (11kb)
[v3] Tue, 4 Dec 2007 17 :43 :40 GMT (15kb)
[v4] Tue, 5 Aug 2008 21 :24 :23 GMT (15kb)
[v5] Thu, 12 Mar 2009 14 :08 :48 GMT (15kb)
[v6] Tue, 28 Jul 2009 18 :49 :17 GMT (15kb)
[v7] Fri, 29 Mar 2013 12 :36 :56 GMT (0kb,I) (withdrawn)

Comments : Most of the (correct) portion of this paper has been
incorporated into the paper "On the Markoff equation"
https://arxiv.org/abs/1208.4032

On the Markoff Equation

Norbert Riedel

[v1] Mon, 20 Aug 2012 15 :05 :47 GMT (51kb)
[v2] Mon, 1 Apr 2013 10 :21 :39 GMT (52kb)
[v3] Wed, 15 May 2013 17 :39 :35 GMT (52kb)
[v4] Mon, 8 Jul 2013 17 :35 :21 GMT (53kb)
[v5] Sun, 13 Oct 2013 21 :23 :28 GMT (50kb)
[v6] Sun, 20 Oct 2013 13 :37 :31 GMT (50kb)
[v7] Mon, 25 Nov 2013 19 :11 :53 GMT (53kb)
[v8] Sun, 12 Jan 2014 14 :59 :52 GMT (53kb)
[v9] Mon, 15 Dec 2014 18 :50 :40 GMT (58kb)
[v10] Fri, 6 Feb 2015 20 :16 :28 GMT (59kb)
[v11] Sat, 5 Sep 2015 12 :30 :49 GMT (59kb)
[v12] Thu, 25 Aug 2022 14 :25 :51 UTC (61 KB)

Serge Perrine

http://www.tessier-ashpool.fr/html/markoff.html

La théorie de Markoff
et ses développements
Tessier et Ashpool, 2002.
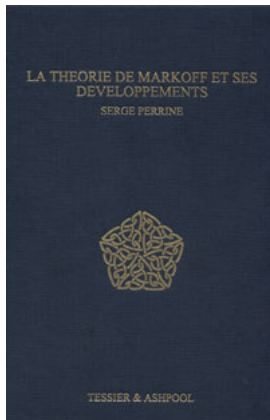
From Frobenius to Riedel :
analysis of the solutions of the Markoff equation

http://hal.archives-ouvertes.fr/hal-00406601/fr/

Comments by Serge Perrine

On the version 2 on November 7, 2007, 32 p.
On the version 3 on April 25, 2008, 57 p.
On the version 4 on March 10, 2009, 103 p.
On the versions 5 and 6 on July 29, 2009, 145 p.

Sur la conjecture de Frobenius relative aux solutions de l'équation de Markoff (Première partie)

Serge Perrine

Version 1
Submitted January 5, 2015
Last modified March 31, 2021

Why the coefficient 3?

Let n be a positive integer.

If the equation $x^2 + y^2 + z^2 = nxyz$ has a solution in positive integers, then

either $n = 3$ and x, y, z are relatively prime,

or $n = 1$ and the GCD of the numbers x, y, z is 3.

Friedrich Hirzebruch
1927 – 2012

Don Zagier

Friedrich Hirzebruch & Don Zagier,
The Atiyah–Singer Theorem and elementary number theory,
Publish or Perish (1974)

Markoff type equations

Bijection between the solutions for $n = 1$ and those for $n = 3$:

- if $x^2 + y^2 + z^2 = 3xyz$, then $(3x, 3y, 3z)$ is solution of $X^2 + Y^2 + Z^2 = XYZ$, since $(3x)^2 + (3y)^2 + (3z)^2 = (3x)(3y)(3z)$.

- if $X^2 + Y^2 + Z^2 = XYZ$, then X, Y, Z are multiples of 3 and $(X/3)^2 + (Y/3)^2 + (Z/3)^2 = 3(X/3)(Y/3)(Z/3)$.

The squares modulo 3 are 0 and 1. If neither X, Y nor Z is a multiple of 3, then $X^2 + Y^2 + Z^2$ is a multiple of 3.

If one or two (not three) integers among X, Y, Z are multiples of 3, then $X^2 + Y^2 + Z^2$ is not a multiple of 3.

Equations $x^2 + ay^2 + bz^2 = (1 + a + b)xyz$

If we insist that $(1, 1, 1)$ is a solution, then up to permutations there are only two more Diophantine equations of the type

$$x^2 + ay^2 + bz^2 = (1 + a + b)xyz$$

having infinitely many integer solutions, namely those with $(a, b) = (1, 2)$ and $(2, 3)$:

$$x^2 + y^2 + 2z^2 = 4xyz \quad \text{and} \quad x^2 + 2y^2 + 3z^2 = 6xyz$$

- $x^2 + y^2 + z^2$: tessalation of the plane by equilateral triangles
- $x^2 + y^2 + 2z^2 = 4xyz$: tessalation of the plane by isoceles rectangle triangles
- $x^2 + 2y^2 + 3z^2 = 6xyz$: tessalation ?

Hurwitz's equation (1907)

For each $n \geq 2$ the set K_n of positive integers k for which the equation

$$x_1^2 + x_2^2 + \cdots + x_n^2 = k x_1 \cdots x_n$$

has a solution in positive integers is finite.

Adolf Hurwitz
1859 − 1919

The largest value of k in K_n is n with the solution

$$(1, 1, \ldots, 1).$$

Examples :

$$K_2 = \{2\},$$
$$K_3 = \{1, 3\},$$
$$K_4 = \{1, 4\},$$
$$K_7 = \{1, 2, 3, 5, 7\}.$$

Hurwitz's equation
$$x_1^2 + x_2^2 + \cdots + x_n^2 = k x_1 \cdots x_n$$

When there is a solution in positive integers, there are infinitely many solutions, which can be organized in finitely many trees.

A. Baragar proved that *there exists such equations which require an arbitrarily large number of trees :*
J. Number Theory (1994), **49** No 1, 27-44.

The analog for the rank of elliptic curves over the rational number field is yet a conjecture.

Markoff and Diophantine approximation

J.W.S. Cassels,
An introduction to
Diophantine approximation,
Cambridge Univ. Press (1957)

John William Scott Cassels
1922 – 2015

Connection with Hurwitz's Theorem

Don Zagier,

On the number of Markoff numbers below a given bound.
Mathematics of Computation, **39** 160 (1982), 709–723.

the part of the *Markoff spectrum* (the set of all $\mu(Q)$) lying above $\frac{1}{3}$ is described exactly by the Markoff numbers. An equivalent theorem is that, under the action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathbf{R} \cup \{\infty\}$ given by $x \to (ax + b)/(cx + d)$, the $\mathrm{SL}_2(\mathbf{Z})$-equivalence classes of real numbers x for which the approximation measure

$$\mu(x) = \limsup_{q \to \infty} \left(q \cdot \min_{p \in \mathbf{Z}} |qx - p| \right)$$

is $> \frac{1}{3}$ are in $1:1$ correspondence with the Markoff triples, the spectrum being the same as above (e.g. $\mu(x) = 5^{-1/2}$ for x equivalent to the golden ratio and $\mu(x) \leqslant 8^{-1/2}$ for all other x). Thus the Markoff numbers are important both in the theory of quadratic forms and in the theory of Diophantine approximation. They have also

M.W., *Open Diophantine Problems*,
Moscow Mathematical Journal **4** N°1, 2004, 245–305.

$$\limsup = \infty, \qquad \liminf \leqslant 1/\sqrt{5}.$$

Historical origin : rational approximation

Hurwitz's Theorem (1891) :
For any real irrational number x, there exist infinitely many rational numbers p/q such that

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{\sqrt{5}q^2}.$$

Golden ratio
$\Phi = (1 + \sqrt{5})/2 = 1.618\,033\,9\ldots$
Hurwitz's result is optimal.

Adolf Hurwitz
1859 – 1919

$$F_n^2 \left| \Phi - \frac{F_{n+1}}{F_n} \right| \to 1/\sqrt{5}.$$

Lagrange's constant

For $x \in \mathbf{R} \setminus \mathbf{Q}$ denote by $\lambda(x) \in [\sqrt{5}, +\infty]$ the least upper bound of the numbers $\lambda > 0$ such that there exist infinitely many $p/q \in \mathbf{Q}$ satisfying

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{\lambda q^2}.$$

This means

$$\frac{1}{\lambda(x)} = \liminf_{q \to \infty} \left(q \min_{p \in \mathbf{Z}} |qx - p| \right).$$

Hurwitz : $\lambda(x) \geq \sqrt{5} = 2.236\,067\,9 \cdots$ for any x and $\lambda(\Phi) = \sqrt{5}$.

Badly approximable numbers

An irrational real number x is *badly approximable* by rational numbers if its Lagrange's constant is finite. This means that there exists $\lambda > 0$ such that, for any $p/q \in \mathbf{Q}$,

$$\left| x - \frac{p}{q} \right| \geq \frac{1}{\lambda q^2}.$$

For instance Liouville's numbers have an infinite Lagrange's constant.

A real number is badly approximable if and only if the sequence $(a_n)_{n \geq 0}$ of partial quotients in its continued fraction expansion

$$x = [a_0, \ a_1, \ a_2, \ \ldots, a_n, \ldots]$$

is bounded.

Badly approximable numbers

Any quadratic irrational real number has a finite Lagrange's constant ($=$ is badly approximable).

It is not known whether there exist real algebraic numbers of degree ≥ 3 which are badly approximable.

It is not known whether there exist real algebraic numbers of degree ≥ 3 which are *not* badly approximable ...

One conjectures that *any irrational real number which is not quadratic and which is badly approximable is transcendental.* This means that one conjectures that no real algebraic number of degree ≥ 3 is badly approximable.

Lebesgue measure

The set of badly approximable real numbers has zero measure for Lebesgue's measure.

Henri Léon Lebesgue
1875–1941

Properties of the Lagrange's constant

We have

$$\lambda(x+1) = \lambda(x): \qquad \left| x + 1 - \frac{p}{q} \right| = \left| x - \frac{p+q}{q} \right|$$

and

$$\lambda(-x) = \lambda(x): \qquad \left| -x - \frac{p}{q} \right| = \left| x + \frac{p}{q} \right|,$$

Also $\lambda(1/x) = \lambda(x):$

$$p^2 \left| \frac{1}{x} - \frac{q}{p} \right| = q^2 \left| \frac{p}{qx} \right| \cdot \left| x - \frac{p}{q} \right|.$$

The modular group

The multiplicative group generated by the three matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is the group $\mathrm{GL}_2(\mathbf{Z})$ of 2×2 matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$ with coefficients in $\mathbf{Z}$

and determinant ± 1.

Jean-Pierre Serre

J-P. SERRE – *Cours d'arithmétique*, Coll. SUP, Presses Universitaires de France, Paris, 1970.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x = \frac{ax + b}{cx + d}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x = x + 1 \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x = -x \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x = \frac{1}{x}$$

$$\lambda(x + 1) = \lambda(x) \qquad \lambda(-x) = \lambda(x) \qquad \lambda(1/x) = \lambda(x)$$

Consequence : *Let $x \in \mathbf{R} \setminus \mathbf{Q}$ and let a, b, c, d be rational integers satisfying $ad - bc = \pm 1$. Set*

$$y = \frac{ax + b}{cx + d}.$$

Then $\lambda(x) = \lambda(y)$.

Hurwitz's work (1891)

The inequality $\lambda(x) \geq \sqrt{5}$ for all real irrational x is optimal for the Golden ratio and for all the *noble* irrational numbers whose continued fraction expansion ends with an infinite sequence of 1's – these numbers are roots of quadratic polynomials having discriminant 5 :

Adolf Hurwitz
1859 – 1919

$$\Phi = [1, 1, 1, \ldots] = [\overline{1}].$$

Notice that $\sqrt{5} = 2 + \frac{1}{2+\sqrt{5}} = [2, \overline{4}]$ is not a noble number.

Noble numbers

A noble number, whose continued fraction expansion ends with an infinite sequence of 1's, is a number related to the Golden ratio Φ by a homography of determinant ± 1 :

$$\frac{a\Phi + b}{c\Phi + d} \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}).$$

The first elements of the spectrum

For all the real numbers which are not noble numbers, a stronger inequality than Hurwitz's inequality

$$\lambda(x) \geq \sqrt{5} = 2,236\ 067\ 977\ldots$$

is valid, namely

$$\lambda(x) \geq 2\sqrt{2} = 2,828\ 427\ 125\ldots$$

This is optimal for

$$\sqrt{2} = 1.414\ 213\ 562\ 373\ 095\ 048\ 801\ 688\ 724\ 209\ 698\ 078\ldots$$

whose continued fraction expansion is

$$[1;\ 2,\ 2,\ \ldots,2,\ \ldots] = [1;\overline{2}].$$

Lagrange Spectrum

The constant $\sqrt{5}$ is best possible in Hurwitz's Theorem.

If α is not $\mathrm{GL}(2, \mathbf{Z})$ equivalent to the Golden ratio, the constant $\sqrt{5}$ improves to $\sqrt{8}$.

If we also exclude the numbers which are $\mathrm{GL}(2, \mathbf{Z})$ equivalent to $1 + \sqrt{2}$, then we can take $\sqrt{221}/5$ as the constant.

The sequence

$$L_1 = \sqrt{5} = 2.236\ldots, \quad L_2 = \sqrt{8} = 2\sqrt{2} = 2.828\ldots,$$

continues as follows :

$$L_3 = \frac{\sqrt{221}}{5} = 2.973\ldots, \qquad L_4 = \frac{\sqrt{1517}}{13}, \qquad L_5 = \frac{\sqrt{7565}}{29}, \ldots$$

Lagrange Spectrum

There is an infinite increasing sequence of real numbers $(L_i)_{i \geq 1}$ with limit 3, and an associated sequence of quadratic irrationalities $(\theta_i)_{i \geq 1}$, such that for any $i \geq 1$, if α is not $\mathrm{GL}(2, \mathbf{Z})$ equivalent to any of $\theta_1, \ldots, \theta_{i-1}$, then there is infinitely many rational numbers that satisfy

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{L_i q^2},$$

and this is best possible when α is $\mathrm{GL}(2, \mathbf{Z})$ equivalent to θ_i.

Lagrange Spectrum

Denoting by $m_1, m_2, \ldots$ the sequence of Markoff numbers, one has

$$L_i = \sqrt{9 - (4/m_i{}^2)}$$

and

$$\theta_i = \frac{-3m_i + 2k_i + \sqrt{9m_i{}^2 - 4}}{2m_i},$$

where k_i is an integer satisfying $a_i k_i \equiv b_i \pmod{m_i}$ and (a_i, b_i, m_i) is a solution of Markoff's equation with $m_i \geq \max\{a_i, b_i\}$.

Here one assumes Markoff's Conjecture to get unicity of (a_i, b_i).

First values : Markoff (1880)

$$L_i = \sqrt{9 - (4/m_i{}^2)}, \qquad \theta_i = \frac{-3m_i + 2k_i + \sqrt{9m_i{}^2 - 4}}{2m_i}$$

$$F_i(X, Y) = m_i X^2 + (3m_i - 2k_i)XY + \frac{k_i{}^2 - 3k_i m_i + 1}{m_i} Y^2.$$

i	m_i	k_i	L_i	θ_i	F_i
1	1	1	$\sqrt{5}$	$(-1 + \sqrt{5})/2$	$X^2 + XY - Y^2$
2	2	1	$\sqrt{8}$	$-1 + \sqrt{2}$	$2(X^2 + 2XY - Y^2)$
3	5	2	$\sqrt{221}/5$	$(-11 + \sqrt{221})/10$	$5X^2 + 11XY - 5Y^2$

Minima of quadratic forms

Let $f(X,Y) = aX^2 + bXY + cY^2$ be a quadratic form with real coefficients. Denote by $\Delta(f)$ its discriminant $b^2 - 4ac$.

Consider the minimum $m(f)$ of $|f(x,y)|$ on $\mathbf{Z}^2 \setminus \{(0,0)\}$. Assume $\Delta(f) \neq 0$ and set

$$C(f) = m(f)/\sqrt{|\Delta(f)|}.$$

Let α and α' be the roots of $f(X,1)$:

$$f(X,Y) = a(X - \alpha Y)(X - \alpha' Y),$$

$$\{\alpha, \alpha'\} = \left\{ \frac{1}{2a}\left(-b \pm \sqrt{\Delta(f)}\right) \right\}.$$

Example with $\Delta < 0$

The form
$$f(X, Y) = X^2 + XY + Y^2$$

has discriminant $\Delta(f) = -3$ and minimum $m(f) = 1$, hence

$$C(f) = \frac{m(f)}{\sqrt{|\Delta(f)|}} = \frac{1}{\sqrt{3}}.$$

For $\Delta < 0$, the form

$$f(X, Y) = \sqrt{\frac{|\Delta|}{3}}(X^2 + XY + Y^2)$$

has discriminant Δ and minimum $\sqrt{|\Delta|/3}$. Again

$$C(f) = \frac{1}{\sqrt{3}}.$$

Definite quadratic forms ($\Delta < 0$)

If the discriminant is negative, J.L. Lagrange and Ch. Hermite (letter to Jacobi, August 6, 1845) proved $C(f) \le 1/\sqrt{3}$ with equality for $f(X, Y) = X^2 + XY + Y^2$. For each $\varrho \in (0, 1/\sqrt{3}]$, there exists such a form f with $C(f) = \varrho$.

Joseph-Louis Lagrange
1736 – 1813

Charles Hermite
1822 – 1901

Carl Jacobi
1804 – 1851

Example with $\Delta > 0$

The form

$$f(X, Y) = X^2 - XY - Y^2$$

has discriminant $\Delta(f) = 5$ and minimum $m(f) = 1$, hence

$$C(f) = \frac{m(f)}{\sqrt{\Delta(f)}} = \frac{1}{\sqrt{5}}.$$

For $\Delta > 0$, the form

$$f(X, Y) = \sqrt{\frac{\Delta}{5}}(X^2 - XY - Y^2)$$

has discriminant Δ and minimum $\sqrt{\Delta/5}$. Again

$$C(f) = \frac{1}{\sqrt{5}}.$$

Indefinite quadratic forms ($\Delta > 0$)

Assume $\Delta > 0$

A. Korkine and E.I.. Zolotarev proved in 1873 $C(f) \leq 1/\sqrt{5}$ with equality for $f_0(X, Y) = X^2 - XY - Y^2$. For all forms which are not equivalent to f_0 under $\mathrm{GL}(2, \mathbf{Z})$, they prove $C(f) \leq 1/\sqrt{8}$.

Egor Ivanovich Zolotarev
1847–1878

$1/\sqrt{5} = 0.447\,213\,595\ldots$ $1/\sqrt{8} = 0.353\,553\,391\ldots$

Indefinite quadratic forms ($\Delta > 0$).

The works by Korkine and Zolotarev inspired Markoff who pursued the study of this question in 1879 and 1880. He produced infinitely many values $C(f_i)$, $i = 0, 1, \ldots$, between $1/\sqrt{5}$ and $1/3$, with the same property as f_0.

Andrei Andreyevich Markoff
1856 – 1922

These values form a sequence which converges to $1/3$. He constructed them by means of the tree of solutions of the Markoff equation.

Indefinite quadratic forms ($\Delta > 0$)

Assume $f(X, Y) = aX^2 + bXY + cY^2 \in \mathbf{R}[X, Y]$ with $a > 0$ has discriminant $\Delta > 0$.

If $|f(x, y)|$ is small with $y \neq 0$, then x/y is close to a root of $f(X, 1)$, say α.

Then
$$|x - \alpha'y| \sim |y| \cdot |\alpha - \alpha'|$$
and $\alpha - \alpha' = \sqrt{\Delta}/a$.

Hence

$$|f(x, y)| = |a(x - \alpha y)(x - \alpha'y)| \sim y^2\sqrt{\Delta}\left|\alpha - \frac{x}{y}\right|.$$

Lagrange spectrum and Markoff spectrum

Markoff spectrum = set of values taken by

$$\frac{1}{C(f)} = \sqrt{\Delta(f)}/m(f)$$

when f runs over the set of quadratic forms $ax^2 + bxy + cy^2$ with real coefficients of discriminant $\Delta(f) = b^2 - 4ac > 0$ and $m(f) = \inf_{(x,y)\in \mathbf{Z}^2\setminus\{0\}} |f(x,y)|$.

Lagrange spectrum = set of values taken by Lagrange's constant.

$$\lambda(x) = 1/\liminf_{q\to\infty} q(\min_{p\in\mathbf{Z}} |qx - p|)$$

when x runs over the set of real numbers.

The Markoff spectrum contains the Lagrange spectrum.
The intersection with the intervall $[\sqrt{5}, 3)$ is the same for both of them, and is a discrete sequence.

Chronologie

J. Liouville, 1844

J-L. Lagrange et Ch. Hermite, 1845

A. Korkine et E.I. Zolotarev, 1873

A. Markoff, 1879

F. Frobenius, 1915

L. Ford, 1917, 1938

R. Remak, 1924

J.W.S. Cassels, 1949

J. Lehner, 1952, 1964

H. Cohn, 1954, 1980

R.A. Rankin, 1957

A. Schmidt, 1976

S. Perrine, 2002

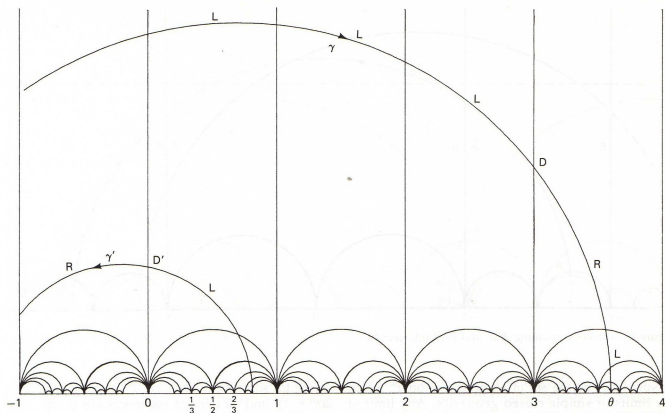# Continued fraction and hyperbolic geometry

Figure 8. Reading off the continued fraction expansion of $\mathfrak{O}$ from $\mathfrak{I}$: $\mathfrak{O} = 3 + \frac{1}{1 + \cdots}$.

Référence : Caroline Series

The Geometry of Markoff Numbers

Caroline Series

Caroline Series,
The Geometry of Markoff Numbers,
The Mathematical Intelligencer **7** N.3 (1985), 20–29.
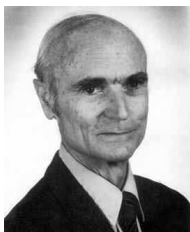
Fuchsian groups and hyperbolic Riemann surfaces

Markoff's tree can be seen as the dual of the triangulation of the hyperbolic upper half plane by the images of the fundamental domain of the modular invariant under the action of the modular group.

Lazarus Immanuel Fuchs
1833 – 1902

Triangulation of polygons, metric properties of polytopes

Harold Scott MacDonald Coxeter
1907 – 2003

Robert Alexander Rankin
1915 – 2001

John Horton Conway
1937 – 2020

Fricke groups

The subgroup Γ of $\mathrm{SL}_2(\mathbf{Z})$ generated by the two matrices

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

is the free group with two generators.

The Riemann surface quotient of the Poincaré upper half plane by Γ is a *punctured torus*. The minimal lengths of the closed geodesics are related to the $C(f)$, for f indefinite quadratic form.

Robert Karl Emanuel Fricke (1861 – 1930)

https://mathshistory.st-andrews.ac.uk/Biographies/Fricke/

Free groups.

Fricke proved that if A and B are two generators of Γ, then their traces satisfy

$$(\operatorname{tr} A)^2 + (\operatorname{tr} B)^2 + (\operatorname{tr} AB)^2 = (\operatorname{tr} A)(\operatorname{tr} B)(\operatorname{tr} AB)$$

Harvey Cohn showed that quadratic forms with a Markoff constant $C(f) \in (1/3, 1/\sqrt{5}]$ are equivalent to

$$cx^2 + (d - a)xy - by^2$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a generator of Γ.

Fundamental domain of a punctured disc
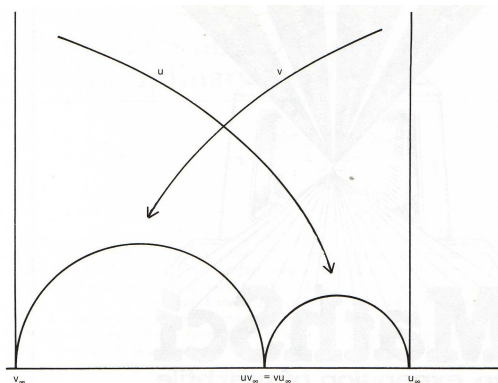
Figure 13. Fundamental region for the punctured torus.

Caroline Series, The Geometry of Markoff Numbers, The Mathematical Intelligencer **7** N.3 (1985), 20–29.
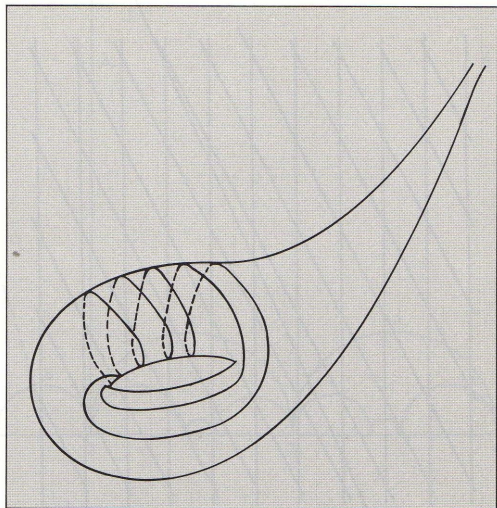
A simple curve on a punctured disc

Figure 1. A simple curve on the punctured torus.

Caroline Series, The Geometry of Markoff Numbers, The Mathematical Intelligencer **7** N.3 (1985), 20–29.

Hyperbolic geometric aspects of the Markoff spectrum (24/01/2000)

Late in the 19th century, A. A. Markoff initiated an extensive theory of the minima of indefinite binary quadratic forms, or, what is the same, extending Hurwitz's Theorem of diophantine approximation. He showed in particular that these minima begin with a countable discrete spectrum which monotonically increases to 3. Early the 20th century, work of L. E. Ford then implies that these values are related to the geometry of the modular surface.

Some forty years later, H. Cohn recognized a connection between these initial values of Markoff's spectrum and certain closed geodesics on the so-called homology cover of the modular surface. In particular, the Markoff numbers, which comprise this initial countable set of values of the spectrum, correspond one-to-one to the simple closed geodesics on a hyperbolic once-punctured torus $\Gamma' \backslash \mathcal{H}$ which is a six-fold cover of the modular surface.

https://www.imj-prg.fr/tn/STN/00/resume-Sheingorn.html

Geodesics on a hyperbolic once-punctured torus

Thomas A. Schmidt

Mark Sheingorn

Schmidt, Thomas A. ; Sheingorn, Mark.
Low height geodesic on $\Gamma^3/\mathcal{H}$ height formulas and examples.
Int. J. Number Theory **3**, No. 3 (2007), 475–501.

Ford circles

The Ford circle associated to the irreducible fraction p/q is tangent to the real axis at the point p/q and has radius $1/2q^2$.
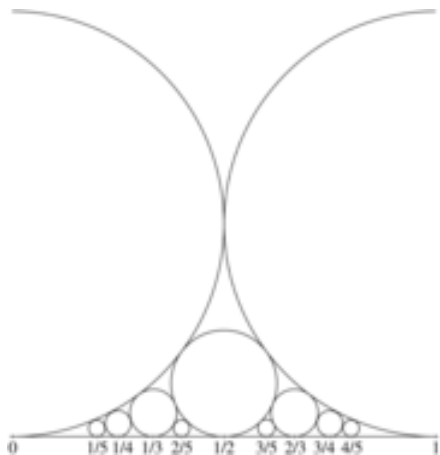
Ford circles associated to two consecutive elements in a Farey sequence are tangent.

Lester Randolph Ford
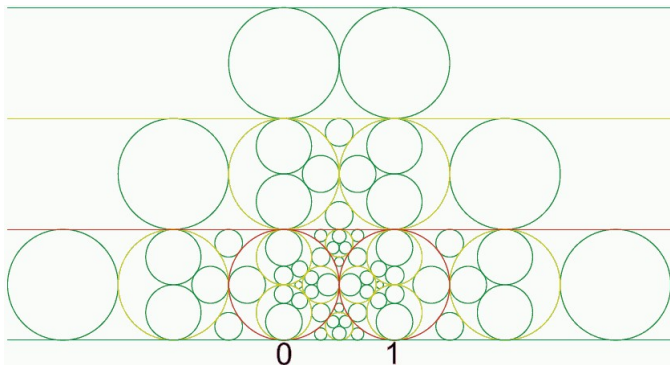(1886–1967)
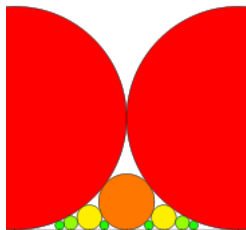
Amer. Math. Monthly (1938).

Farey sequence of order 5

$$\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}$$

Complex continued fraction

The third generation of Asmus Schmidt's complex continued fraction method.

Laurent's phenomenon

Connection with Laurent polynomials (S. Fomin and A. Zelevinsky https://arxiv.org/abs/math/0104241).

If f and g are Laurent polynomials in one variable x, i.e., polynomials in x, x^{-1}, in general $g\big(f(x)\big)$ is not a Laurent polynomial :

$$f(x) = \frac{x^2 + 1}{x} = x + \frac{1}{x},$$

$$f\big(f(x)\big) = \frac{\left(x + \dfrac{1}{x}\right)^2 + 1}{x + \dfrac{1}{x}} = \frac{x^4 + 3x^2 + 1}{x(x^2 + 1)}.$$

James Propp, *The combinatorics of frieze patterns and Markoff numbers*,
https://arxiv.org/abs/math/0511633

Simultaneous rational approximation and Markoff spectrum

Relation between Markoff numbers and extremal numbers : simultaneous approximation of x and x^2 by rational numbers with the same denominator.

Damien Roy

Markoff–Lagrange spectrum and extremal numbers, arXiv.0906.0611 [math.NT] 2 June 2009

Greatest prime factor of Markoff pairs

Pietro Corvaja and Umberto Zannier, 2006 :
The greatest prime factor of the product xy when x, y, z is a solution of Markoff's equation tends to infinity with $\max\{x, y, z\}$.

Equivalent statement :
If S denotes a finite set of prime numbers, the equation

$$x^2 + y^2 + z^2 = 3xyz$$

has only finitely many solutions in positive integers x, y, z, such that xy has no prime divisor outside S.
(The integers x and y are called S–units.)

Oregon State University, College of Science, Department of Mathematics
Algebra & Number Theory Seminar
On line

On Markoff Numbers

Michel Waldschmidt

Professeur Émérite, Sorbonne Université,
Institut de Mathématiques de Jussieu, Paris
http://www.imj-prg.fr/~michel.waldschmidt/