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The Matrix Coefficient Conjecture of Dasgupta and Kakde

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ICCGNFRT 2017 - 2025

2017 September 4: ICCGNFRT2017

Harish-Chandra Research Institute (HRI), Allahabad. *Families of Thue equations associated with a rank one subgroup of the unit group of a number field.*

2018 October 8 - 11: ICCGNFRT2018.

Harish-Chandra Research Institute (HRI), Allahabad.

On the Landau–Ramanujan constant.

2019 October 16 - 19: ICCGNFRT2019

Harish-Chandra Research Institute (HRI), Allahabad. .

On the representation of integers by cyclotomic binary forms of large degree.

2021 October 21-24, 2021, (online) ICCGNFRT2021

The Kerala School of Mathematics (KSoM), Calicut (Kozhikode).

A conference celebrating Michel Waldschmidt's 75th birth year honoring his contribution to Mathematics and Education

2022 November 21 - 24, ICCGNFRT2022.

The Kerala School of Mathematics (KSoM), Calicut (Kozhikode).

On the number of integers represented by families of binary forms.

2023 October 26 - 30, ICCGNFRT2023

The Kerala School of Mathematics (KSoM), Calicut (Kozhikode).

On the number of integers represented by families of binary binomial forms.

2024 November 20 - 24 ICCGNFRT2024

Indian Institute of Science Education and Research (IISER), Berhampur .

On Binary quadratic forms with the same value set, following E. Fouvry & P. Koymans.

2025 October 6 - 10 ICCGNFRT2025.

SRM University AP *On multicyclotomic polynomials and binary forms.*

My previous visits to Guwahati

- November 5, 2007, IITG, French Science Today
Introduction to cryptography and to error correcting codes.
- December 6, 2008, on the way back from Bhutan going to HRI Allahabad.
- November 18 - 29, 2013, CIMPA school, Shillong, North Eastern Hill University.
Fourier analysis of groups in combinatorics.
- ◇ November 14, 2013, Manipur University.
- ◇ November 15, 2013, Tezpur University.
- February 26 - 27, 2023, IITG (Anupam Saikia)
February 27, lecture: *On the abc Conjecture.*

Abstract

In their recent paper (Advances in Mathematics, 2026), [Samit Dasgupta](#) and [Mahesh Kakde](#) propose their *matrix coefficient conjecture* which is inspired by earlier results on the rank of matrices, the entries of which are logarithms of algebraic numbers.

The matrix coefficient conjecture is a consequence of the conjecture on algebraic independence of logarithms of algebraic numbers, and it implies a number of conjectures on nonvanishing of regulators ([Leopoldt](#), [Gross–Kuz'min](#)).

The matrix coefficient conjecture

Conjecture.

Let M be a square $d \times d$ matrix with entries logarithms of algebraic numbers and zero determinant. Then there exist u and v in $\mathbb{Q}^d \setminus \{0\}$ such that $\langle u, Mv \rangle = 0$.

If a square matrix of logarithms (complex or p -adic) of algebraic numbers has vanishing determinant, then after a rational change of basis on the left and right, it can be made to have a vanishing coefficient.



Dasgupta, Samit

Ranks of matrices of logarithms of algebraic numbers. I: The theorems of Baker and Waldschmidt-Masser. *Essent. Number Theory* 2, No. 1, 93-138 (2023).

Zbl 1543.11059



Dasgupta, Samit; Kakde, Mahesh

Ranks of matrices of logarithms of algebraic numbers. II: The matrix coefficient conjecture. *Adv. Math.* 487, Article ID 110753, 19 p. (2026).

Zbl 08155756

Hermite – Lindemann – Gel'fond – Schneider

Hermite 1873 transcendence of e

Lindemann 1882 transcendence of π

Theorem (Hermite – Lindemann).

Let α be a non-zero algebraic complex number and let $\log \alpha$ be a non-zero logarithm of α . Then $\log \alpha$ is transcendental.

Theorem (Gel'fond – Schneider 1934).

Let α_1, α_2 be two algebraic numbers and let $\log \alpha_1, \log \alpha_2$ be logarithms of α_1 and α_2 respectively. Assume $\log \alpha_1, \log \alpha_2$ are linearly independent over $\mathbb{Q}$. Then $\log \alpha_1, \log \alpha_2$ are linearly independent over the field of algebraic numbers.

Linear independence of logarithms

Theorem (Baker 1967).

Let $\ell_1, \dots, \ell_n$ be $\mathbb{Q}$ -linearly independent complex numbers such that $\alpha_1 = e^{\ell_1}, \dots, \alpha_n = e^{\ell_n}$ are algebraic. Then the numbers $1, \ell_1, \dots, \ell_n$ are linearly independent over the field $\overline{\mathbb{Q}}$ of algebraic numbers.

Notation: Let $\mathcal{L}$ denote the set of complex numbers ℓ such that $e^\ell \in \overline{\mathbb{Q}}$. This set $\mathcal{L}$ is a $\mathbb{Q}$ -vector subspace of $\mathbb{C}$.

Serre, Jean-Pierre

L'injection de $\mathcal{L}$ dans $\mathbb{C}$ se prolonge en une application $\overline{\mathbb{Q}}$ -linéaire ι de $\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathcal{L}$ dans $\mathbb{C}$, et le théorème revient à dire que ι est injective.



Serre, Jean-Pierre

Travaux de Baker.

Sémin. Bourbaki 1969/70, No. 368, 73–86 (1971).

Zbl 0212.07201

Consequences of Baker's Theorem

- Nonvanishing of regulators : abelian extensions (Ax – Brumer).
- Transcendence of numbers like $\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$, $e^{\beta_0} \alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n}$.
- Siegel integral

$$\int_0^1 \frac{dx}{1+x^3} = \frac{1}{3} \left(\log 2 + \frac{\pi}{\sqrt{3}} \right)$$



Siegel, Carl Ludwig

Transcendental numbers. Annals of Mathematics Studies. 16.

Princeton, N. J.: Princeton University Press. 102 p. (1949).

Reprint of the 1949 edition. Texts and Readings in Mathematics 60.

New Delhi: Hindustan Book Agency 102 p. (2012).

Zbl 0039.04402

Zbl 1283.11003

Algebraic independence of logarithms

Conjecture (Algebraic independence of logarithms).

$\mathbb{Q}$ -linearly independent logarithms of algebraic numbers are algebraically independent.

This is a special case of Schanuel's Conjecture.

Conjecture (Schanuel's conjecture).

If $x_1, \dots, x_n$ are linearly independent complex numbers, there are at least n algebraically independent numbers among the $2n$ numbers

$$x_1, \dots, x_n, \exp(x_1), \dots, \exp(x_n).$$

Open problem

Prove that the field generated by all logarithms of all nonzero algebraic numbers has transcendence degree ≥ 2 .

Regulators

Many questions in number theory concern the nonvanishing of determinants of square matrices of logarithms (complex or p -adic) of algebraic numbers.

Question. Let ℓ_{ij} ($1 \leq i \leq d$, $1 \leq j \leq \ell$) be elements in $\mathcal{L}$. What is the rank of the $d \times \ell$ matrix

$$M := (\ell_{ij})_{\substack{1 \leq i \leq d \\ 1 \leq j \leq \ell}}?$$

Example: *Gruppendeterninant*

Let G be a finite group of order g and X_s ($s \in G$) be g independent variables. The determinant of the group G is

$$\Delta_G = \det(X_{st^{-1}})_{(s,t) \in G \times G}.$$

Gruppendeterminant

With $G = \{e, s_1, \dots, s_{g-1}\}$,

$$\Delta_G = \det \begin{pmatrix} X_e & X_{s_1} & \cdots & X_{s_{g-2}} & X_{s_{g-1}} \\ X_{s_1^{-1}} & X_e & \cdots & X_{s_{g-2}s_1^{-1}} & X_{s_{g-1}s_1^{-1}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ X_{s_{g-2}^{-1}} & X_{s_1 s_{g-2}^{-1}} & \cdots & X_e & X_{s_{g-2}s_{g-1}^{-1}} \\ X_{s_{g-1}^{-1}} & X_{s_1 s_{g-1}^{-1}} & \cdots & X_{s_{g-2}s_{g-1}^{-1}} & X_e \end{pmatrix}.$$

Bourbaki, historical note

Dedekind, (1880 – 1886), Frobenius (1896)

Let $\varrho_1, \dots, \varrho_r$ be the irreducible representations of G , f_n the degree of ϱ_n ($1 \leq n \leq r$). Then

$$\Delta_G = \prod_{n=1}^r \Phi_n^{f_n}$$

where Φ_n is an irreducible homogeneous polynomial of degree f_n .

For instance if G is abelian ($f_n = 1$) with character group $\{\chi\}$, then Δ_G is a product of linear forms with algebraic coefficients

$$\Delta_G = \prod_{\chi} \sum_{s \in G} \chi(s) X_s.$$

In the cyclic case this is a *circulant determinant*.

Regulators

In general the linear form $\sum_{s \in G} X_s$ divides Δ_G : with $G = \{e, s_1, \dots, s_{g-1}\}$,

$$\Delta_G = \left(\sum_{s \in G} X_s \right) \det \begin{pmatrix} 1 & X_{s_1} & \cdots & X_{s_{g-1}} \\ 1 & X_e & \cdots & X_{s_{g-1}s_1^{-1}} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & X_{s_1 s_{g-1}^{-1}} & \cdots & X_e \end{pmatrix}.$$

The p -adic version of [Baker's Theorem](#) is true ([Brumer](#)). According to [Ax](#), this implies that the p -adic regulator of a number field which is abelian over $\mathbb{Q}$ or over an imaginary quadratic field does not vanish.

p -adic regulator

Let K be a Galois real number field, with Galois group G , p a prime number, ϵ a Minkowski unit of K , σ_0 an element of G and

$$R_p = \det \left(\log_p(\sigma\tau^{-1}\epsilon) \right)_{\substack{\sigma \neq \sigma_0 \\ \tau \neq \sigma_0}}$$

the p -adic regulator of K .

Then

$$R_p = \frac{1}{g} P(\log_p(s\epsilon))_{s \in G} \quad \text{with} \quad P(X_s)_{s \in G} = \frac{\Delta_G}{\sum_{s \in G} X_s}.$$

2×2 matrices

Let $\ell_{ij} \in \mathcal{L}$ ($1 \leq i \leq d$, $1 \leq j \leq \ell$). Write $\ell_{ij} = \log \alpha_{ij}$ where $\alpha_{ij} = e^{\ell_{ij}}$.

Consider the $d \times \ell$ matrix

$$M := (\ell_{ij})_{\substack{1 \leq i \leq d \\ 1 \leq j \leq \ell}}.$$

Example: $\ell = d = 2$:

$$M = \det \begin{pmatrix} \log \alpha_1 & \log \alpha_2 \\ \log \alpha_3 & \log \alpha_4 \end{pmatrix} = \log \alpha_1 \log \alpha_4 - \log \alpha_2 \log \alpha_3.$$

Conjecture.

Assume that the two rows of M are $\mathbb{Q}$ -linearly independent, and that the two columns of M are $\mathbb{Q}$ -linearly independent. Then M has rank 2.

The Four Exponentials Conjecture

The conjecture on the rank of a 2×2 matrix with entries in $\mathcal{L}$ is a consequence of the conjecture on algebraic independence of $\mathbb{Q}$ -linearly independent logarithms of algebraic number.

Let V be the affine variety of equation $X_1X_4 = X_2X_3$ in $\mathbb{C}^4$ (ruled surface – Segre).

Lemma.

The only lines in $\mathbb{C}^4$ contained in the variety V are given by

$$aX_1 = bX_2, \quad aX_3 = bX_4$$

and

$$aX_1 = bX_3, \quad aX_2 = bX_4$$

with $(a, b) \in \mathbb{C}^2 \setminus \{(0, 0)\}$.

Schneider's problem 1

Wir nennen die folgenden Probleme:

1. Es seien α, β, γ algebraische Zahlen und $\alpha \neq 0, 1$ sowie $\frac{\log \beta}{\log \alpha}$ und $\frac{\log \gamma}{\log \alpha}$ irrational. Es ist $\delta = \exp\left(\frac{\log \beta \log \gamma}{\log \alpha}\right)$ auf Transzendenz zu untersuchen.



Theodor Schneider.

Einführung in die transzendenten Zahlen.

Die Grundlehren der Mathematischen Wissenschaften. Band **81**.

Berlin-Göttingen-Heidelberg: Springer-Verlag (1957).

Zbl 0077.04703

Introduction aux nombres transcendents.

Traduit par P. Eymard. Paris: Gauthier-Villars. (1959).

Zbl 0098.26304

Four Exponentials Conjecture

Conjecture.

Let x_1, x_2 be two $\mathbb{Q}$ -linearly independent complex numbers and y_1, y_2 be two $\mathbb{Q}$ -linearly independent complex numbers. Then one at least of the four numbers

$$e^{x_1 y_1}, e^{x_1 y_2}, e^{x_2 y_1}, e^{x_2 y_2}$$

is transcendental.

Same as Schneider's first problem:

$$x_1 = \log \alpha, x_2 = \log \beta, y_1 = 1, y_2 = \frac{\log \gamma}{\log \alpha} = \frac{\log \delta}{\log \beta}$$

Open problem:

Assume 2^t and 3^t are integers. Prove that t is an integer.

Six Exponentials and Four Exponentials

Let $x_1, \dots, x_d$ be $\mathbb{Q}$ -linearly independent complex numbers and $y_1, \dots, y_\ell$ be $\mathbb{Q}$ -linearly independent complex numbers.

Theorem (Six Exponentials Theorem: Siegel – Schneider – Lang – Ramachandra).

If $\ell d > \ell + d$ then one at least of the ℓd numbers

$$e^{x_i y_j} \quad (1 \leq i \leq d, 1 \leq j \leq \ell)$$

is transcendental. The relevant case is $\{\ell, d\} = \{2, 3\}$.

Compare with the **Four Exponentials Conjecture**:

If $\ell d \geq \ell + d$ then one at least of the ℓd numbers

$$e^{x_i y_j} \quad (1 \leq i \leq d, 1 \leq j \leq \ell)$$

is transcendental.

The relevant case is $\ell = d = 2$.

References



Waldschmidt, Michel

Variations on the six exponentials theorem.

Tandon, Rajat (ed.), Algebra and number theory. Proceedings of the silver jubilee conference, Hyderabad, India, December 11–16, 2003. New Delhi: Hindustan Book Agency 338-355 (2005).

Zbl 1176.11033



Waldschmidt, Michel

Further variations on the six exponentials theorem.

Hardy-Ramanujan J. 28, 1-9 (2005).

Zbl 1116.11054



Waldschmidt, Michel

The four exponentials problem and Schanuel's conjecture.

Morel, Jean-Michel (ed.) et al., Mathematics going forward.

Collected mathematical brushstrokes. Springer. Lect. Notes Math. 2313, 579-592 (2023).

Zbl 1543.11060

Resonance



K. Senthil Kumar, R. Thangadurai & Veekesh Kumar
On a Problem of Alaoglu and Erdős
Resonance, 23, 749–758 (2018),
doi 10.1007/s12045-018-0676-1



Waldschmidt, Michel
Six Exponentials Theorem — Irrationality
Resonance, 27, 599–607 (2022)
doi 10.1007/s12045-022-1351-0

Larger rank

Linear independence of rows and columns is not sufficient: the rows as well as the columns of the rank 2 matrix

$$\begin{pmatrix} 0 & -\log 2 & \log 3 \\ \log 2 & 0 & -\log 5 \\ -\log 3 & \log 5 & 0 \end{pmatrix}$$

are linearly independent.

Antisymmetric matrix:

$$\begin{pmatrix} 0 & -X_3 & X_2 \\ X_3 & 0 & -X_1 \\ -X_2 & X_1 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Structural rank

Let ℓ_{ij} ($1 \leq i \leq d$, $1 \leq j \leq \ell$) be elements in $\mathcal{L}$. What is the rank of the $d \times \ell$ matrix

$$M := (\ell_{ij})_{\substack{1 \leq i \leq d \\ 1 \leq j \leq \ell}}?$$

Let $\lambda_1, \dots, \lambda_k$ be a basis of the $\mathbb{Q}$ -vector space spanned by the entries ℓ_{ij} of the matrix and let b_{ijh} be the rational numbers such that

$$\ell_{ij} = \sum_{h=1}^k b_{ijh} \lambda_h, \quad 1 \leq i \leq d, \quad 1 \leq j \leq \ell.$$

Damien Roy defines the *structural rank* $\text{rstr}(M)$ of the matrix M as the rank of the matrix

$$\left(\sum_{h=1}^k b_{ijh} X_h \right)_{\substack{1 \leq i \leq d \\ 1 \leq j \leq \ell}} \in \text{Mat}_{d \times \ell}(\mathbb{Q}(X_1, \dots, X_k)).$$

The Structural Rank Conjecture

Let

$$M := (\ell_{ij})_{\substack{1 \leq i \leq d \\ 1 \leq j \leq \ell}}$$

be a $d \times \ell$ matrix with entries logarithms of algebraic numbers.

Conjecture (Structural Rank Conjecture).

The rank of M is $\text{rstr}(M)$.

Same in the p -adic case.

The Structural Rank Conjecture is a consequence of the conjecture on algebraic independence of logarithms.

Converse

Damien Roy: the conjecture on algebraic independence of logarithms is equivalent to the equality $\text{rank}(M) = \text{rstr}(M)$ for a matrix M with coefficients in the $\mathbb{Q}$ -space spanned by 1 and the logarithms of algebraic numbers.

Lemma.

Any polynomial in $\mathbb{Q}[X_1, \dots, X_n]$ is the determinant of some matrix with coefficients in $\mathbb{Q} + \mathbb{Q}X_1 + \dots + \mathbb{Q}X_n$.

The equality $\text{rank}(M) = \text{rstr}(M)$ for matrices with entries in $\mathcal{L}$ is equivalent to the conjecture on *homogeneous* algebraic independence of logarithms of algebraic numbers.

Lower bound for the rank (1981)

The best known lower bound for the rank of a square matrix with entries in the $\mathbb{Q}$ -vector space $\mathcal{L}$ is

$$\text{rank}(M) \geq \frac{1}{2} \text{rstr}(M).$$



Masser, D. W.

On polynomials and exponential polynomials in several complex variables. Invent. Math. 63, 81-95 (1981).

Zbl 0436.32005



Waldschmidt, Michel

Transcendance et exponentielles en plusieurs variables. Invent. Math. 63, 97-127 (1981).

Zbl 0454.10020



Roy, Damien

Points whose coordinates are logarithms of algebraic numbers on algebraic varieties. Acta Math. 175, No. 1, 49-73 (1995).

Zbl 0833.11031

Entries are linear forms in logarithms

This lower bound $\text{rank}(M) \geq \frac{1}{2}\text{rstr}(M)$ has been extended by Damien Roy to the matrices with entries in the $\overline{\mathbb{Q}}$ -vector space $\tilde{\mathcal{L}}$ spanned by 1 and $\mathcal{L}$.

The equality $\text{rank}(M) = \text{rstr}(M)$ for matrices with entries in $\tilde{\mathcal{L}}$ is equivalent to the conjecture on algebraic independence of logarithms of algebraic numbers.



Roy, Damien

Matrices with linear forms as coefficients.

Sémin. Théor. Nombres, Paris/Fr. 1987-88, Prog. Math. 81, 273-281 (1990).

Zbl 0712.11042



Roy, Damien

Matrices whose coefficients are linear forms in logarithms.

J. Number Theory 41, No. 1, 22-47 (1992).

Zbl 0763.11030

Strict lower bound

Even the tiny improvement

$$\text{rank}(M) > \frac{1}{2} \text{rstr}(M)$$

can be proved only under the (unlikely) assumption that the field generated by the coefficients has transcendence degree 1.

Example: **Four exponentials in transcendence degree 1:**

If K is a field of transcendence degree one, x_1, x_2 are $\mathbb{Q}$ -linearly independent elements of K and y_1, y_2 are also $\mathbb{Q}$ -linearly independent of K , then one at least of the four numbers

$$e^{x_1 y_1}, e^{x_1 y_2}, e^{x_2 y_1}, e^{x_2 y_2}$$

is transcendental.

One of the two numbers e^e , e^{e^2} is transcendental

Theorem.

Let x_1, x_2 be $\mathbb{Q}$ -linearly independent complex numbers and y_1, y_2 be also $\mathbb{Q}$ -linearly independent complex numbers. Assume that the two numbers $e^{x_1 y_1}$ and $e^{x_1 y_2}$ are algebraic. Then two at least of the 6 numbers

$$x_1, x_2, y_1, y_2, e^{x_2 y_1}, e^{x_2 y_2}$$

are algebraically independent.



Brownawell, Dale

Schneider's eighth problem. Proc. Number Theory Conf., Univ. Colorado, Boulder 20-24 (1972).

The algebraic independence of certain numbers related by the exponential function. J. Number Theory 6, 22-31 (1974).

Zb1 0324.10027 Zb1 0275.10020



Waldschmidt, Michel

Solution du huitième problème de Schneider.

J. Number Theory 5, 191-202 (1973).

Zb1 0262.10021

Sketch of proof of a lower bound for the rank

Let

$$M = (\ell_{ij})_{\substack{1 \leq i \leq d \\ 1 \leq j \leq \ell}}$$

be a $d \times \ell$ matrix with entries ℓ_{ij} logarithms of algebraic numbers:

$$\gamma_{ij} := e^{\ell_{ij}} \in \overline{\mathbb{Q}}.$$

Let n be the rank of the matrix M : there exists $x_1, \dots, x_d$ and $y_1, \dots, y_\ell$ in $\mathbb{C}^n$ satisfying

$$\ell_{ij} = \langle x_i, y_j \rangle \quad (1 \leq i \leq d, 1 \leq j \leq \ell),$$

with the standard scalar product in $\mathbb{C}^n$.

Set $Y := \mathbb{Z}y_1 + \dots + \mathbb{Z}y_\ell \subset \mathbb{C}^n$. The subgroup of $(\mathbb{C}^\times)^d$ generated by

$$\gamma_j := (\gamma_{1j}, \dots, \gamma_{dj}) \quad (j = 1, \dots, \ell)$$

is

$$\Gamma = \langle \gamma_1, \dots, \gamma_\ell \rangle = \{ (e^{x_1 y}, \dots, e^{x_d y}) \mid y \in Y \},$$

Transcendence construction

For S a positive integer, denote

$$\mathbb{Z}^\ell(S) = \{(s_1, \dots, s_\ell) \mid 0 \leq s_j < S \ (1 \leq j \leq \ell)\}$$

and

$$\Gamma(S) = \{\gamma_1^{s_1} \cdots \gamma_\ell^{s_\ell} \mid (s_1, \dots, s_\ell) \in \mathbb{Z}^\ell(S)\} \subset (\mathbb{C}^\times)^d.$$

Assume $n < \ell d / (\ell + d)$. Then, for sufficiently large N , there exists a nonzero polynomial $P \in \mathbb{C}[X_1, \dots, X_d]$ of total degree $< N^{\ell/d}$ which vanishes on the set $\Gamma(dN)$.

Constraints: for a polynomial of degree T , the system of equations with $\Gamma(S)$ is non trivial only if $S^\ell > T^d$. The transcendence argument requires $ST < S^{\ell/n}$. Hence the condition $n(\ell + d) < \ell d$.

Linear algebra: the first upper bound

Let γ_{ij} be nonzero complex numbers, $\gamma_j = (\gamma_{1j}, \dots, \gamma_{dj})$, $\Gamma = \langle \gamma_1, \dots, \gamma_\ell \rangle \subset (\mathbb{C}^\times)^d$. For S a positive integer, let

$$\Gamma(S) = \{\gamma_1^{s_1} \cdots \gamma_\ell^{s_\ell} \mid 0 \leq s_j < S \ (1 \leq j \leq \ell)\}.$$

Let $\omega(\Gamma(S))$ be the least degree of a nonzero polynomial vanishing on $\Gamma(S)$.

Linear algebra: if $T^d > S^\ell$, then there is a nonzero polynomial of partial degrees $< T$ vanishing on the S^ℓ elements of $\Gamma(S)$.

Hence

$$\omega(\Gamma(S)) \leq dS^{\ell/d}.$$

Linear algebra: refined upper bound

Example : assume $\gamma_{ij} = 1$ for $d_1 < i \leq d, 1 \leq j \leq \ell_1$. Then there exists $P \in \mathbb{C}[X_{d_1+1}, \dots, X_d]$ of degree

$$\leq (d - d_1)S^{(\ell - \ell_1)/(d - d_1)}$$

which vanishes on $\Gamma_1(S)$ where

$$\Gamma_1 = \langle \gamma_{\ell_1+1}, \dots, \gamma_\ell \rangle \subset \Gamma.$$

Since

$$\prod_{j=1}^{\ell} \gamma_{ij}^{s_j} = \prod_{j=\ell_1+1}^{\ell} \gamma_{ij}^{s_j} \quad \text{for } d_1 < i \leq d$$

it follows that P vanishes on $\Gamma(S)$. Hence

$$\omega(\Gamma(S)) \leq cS^{(\ell - \ell_1)/(d - d_1)}.$$

Upper bound

Let $G = (\mathbb{C}^\times)^d$ and $\Gamma = \langle \gamma_1, \dots, \gamma_\ell \rangle \subset G$ of rank ℓ .

When $\gamma_{ij} = 1$ for $d_1 < i \leq d$, $1 \leq j \leq \ell_1$, for $G_1 = (\mathbb{C}^\times)^{d_1} \times \{1\}^{d-d_1}$ and $\Gamma_1 = \langle \gamma_1, \dots, \gamma_{\ell_1} \rangle$, we have $\Gamma_1 \subset G_1$, $d_1 = \dim G_1$ and $\text{rank}_{\mathbb{Z}}(\Gamma \cap G_1) \geq \ell_1$.

More generally, if G_1 is an algebraic subgroup of G with $G_1 \neq G$, if ℓ_1 is the rank of $\Gamma \cap G_1$, then

$$\omega(\Gamma(S)) \leq cS^{(\ell-\ell_1)/(d-d_1)}.$$

Define

$$\mu(\Gamma) = \min_{G_1 \subsetneq G} \frac{\ell - \text{rank}_{\mathbb{Z}}(\Gamma \cap G_1)}{d - \dim G_1}.$$

Masser's Zero Estimate

Hence

$$\omega(\Gamma(S)) \leq c_1 S^{\mu(\Gamma)}$$

Theorem (Masser's Zero Estimate).

$$\omega(\Gamma(S)) \geq c_2 S^{\mu(\Gamma)}.$$



Masser, D. W.

On polynomials and exponential polynomials in several complex variables. Invent. Math. 63, 81-95 (1981).

Zbl 0436.32005

Conclusion

If a $d \times \ell$ matrix M with entries logarithms of algebraic numbers has rank $< \ell d / (d + \ell)$, then there exist two regular matrices P and Q with rational coefficients such that

$$PMQ = \underbrace{\begin{pmatrix} M_1 & 0 \\ M_2 & M_3 \end{pmatrix}}_{\ell_1} \underbrace{\begin{matrix} \} \\ \} \end{matrix}}_{\ell - \ell_1} \begin{matrix} d_1 \\ d - d_1 \end{matrix}$$

where M_1 is a $d_1 \times \ell_1$ matrix of rank n_1 with

$$\frac{d_1}{n_1} > \frac{d}{n} \quad \text{and} \quad \ell_1 d_1 \leq n_1 (\ell_1 + d_1).$$

Square case:

A matrix M with entries logarithms of algebraic numbers has rank $\geq \text{rstr}(M)/2$.

The matrix coefficient conjecture ($\ell = d$)

Let M be a square $d \times d$ matrix with entries logarithms of algebraic numbers and zero determinant. Then there exist u and v in $\mathbb{Q}^d \setminus \{0\}$ such that

$$(u_1 \quad \cdots \quad u_d) M \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix} = 0$$



Dasgupta, Samit

Ranks of matrices of logarithms of algebraic numbers. I: The theorems of Baker and Waldschmidt-Masser. Essent. Number Theory 2, No. 1, 93-138 (2023).

Zbl 1543.11059



Dasgupta, Samit; Kakde, Mahesh

Ranks of matrices of logarithms of algebraic numbers. II: The matrix coefficient conjecture. Adv. Math. 487, Article ID 110753, 19 p. (2026).

Zbl 08155756

The matrix coefficient conjecture

- In the case of a square 2×2 matrix, the Matrix Coefficient Conjecture is equivalent to the Four Exponentials Conjecture.
- The Matrix Coefficient Conjecture is a consequence of the Structural Rank Conjecture.
- The Matrix Coefficient Conjecture (p -adic version) implies Leopoldt's conjecture (and the Gross–Kuz'min conjecture).

New tool: Instead of the degree, Samit Dasgupta and Mahesh Kakde introduce the number of nonzero coefficients of the polynomial. They replace Masser's Zero estimate with a result involving a theorem of Bernstein – Kushnirenko.

Bernstein – Kushnirenko number of a polynomial

For a polynomial $P \in \mathbb{C}[X_1, \dots, X_d] \setminus \{0\}$, we denote by $S(P)$ the support of f , i.e. the set of tuples $(a_1, \dots, a_d) \in \mathbb{Z}^d$ such that the coefficient of $X_1^{a_1} \cdots X_d^{a_d}$ in P is nonzero.

Let $\Delta(P) \subset \mathbb{R}^d$ denote the convex hull of $S(P)$.

The Bernstein – Kushnirenko number of P is defined to be $\text{BK}(P) := d! \text{Vol}(\Delta(P))$. More generally, given $f_1, \dots, f_d \in \mathbb{C}[X_1, \dots, X_d]$, we define

$$\text{BK}(f_1, \dots, f_d) = d! \text{Vol}(\Delta(f_1), \dots, \Delta(f_d))$$

where the right side denotes mixed volume.

Bernstein – Kushnirenko Theorem and degree

Theorem (Bernstein – Kushnirenko).

Let $f_1, \dots, f_d \in \mathbb{C}[X_1, \dots, X_d]$ have finitely many common zeroes in $(\mathbb{C}^\times)^d$. Then the number of common zeroes of the f_i in $(\mathbb{C}^\times)^d$ is bounded by $\text{BK}(f_1, \dots, f_d)$.

Let $P \in \mathbb{C}[X_1, \dots, X_d]$, $P \neq 0$. Define the BK-degree of P , denoted $\text{BKd}(P)$, to be the maximum as $i = 1, \dots, d$ of $\text{BK}(P, \dots, P, \ell, \dots, \ell)$, where ℓ denotes a generic linear polynomial and there are i copies of P and $(d - i)$ copies of ℓ .

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The two following properties are equivalent :

- (i) For N large enough, there exists a nonzero polynomial $P(X_1, \dots, X_d)$ such that $\text{BKd}(P) < N^d$ and $P(\gamma) = 0$ for all $\gamma \in \Gamma(dN)$.
- (ii) There are nonzero elements s, t in $\mathbb{Z}^d$ such that

$$\prod_{i=1}^d \prod_{j=1}^d \gamma_{ij}^{s_i t_j} = 1.$$

In order to prove the Four Exponentials Conjecture, it would be sufficient to prove that given a rank one 2×2 matrix M with entries logarithms of algebraic numbers, for N large enough, there exists a polynomial $P(X_1, X_2)$ such that $|S(P)| < N^2$ and $P(x) = 0$ for all $x \in X(2N)$.

The Algebraic Matrix Coefficient Conjecture

Damien Roy generalized the results on the rank of matrices with entries in the $\mathbb{Q}$ -vector space $\mathcal{L}$ to the coefficients in the $\overline{\mathbb{Q}}$ -vector space spanned by 1 and $\mathcal{L}$

In this direction, Samit Dasgupta and Mahesh Kakde deduce from the Structural Rank Conjecture (consequence of the conjecture on algebraic independence of logarithms) the following statement

Conjecture (The Algebraic Matrix Coefficient Conjecture).

Let M be a square matrix with zero determinant with entries in the $\overline{\mathbb{Q}}$ -vector space spanned by $\mathcal{L}$. Then there exist u and v in $\overline{\mathbb{Q}}^d \setminus \{0\}$ such that $\langle u, Mv \rangle = 0$.

References: Strategies of transcendence proofs

<http://www.imj-prg.fr/~michel.waldschmidt/articles/pdf/StrategyTrancendence.pdf>

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1. The method of Gel'fond – Schneider
2. Several variables
3. Algebraic independence. Modular functions. Mahler's method
4. Schanuel's Conjecture

Slides: 179 p

Four videos on youtube.

<https://www.youtube.com/playlist?list=PLVOHgkNxjZIo>

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Charles Hermite

1822 – 1901



Ferdinand von Lindemann

1852 – 1939



Rodion Kuz'min

1891 – 1949



Heinrich-Wolfgang Leopoldt

1927 - 2011



Stephen Schanuel

1933 – 2014



Alan Baker

1939 – 2018



Benedict Gross

1950 – 2025



W. Dale Brownawell



David Masser



Damien Roy



Samit Dasgupta



Mahesh Kakde

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The Matrix Coefficient Conjecture of Dasgupta and Kakde

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